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Berezin-Toeplitz quantization and applications

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Résumé

Dans cette thèse, nous étudions la quantification de Berezin-Toeplitz des variétés symplectiques préquantifiées en général, se concentrant sur deux généralisations naturelles du cas Kählerien. Nous considérons la quantification d'un certain nombre d'objets classiques associés à la variété symplectique sous-jacente et étudions leurs propriétés asymptotiques à la limite semi-classique, lorsque la puissance tensorielle du fibré préquantifiant tend vers l'infini.

Dans le premier chapitre, nous étudions la quantification de Berezin-Toeplitz en utilisant comme espace quantique le kernel de l'opérateur de Dirac spin^c, dans lequel les états quantiques peuvent être des combinaisons de formes de tout degré. Nous calculons le second coefficient de l'expansion asymptotique de la composition de deux opérateurs de Berezin-Toeplitz, exhibant la même formule que dans le cas Kählerien.

Dans le second chapitre, nous étudions la quantification de Berezin-Toeplitz en utilisant comme espace quantique les états propres du laplacien de Bochner renormalisé correspondant aux valeurs propres localisées près de l'origine. Nous montrons que cette quantification possède le comportement semi-classique attendu, établissant en particulier l'expansion asymptotique de la composition de deux opérateurs de Berezin-Toeplitz. Nous calculons ensuite le second coefficient de cette expansion en suivant la méthode du premier chapitre.

Dans le troisième chapite, nous étudions les états quantiques associés aux sous-variétés isotropes d'une variété symplectique préquantifiée dans le contexte de la quantification de Berezin-Toeplitz établie au second chapitre, établissant leurs propriétés semi-classiques. Nous montrons ensuite comment ces résultats s'étendent au cas des variétés orbifoldes complètes, puis donnons une application aux séries de Poincaré relatives en théorie des formes automorphes.

Mots-clefs : Quantification de Berezin-Toeplitz, limite semi-classique, expansion asymptotique, noyau de Bergman généralisé, sous-variétés de Bohr-Sommerfeld.

Abstract

In this thesis, we study the Berezin-Toeplitz quantization of general prequantified symplectic manifolds, focusing on two natural generalisations of the Kähler case. We consider the quantum counterpart of classical objects associated with the underlying symplectic manifold and study their asymptotic properties at the semi-classical limit, when the tensor power of the prequantifying line bundle tends to infinity.

In the first chapter, we study Berezin-Toeplitz quantization using as quantum space the kernel of the spin^c Dirac operator, in which quantum states can be a combination of forms of any degree. We compute the second coefficient of the asymptotic expansion of the composition of two Berezin-Toeplitz operators, exhibiting the same formula as in the Kähler case.

In the second chapter, we study Berezin-Toeplitz quantization using as quantum space the eigenstates of the renormalized Bochner Laplacian corresponding to eigenvalues localized near the origin. We show that this quantization has the correct semiclassical behavior, establishing in particular the asymptotic expansion of the composition of two Berezin-Toeplitz operators. We then compute the second coefficient of this expansion following the method of the first chapter.

In the third chapter, we study the quantum states associated with isotropic submanifolds of a prequantified symplectic manifold in the context of the Berezin-Toeplitz quantization established in the second chapter, and study their semi-classical properties. We then show how these results extend to the case of complete orbifolds, and we give an application to relative Poincaré series in the theory of automorphic forms.

Key words: Berezin-Toeplitz quantization, semi-classical limit, asymptotic expansion, generalized Bergman kernel, Bohr-Sommerfeld submanifolds.

Contents

Introduction

L'objectif de la quantification est d'associer à un espace de phases classique, ici une variété symplectique (X, ω), un espace de Hilbert d'états quantiques $\mathscr{H}$. Cette association doit faire correspondre l'espace des observables classiques, ici l'espace de fonctions $\mathscr{C}^\infty(X)$, avec l'espace $B(\mathscr{H})$ des opérateurs bornés de $\mathscr{H}$. En *quantification géométrique*, on suppose de plus que (X, ω) est muni d'un fibré en droite hermitien (L, h^L) avec connexion hermitienne ∇^L satisfaisant la condition suivante, dite de *préquantification*,

$$\omega = \frac{\sqrt{-1}}{2\pi} R^L, \tag{0.0.1}$$

où $R^L = (\nabla^L)^2$ est la courbure de ∇^L. On dira que (L, h^L, ∇^L) est *préquantifiant*.

Si (X, ω) est compacte et munie d'une structure complexe J compatible avec ω, faisant de (X, J, ω) une *variété Kählerienne*, et si (L, h^L) est holomorphe tel que sa connexion de Chern ∇^L (c'est à dire l'unique connexion hermitienne compatible avec sa structure holomorphe) satisfasse (0.0.1), alors la *quantification holomorphe* de (X, J) associée à (L, h^L) est l'espace $\mathscr{H} = H^0(X, L)$ des sections holomorphes de L, muni du produit hermitien L^2 associé. On considère plus généralement les espaces $\mathscr{H}_p = H^0(X, L^p)$ de quantification holomorphe de (X, J) associés à la p-ième puissance tensorielle (L^p, h^{L^p}) de (L, h^L), pour tout $p \in \mathbb{N}^*$. Dans ce contexte, un résultat asymptotique lorsque p tend vers l'infini décrit la *limite semi-classique*, lorsque l'échelle devient si grande que l'on récupère les lois de la mécanique classique à partir des lois de la mécanique quantique. En particulier, on veut retrouver la structure d'algèbre de Poisson canonique sur l'espace des observables classiques $\mathscr{C}^\infty(X)$ à partir de la structure d'algèbre de l'espace des observables quantiques $B(\mathscr{H}_p)$ lorsque p tend vers l'infini, faisant correspondre la multiplication de fonctions avec la composition d'opérateurs et le crochet de Poisson induit par la structure symplectique de (X, ω) sur les fonctions avec le crochet de Lie usuel des opérateurs.

Soit g^{TX} la *métrique de Kähler* de (X, J, ω), définie sur TX par $g^{TX}(\cdot, \cdot) = \omega(\cdot, J\cdot)$, et soit dv_X la forme volume induite. Pour tout $p \in \mathbb{N}^*$, on note P_p la projection orthogonale de $\mathscr{C}^\infty(X, L^p)$ sur $\mathscr{H}_p$ pour le *produit hermitien* L^2, noté $\langle \cdot, \cdot \rangle_p$ et défini pour tout $s_1,\ s_2 \in \mathscr{C}^\infty(X, L^p)$ par

$$\langle s_1, s_2 \rangle_p = \int_X \langle s_1(x), s_2(x) \rangle_{L^p} dv_X(x), \tag{0.0.2}$$

où $\langle \cdot, \cdot \rangle_{L^p}$ est le produit hermitien associé à h^{L^p}. Pour tout $f \in \mathscr{C}^\infty(X, \mathbb{C})$, la *quantifi-*

cation de Berezin-Toeplitz de f est la famille $\{T_{f,p} \in \mathrm{End}(\mathscr{H}_p)\}_{p\in\mathbb{N}^*}$ définie sur l'espace des sections lisses de L^p pour tout $p \in \mathbb{N}^*$ par

$$T_{f,p} = P_p f P_p : \mathscr{C}^\infty(X, L^p) \to \mathscr{C}^\infty(X, L^p), \tag{0.0.3}$$

où f est l'opérateur de multiplication par la fonction f. On a alors la formule asymptotique suivante, pour tout $f, g \in \mathscr{C}^\infty(X, \mathbb{C})$ et lorsque $p \to \infty$ au sens de la norme d'opérateur induite par $\langle\cdot,\cdot\rangle_p$ sur $B(\mathscr{H}_p)$,

$$T_{f,p}T_{g,p} = \sum_{r=0}^{\infty} p^{-r} T_{C_r(f,g),p} + O(p^{-\infty}), \tag{0.0.4}$$

où les C_r sont des opérateurs bidifférentiels pour tout $r \in \mathbb{N}$, avec $C_0(f,g) = fg$. En particulier, on a

$$T_{f,p}T_{g,p} = T_{fg,p} + O(p^{-1}), \tag{0.0.5}$$

au sens de la norme d'opérateur, ce qui montre que la composition des quantifications de f et g tend vers la quantification de leur produit fg à la limite semi-classique. De plus, si $\{\cdot,\cdot\}$ est le produit de Poisson associé à la forme symplectique $2\pi\omega$, on a

$$[T_{f,p}, T_{g,p}] = \sqrt{-1}p^{-1}T_{\{f,g\},p} + O(p^{-2}). \tag{0.0.6}$$

On voit ainsi que la quantification de Berezin-Toeplitz satisfait les conditions requises pour une quantification à la limite semi-classique.

Les formules (0.0.4)-(0.0.6) ont été établies par Bordemann, Meinrenken et Schlichenmaier [10], [54] en utilisant les travaux de Boutet de Monvel et Sjöstrand [15] sur le noyau de Szegö et ceux de Boutet de Monvel et Guillemin [14] sur les structures de Toeplitz. Une vaste généralisation de ce programme a été établie par Ma et Marinescu [51], en utilisant les techniques de localisation analytique de Bismut-Lebeau inspirées de la théorie de l'indice local. Les travaux présentés dans cette thèse s'inscrivent dans ce cadre. Ils sont séparés en trois chapitres, et nous donnons ci-dessous une introduction détaillée pour chacun d'entre eux.

0.1 Composition d'opérateurs de Berezin-Toeplitz pour l'opérateur de Dirac spin^c

Cette partie de ma thèse est contenue dans [35].

Soit (X,ω) une variété symplectique compacte de dimension $2n$ et soit (L, h^L) un fibré en droite hermitien au-dessus de X, muni d'une connexion hermitienne ∇^L satisfaisant (0.0.1). On choisit une structure presque complexe J compatible avec ω ainsi qu'une métrique riemannienne J-invariante g^{TX} sur le fibré tangent TX de X, et on note ∇^{TX} la connexion de Levi-Civita associée. On note $TX_\mathbb{C} = TX \otimes_\mathbb{R} \mathbb{C}$ la complexification de TX. La structure presque complexe J induit une décomposition

$$TX_\mathbb{C} = T^{(1,0)}X \oplus T^{(0,1)}X \tag{0.1.1}$$

en espaces propres de J associés aux valeurs propres $\sqrt{-1}$ et $-\sqrt{-1}$. On note $\Lambda(T^{*(0,1)}X)$ le produit extérieur total du dual $T^{*(0,1)}X$ de $T^{(1,0)}X$. Pour tout $x \in X$, on définit alors un endomorphisme hermitien positif $\dot{R}^L_x \in \mathrm{End}(T^{(1,0)}_x X)$ pour tout $v, w \in T^{(1,0)}_x X$ par la formule

$$g^{TX}(\dot{R}^L_x v, \overline{w}) = R^L(v, \overline{w}). \tag{0.1.2}$$

Pour tout $x \in X, v \in T_x X$, et en notant $v = v^{(1,0)} + v^{(0,1)}$ la décomposition de v selon (0.1.1), son *action de Clifford* $c(v) \in \mathrm{End}(\Lambda(T^{*(0,1)}X))_x$ est définie par

$$c(v) = \sqrt{2}(v^{(1,0),*} - i_{v^{(0,1)}}), \tag{0.1.3}$$

où on note $v^{(1,0),*}$ le produit extérieur par le dual métrique de $v^{(1,0)}$ dans $T^{*(0,1)}X \simeq T^{(1,0)}X$, et $i_{v^{(0,1)}}$ la contraction par $v^{(0,1)} \in T^{(0,1)}X$. Soit $\nabla^{T^{(1,0)}X}$ la connexion sur $T^{(1,0)}X$ définie par $\nabla^{T^{(1,0)}X} = P^{T^{(1,0)}X}\nabla^{TX}$, où $P^{T^{(1,0)}X}$ est la projection de $TX_{\mathbb{C}}$ sur $T^{(1,0)}X$ via (0.1.1), et soit $\nabla^{\det}$ la connexion induite par $\nabla^{T^{(1,0)}X}$ sur $\det(T^{(1,0)}X)$. On note ∇^{Cl} la *connexion de Clifford* sur $\Lambda(T^{*(0,1)}X)$ associée à ∇^{TX} et $\nabla^{\det}$ définie comme dans [49, (1.3.5)] par

$$\nabla^{Cl} = d + \frac{1}{4}\sum_{i,j=1}^{2n} g^{TX}\left(\Gamma^{TX}e_i, e_j\right) c(e_i)c(e_j) + \frac{1}{2}\Gamma^{det}, \tag{0.1.4}$$

où $\{e_i\}_{i=1}^{2n}$ est une base orthonormée locale de TX et $\Gamma^{TX}, \Gamma^{det}$ sont les formes de connexions de $\nabla^{TX}, \nabla^{det}$ dans une carte locale.

Soit (E, h^E) un fibré vectoriel hermitien auxiliaire au-dessus de X muni d'une connexion hermitienne ∇^E. Pour tout $p \in \mathbb{N}^*$, on définit

$$\mathbf{E}_p = L^p \otimes \Lambda(T^{*(0,1)}X) \otimes E, \tag{0.1.5}$$

muni de la métrique $h^{\mathbf{E}_p}$ et de la connexion $\nabla^{\mathbf{E}_p}$ induites par h^L, g^{TX}, h^E et ∇^L, ∇^{Cl}, ∇^E. On définit alors *l'opérateur de Dirac spin*^{c} D^c_p par la formule

$$D^c_p = \sum_{j=1}^{2n} c(e_j)\nabla^{\mathbf{E}_p}_{e_j} : \mathscr{C}^\infty(X, \mathbf{E}_p) \to \mathscr{C}^\infty(X, \mathbf{E}_p), \tag{0.1.6}$$

où $\{e_j\}_{j=1}^{2n}$ est une base orthonomale locale de TX pour g^{TX}. En particulier, D^c_p est un opérateur différentiel elliptique d'ordre 1 sur $\mathscr{C}^\infty(X, \mathbf{E}_p)$, formellement auto-adjoint pour le produit hermitien L^2 défini comme en (0.0.2) avec $h^{\mathbf{E}_p}$ au lieu de h^{L^p}. La théorie des opérateurs elliptiques nous dit alors que $\dim \mathrm{Ker}(D^c_p) < +\infty$.

En fait, d'après un résultat de Ma et Marinescu [47, Th.2.5], le *théorème de l'indice d'Atiyah-Singer* (voir [49, Th.1.3.9]) nous donne la formule de *Riemann-Roch-Hirzebruch* suivante, pour tout $p \in \mathbb{N}$ suffisament grand,

$$\dim \mathrm{Ker}(D^c_p) = \int_X \mathrm{Td}(T^{(1,0)}X)\,\mathrm{ch}(E)\exp(p\omega), \tag{0.1.7}$$

où $\mathrm{Td}(T^{(1,0)}X)$ représente la classe de Todd de $T^{(1,0)}X$ et $\mathrm{ch}(E)$ représente le caractère de Chern de E, comme expliqué par exemple dans [47, § 1.3.4]. Ainsi, la dimension de

$\mathrm{Ker}(D^c_p)$ est en fait donnée par un polynôme de degré n en p, pour tout $p \in \mathbb{N}$ suffisament grand, dont les coefficients sont explicitement calculables en termes d'invariants topologiques de X, E et L.

Dans le cas où J est intégrable, faisant de (X, J, ω) une variété Kählerienne, et si $(L, h^L), (E, h^E)$ sont holomorphes munis de leur connexion de Chern ∇^L, ∇^E, l'opérateur de Dirac spin^c devient

$$D^c_p = \sqrt{2}(\overline{\partial}^{L^p\otimes E} + \overline{\partial}^{L^p\otimes E,*}), \tag{0.1.8}$$

où $\overline{\partial}^{L^p\otimes E}$ est l'opérateur $\overline{\partial}$ holomorphe de $L^p \otimes E$ agissant sur le complexe de Dolbeault $\oplus_{q=1}^n \Omega^{0,q}(X, L^p \otimes E) = \mathscr{C}^\infty(X, \mathbf{E}_p)$ et $\overline{\partial}^{L^p\otimes E,*}$ est son adjoint formel pour le produit hermitien L^2. En particulier, son carré $(D^c_p)^2$ préserve le degré, et coincide avec 2 fois le *Laplacien de Kodaira* de $L^p \otimes E$. D'après la théorie de Hodge, on a un isomorphisme naturel

$$\mathrm{Ker}(D^c_p) \simeq \oplus_{q=1}^n H^q(X, L^p \otimes E), \tag{0.1.9}$$

où $H^q(X, L^p \otimes E)$ est le q-ième groupe de cohomologie de Dolbeault associé à $L^p \otimes E$ pour tout $1 \leqslant q \leqslant n$. Par (0.0.1), (L, h^L) est positif, et le théorème d'annulation de Kodaira-Serre implique

$$H^q(X, L^p \otimes E) = 0, \tag{0.1.10}$$

pour tout $q > 0$ et pour tout p suffisament grand. Par l'identification (0.1.9) et l'annulation (0.1.10), on a donc $\mathrm{Ker}(D^c_p) = H^0(X, L^p \otimes E)$ pour tout p suffisament grand. On voit ainsi que le noyau $\mathrm{Ker}(D^c_p)$ de l'opérateur de Dirac spin^c généralise les espaces de quantification holomorphe usuels au cas symplectique. On voit de plus que cette généralisation inclut le cas d'une métrique riemanienne J-invariante g^{TX} quelconque, ainsi que le cas où L^p est tordu par un fibré auxiliaire E. Dans cette partie, on considère ainsi les espaces de quantification $\mathscr{H}_p := \mathrm{Ker}(D^c_p)$, pour tout $p \in \mathbb{N}^*$.

On note encore P_p la projection orthogonale de $\mathscr{C}^\infty(X, \mathbf{E}_p)$ sur $\mathrm{Ker}(D^c_p)$ par rapport au produit hermitien L^2. Par stricte analogie avec le cas Kählerien, on peut alors définir la quantification de Berezin-Toeplitz d'un endomorphisme $f \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ de E par la formule

$$T_{f,p} = P_p f P_p : \mathscr{C}^\infty(X, \mathbf{E}_p) \to \mathscr{C}^\infty(X, \mathbf{E}_p), \tag{0.1.11}$$

où f est cette fois l'opérateur donné par l'application de f sur E. Ma et Marinescu [51] montrent alors le résultat suivant.

Theorem 0.1.1. *[51, Th.1.1] Soit (X, J, ω) une variété symplectique compacte, soient J une structure presque complexe compatible avec ω et g^{TX} une métrique Riemanienne J-invariante sur TX. On suppose (X, ω) munie de (L, h^L, ∇^L) préquantifiant, et on considère un fibré auxiliaire (E, h^E, ∇^E) comme ci-dessus. Alors pour tout $f, g \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ et pour tout $k \in \mathbb{N}$, il existe $C_k > 0$ tel que pour tout $p \in \mathbb{N}$,*

$$\left\| T_{f,p}T_{g,p} - \sum_{r=0}^{k} p^{-r} T_{C_r(f,g),p} \right\|_p < C_k p^{-k}, \tag{0.1.12}$$

où $\|\cdot\|_p$ *est la norme d'opérateur induite par* $\langle\cdot,\cdot\rangle_p$ *sur les endomorphismes de* $\mathbf{E}_p$ *et où les* C_r *sont des opérateurs bidifférentiels sur* $\mathscr{C}^\infty(X, \mathrm{End}(E))$ *pour tout* $r \in \mathbb{N}$*, avec* $C_0(f,g)$ *égal à la composition de* f *avec* g*.*

De plus, si $f, g \in \mathscr{C}^\infty(X, \mathbb{C})$*, on a*

$$[T_{f,p}, T_{g,p}] = \sqrt{-1}p^{-1}T_{\{f,g\},p} + O(p^{-2}), \tag{0.1.13}$$

au sens de la norme d'opérateur lorsque $p \to \infty$*, où* $\{f,g\} \in \mathscr{C}^\infty(X, \mathbb{C})$ *est le crochet de Poisson associé à* $2\pi\omega$ *de* f *et* g*, agissant sur* E *par multiplication scalaire.*

D'autre part, dans le cas où (X, J, ω, g^{TX}) est Kählerienne et $(L, h^L), (E, h^E)$ sont munis de leur connexion de Chern, Ma et Marinescu [52] précisent (0.1.13) en calculant le deuxième coefficient de l'expansion (0.1.12).

Le résultat principal du chapitre 1 est le théorème 1.1.1, où je calcule le deuxième coefficient de l'expansion (0.1.12) dans le cas général. Soient $\nabla^{1,0}$ et $\nabla^{0,1}$ les projections sur $T^{(1,0)}X$ et $T^{(0,1)}X$ via (0.1.1) de la connexion sur $\mathrm{End}(E)$ induite par ∇^E, et soit $\langle.,.\rangle$ l'accouplement induit par g^{TX} sur $T^*X \otimes \mathrm{End}(E)$ à valeurs dans $\mathrm{End}(E)$.

Theorem 0.1.1. *Supposons que* $g^{TX}(\cdot,\cdot) = \omega(\cdot, J\cdot)$*. Pour tout* $f, g \in \mathscr{C}^\infty(X, \mathrm{End}(E))$*, on a la formule suivante pour le second coefficient* $C_1(f,g)$ *dans l'expansion asymptotique* (0.1.12) *du théorème 0.1.1,*

$$C_1(f,g) = -\frac{1}{2\pi}\langle \nabla^{1,0}f, \nabla^{0,1}g\rangle. \tag{0.1.14}$$

Dans le cas où $f, g \in \mathscr{C}^\infty(X, \mathbb{C})$, le théorème 0.1.1 implique

$$C_1(f,g) - C_1(g,f) = \sqrt{-1}\{f,g\}. \tag{0.1.15}$$

Via (0.1.12), on retrouve ainsi la formule (0.1.13).

L'outil fondamental à la base de ce résultat, et plus généralement à la base des travaux de Ma et Marinescu sur la quantification de Berezin-Toeplitz, est le noyau de Schwartz de la projection orthogonale P_p sur $\mathrm{Ker}(D_p^c)$ par rapport à la forme volume Riemannienne dv_X de (X, g^{TX}), appelé *noyau de Bergman* et noté $P_p(x,y) \in E_{p,x} \otimes E_{p,y}^*$, $x, y \in X$. Comme $\dim \mathrm{Ker}(D_p^c) < +\infty$ et comme P_p est auto-adjoint, celui-ci est lisse en $x, y \in X$. De plus, il décroit rapidement lorsque p tend vers l'infini (c'est à dire plus vite que tout polynôme en p^{-1}) à l'extérieur de tout voisinage de la diagonale, et d'après un résultat de Dai, Liu et Ma [20], il admet une expansion asymptotique dans un voisinage de la diagonale, que nous décrivons ci-dessous.

Choisissons $x_0 \in X$ et $\varepsilon > 0$ suffisament petit. On se place dans des coordonnées normales de rayon ε autour de x_0, dans lesquelles on trivialise toutes les données fibrées par transport parallèle le long des rayons géodésiques. On identifie ensuite L avec $\mathbb{C}$ par le choix d'un vecteur unitaire de L_{x_0}. En particulier, les fibrés considérés ne dépendent plus de $p \in \mathbb{N}$, cette dépendance étant transférée sur la forme des opérateurs dans ces coordonnées. Spécifiquement, on peut montrer (voir [20, Th.4.6]) que la restriction à ces coordonnées du carré $(D_p^c)^2$ de l'opérateur de Dirac spin^c est égal, après un changement d'échelle adéquat en $\sqrt{p} := 1/t$, à un opérateur $\mathscr{L}_t$ satisfaisant

$$\mathscr{L}_t = \mathscr{L}_0 + \sum_{k=1}^{m} t^k \mathcal{O}_k + \mathscr{O}(t^{m+1}), \tag{0.1.16}$$

pour tout $m \in \mathbb{N}$, où $\{\mathcal{O}_k\}_{k\in\mathbb{N}^*}$ est une famille d'opérateurs différentiels d'ordre 2, dont les coefficients sont calculables explicitement en terme de données locales, et où les coefficients de l'opérateur différentiel $\mathscr{O}(t^{m+1})$, ainsi que ses dérivées d'ordre inférieur ou égal à l, sont majorés par $C_l t^{m+1}$, pour tout $l \in \mathbb{N}$.

Soit $\{w_j\}_{j=1}^n$ une base orthonormale de $T^{(1,0)}X$ qui diagonalise $\dot{R}^L_{x_0}$, de sorte que dans cette base,

$$\dot{R}^L_{x_0} = \operatorname{diag}(a_1(x_0), \dots a_n(x_0)) \in \operatorname{End}(T^{(1,0)}_{x_0}X) \tag{0.1.17}$$

avec $a_j := a_j(x_0) > 0$ pour tout $1 \leqslant j \leqslant n$. On identifie $T_{x_0}X$ avec $\mathbb{R}^{2n}$ via la base orthonormale définie par

$$e_{2j-1} = \frac{1}{\sqrt{2}}(w_j + \overline{w}_j) \quad \text{et} \quad e_{2j} = \frac{\sqrt{-1}}{\sqrt{2}}(w_j - \overline{w}_j), \tag{0.1.18}$$

pour tout $1 \leqslant j \leqslant n$. On note $Z = (Z_1, \dots, Z_{2n}) \in \mathbb{R}^{2n}$ les coordonnées réelles induites, et $z = (z_1, \dots, z_n)$ les coordonnées complexes sur $\mathbb{C}^n$ telles que $z_j = Z_{2j-1} + \sqrt{-1}Z_{2j}$ pour tout $1 \leqslant j \leqslant n$.

Dans cette situation, on a une expression explicite pour

$$\mathscr{L}_0 = \mathscr{L} + \sum_{j=1}^{n} 2a_j \overline{w}_j^* \iota_{\overline{w}_j} \tag{0.1.19}$$

agissant sur $L^2(\mathbb{R}^{2n}, \Lambda(T^{*(0,1)}X)_{x_0} \otimes E_{x_0})$, où $\mathscr{L}$ agit sur $L^2(\mathbb{R}^{2n})$. L'*opérateur modèle* $\mathscr{L}|_{L^2(\mathbb{R}^{2n})}$ prend la forme d'une version complexifiée de l'oscillateur harmonique. Il admet un spectre discret, et on peut calculer explicitement une base de fonctions propres. En particulier, une base orthogonale de fonctions propres de $\operatorname{Ker}(\mathscr{L}|_{L^2(\mathbb{R}^n)})$ est donnée par

$$z^\beta \exp\left(-\frac{1}{4}\sum_{j=1}^{n} a_j |z_j|^2\right), \ \beta \in \mathbb{N}^n. \tag{0.1.20}$$

Ceci permet de calculer le noyau de Schwartz de la projection orthogonale $\mathscr{P} : L^2(\mathbb{R}^{2n}) \to \operatorname{Ker}(\mathscr{L})$, qui s'écrit

$$\mathscr{P}(Z, Z') = \prod_{i=1}^{n} \frac{a_i}{2\pi} \exp\left(-\frac{1}{4}\sum_{j=1}^{n} a_j(|z_j|^2 + |z'_j|^2 - 2z_j\bar{z}_j{}')\right). \tag{0.1.21}$$

D'autre part, $\mathscr{L}$ et $\sum_{j=1}^n 2a_j \overline{w}_j^* \iota_{\overline{w}_j}$ sont tous deux positifs, et le noyau de ce dernier est réduit à $\mathbb{C} \subset \Lambda(T^{(0,1)}X)_{x_0}$, d'où $\operatorname{Ker}(\mathscr{L}_0) = \operatorname{Ker}(\mathscr{L}) \otimes \mathbb{C} \otimes E_{x_0}$. On en déduit que le noyau de Schwartz de la projection orthogonale de $L^2(\mathbb{R}^{2n}, \Lambda(T^{(0,1)}X)_{x_0} \otimes E_{x_0})$ vers

$\mathrm{Ker}(\mathscr{L}_0)$ s'écrit $\mathscr{P}(Z,Z')I_{\mathbb{C}\otimes E}$, où $I_{\mathbb{C}\otimes E}$ est la projection canonique de $\Lambda(T^{*(0,1)}X)\otimes E$ sur $\mathbb{C}\otimes E$.

Le résultat de Dai, Liu et Ma [20, Th.4.18'] énonce alors que le noyau de Bergman, que l'on note dans cette trivialisation locale $P_p(Z,Z')$ pour $|Z|,|Z'|<\varepsilon$, satisfait une expansion asymptotique de la forme

$$P_p(Z,Z') \cong \sum_{r=0}^{+\infty} J_r(\sqrt{p}Z,\sqrt{p}Z')\mathscr{P}(\sqrt{p}Z,\sqrt{p}Z') + \mathscr{O}(p^{-\infty}), \tag{0.1.22}$$

uniforme en $x_0 \in X$, avec $J_0(Z,Z') \equiv I_{\mathbb{C}\otimes E}$ et où les $J_r(Z,Z')$, $r\in\mathbb{N}$, sont des polynômes de même parité que r en $Z,Z'\in\mathbb{R}^{2n}$ à valeurs dans $\mathrm{End}(\Lambda(T^{*(0,1)}X)\otimes E)_{x_0}$, explicitement calculables en fonction des coefficients des opérateurs $\mathcal{O}_k$, $k\leqslant r$, apparaissant dans (0.1.16) (la forme exacte de l'expansion (0.1.22) est décrite en Section 1.3.1).

Cette expansion est suffisament précise pour permettre alors de déduire que si deux opérateurs satisfont une telle expansion et décroissent rapidement en $p\to\infty$ en dehors de tout voisinage de la diagonale, alors leur composition satisfait les mêmes propriétés. Plus précisément, étant donnés deux opérateurs bornés T_1 et T_2 admettant des noyaux lisses $T_1(\cdot,\cdot), T_2(\cdot,\cdot)$, on utilise la formule suivante pour le noyau $T_1T_2(\cdot,\cdot)$ de leur composition, évalué en $x,y\in X$,

$$T_1T_2(x,y) = \int_X T_1(x,w)T_2(w,y)dv_X(w). \tag{0.1.23}$$

On en déduit immédiatement la décroissance rapide de $T_1T_2(x,y)$ en dehors de tout voisinage de la diagonale, et seule l'intégrale sur une boule arbitrairement petite autour de x_0 contribue à l'expansion asymptotique. On peut alors utiliser l'expansion (0.1.22) et la décroissance rapide de (0.1.21) en $p\to\infty$ hors de la diagonale pour ramener l'intégrale (0.1.23) à des intégrales sur $\mathbb{C}^n$, et celles ci peuvent être calculées explicitement. En effet, pour tout $F(Z,Z'), G(Z,Z')$ polynômes en $Z,Z'\in\mathbb{R}^{2n}$, il existe $\mathcal{K}[F,G](Z,Z')$, polynôme en $Z,Z'\in\mathbb{R}^{2n}$, tel que

$$\mathcal{K}[F,G](Z,Z')\mathscr{P}(Z,Z') = \int_{\mathbb{R}^{2n}} F(Z,Z'')\mathscr{P}(Z,Z'')G(Z'',Z')\mathscr{P}(Z'',Z')dZ'', \tag{0.1.24}$$

et on peut calculer $\mathcal{K}[F,G]$ en fonction des coefficients de F et de G en utilisant les propriétés de la base de fonctions propres explicite de $\mathscr{L}$. En particulier, pour tout $f\in\mathscr{C}^\infty(X,\mathrm{End}(E))$, on peut appliquer la formule de Taylor à f pour montrer que $f(Z)\mathscr{P}(Z,Z')$ satisfait une expansion du type (0.1.22), ce qui montre que sa quantification de Berezin-Toeplitz $T_{f,p}$ définie en (0.1.11) satisfait elle aussi une expansion du type

$$T_{f,p}(Z,Z') \cong \sum_{r=0}^{+\infty} \mathcal{Q}_r(f)(\sqrt{p}Z,\sqrt{p}Z')\mathscr{P}(\sqrt{p}Z,\sqrt{p}Z') + \mathscr{O}(p^{-\infty}), \tag{0.1.25}$$

uniforme en $x_0\in X$, avec $\mathcal{Q}_0(f)(Z,Z')\equiv f(x_0)$ et où les $\mathcal{Q}_r(f)(Z,Z')$, $r\in\mathbb{N}$, sont des polynômes explicitement calculables de même parité que r en $Z,Z'\in\mathbb{R}^{2n}$. De la même manière, on montre que la composition $T_{f,p}T_{g,p}$ pour $f, g \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ satisfait elle aussi une expansion de ce type pour des polynômes explicitement calculables $\mathcal{Q}_r(f,g)(Z,Z')$, $r \in \mathbb{N}$, de même parité que $r \in \mathbb{N}$ en $Z, Z' \in \mathbb{R}^{2n}$, avec $\mathcal{Q}_0(f,g)(Z,Z') \equiv f(x_0)g(x_0)$.

Finalement, Ma et Marinescu [51, Th.4.9] donnent un critère qui montre l'existence d'une expansion de la forme (0.1.12) à partir de l'existence d'une telle expansion pour $T_{f,p}T_{g,p}$, ainsi que de certaines propriétés générales découlant de (0.1.11). On voit ainsi que les coefficients $C_r(f,g)$ apparaissant dans (0.1.12) sont explicitement calculables par ces méthodes.

Ma et Marinescu [52] ont calculé $C_1(f,g)$ et $C_2(f,g)$ dans le cas Kählerien, établissant en particulier la formule (0.1.14). Dans ce cas, il y a deux simplifications notables dans les calculs ; d'une part, les opérateurs $\mathcal{O}_k$, $k \in \mathbb{N}^*$, dans (0.1.16) font intervenir des dérivées covariantes de la structure presque complexe J par rapport à la connexion de Levi-Civita, qui s'annulent si (X, J, ω, g^{TX}) est Kählerienne. D'autre part, on a vu que pour p suffisament grand, l'espace $\mathrm{Ker}(D_p^c)$ est réduit à $H^0(X, L^p \otimes E)$ dans le cas Kählerien, et en particulier il ne contient pas de formes de degré strictement positif. Cela se traduit localement par le fait que l'on est restreint à des calculs dans $\mathrm{End}(E)_{x_0}$ au lieu de $\mathrm{End}(\Lambda(T^{*(0,1)}X) \otimes E)_{x_0}$. Cela n'est plus vrai en général.

Notons par ailleurs que les coefficients de l'expansion (0.1.12) sont à valeurs dans $\mathrm{End}(E)$, et en particulier ne présentent pas de partie forme. Cela est une indication que les termes de degré strictement positif dans $\Lambda(T^{*(0,1)}X)_{x_0}$ doivent finir par s'annuler dans les calculs. Malheureusement, cela ne signifie pas que l'on peut simplement les ignorer, car ils peuvent contribuer à la partie de degré 0. Malgré cela, je montre certaines annulations fantastiques dans les termes impliqués dans ce calcul, ce qui me permet d'en déduire la formule la plus simple possible pour le deuxième coefficient $C_1(f,g)$ de l'expansion (0.1.12), c'est à dire la même que dans le cas Kählerien.

0.2 Quantification de Berezin-Toeplitz pour les petites valeurs propres du laplacien de Bochner renormalisé

Cette partie de ma thèse est un travail en commun avec Ma, Marinescu et Lu, et est contenue dans [37].

Soit (X, ω) une variété symplectique compacte de dimension $2n$ et soit (L, h^L) un fibré en droite hermitien au-dessus de X, muni d'une connexion hermitienne ∇^L satisfaisant (0.0.1). Etant donné une structure presque complexe J compatible avec ω et une métrique Riemannienne J-invariante g^{TX} sur TX, ainsi qu'un fibré hermitien (E, h^E) au dessus de X muni d'une connexion hermitienne ∇^E, on peut comme en Section 0.1 considérer comme espaces de quantification pour tout $p \in \mathbb{N}^*$ le noyau de l'opérateur de Dirac spin^c agissant sur les sections de $L^p \otimes E$ tensorisé par l'algèbre extérieure $\Lambda(T^{*(0,1)}X)$.

Une autre approche consiste à prendre comme espaces de quantification la somme directe des espaces propres associés aux petites valeurs propres d'un Laplacien de Bochner renormalisé agissant sur les sections de $L^p \otimes E$. Cela a pour avantage que les espaces considérés sont inclus dans $\mathscr{C}^\infty(X, L^p \otimes E)$, de la même manière que les sections holomorphes dans le cas (X, J, ω, g^{TX}) Kählerienne. En revanche, la présence de petites valeurs propres complique sensiblement l'analyse, et la méthode décrite dans la Section 0.1 ne se généralise pas immédiatement. En particulier, l'expansion asymptotique du noyau de Bergman (0.1.22) au sens de la Section 1.3.1 n'est plus une conséquence des travaux de Dai, Liu et Ma [20]. La stratégie consiste à montrer une expansion asymptotique dans un sens plus faible, mais qui suffit pour étendre la quantification de Berezin-Toeplitz à ce cas. Cette approche est décrite ci-dessous.

Soit (E, h^E) un fibré vectoriel hermitien auxiliaire au-dessus de X muni d'une connexion hermitienne ∇^E. Pour tout $p \in \mathbb{N}^*$, et contrairement à (0.1.5), on définit

$$E_p = L^p \otimes E, \tag{0.2.1}$$

muni de la métrique h^{E_p} et de la connexion ∇^{E_p} induites par h^L, h^E et ∇^L, ∇^E. Le *Laplacien de Bochner* agissant sur $\mathscr{C}^\infty(X, E_p)$ est défini par la formule

$$\Delta^{E_p} = -\sum_{j=1}^{2n}\left[(\nabla^{E_p}_{e_j})^2 - \nabla^{E_p}_{\nabla^{TX}_{e_j}e_j}\right], \tag{0.2.2}$$

où $\{e_j\}_{j=1}^{2n}$ est une base orthonormée locale de TX pour g^{TX}. Pour $\Phi \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ hermitien, on définit le *Laplacien de Bochner renormalisé* $\Delta_{p,\Phi}$ agissant sur $\mathscr{C}^\infty(X, E_p)$ par la formule

$$\Delta_{p,\Phi} = \Delta^{E_p} - p\,\mathrm{Tr}[\dot{R}^L] + \Phi. \tag{0.2.3}$$

Dans la suite, on fixe $\Phi \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ et on note $\Delta_p = \Delta_{p,\Phi}$ pour (0.2.3). Alors Δ_p est un opérateur différentiel elliptique d'ordre 2 essentiellement auto-adjoint pour le produit L^2 défini comme en (0.0.2), et a un spectre discret contenu dans $\mathbb{R}$.

Theorem 0.2.1. *[47, Cor.1.2] Il existe $\widetilde{C},\ C > 0$ tels que pout tout $p \in \mathbb{N}^*$,*

$$\mathrm{Spec}(\Delta_p) \subset\ [-\widetilde{C}, \widetilde{C}] \cup\,]2\mu_0 p - C, +\infty[, \tag{0.2.4}$$

où $\mu_0 = \inf_{1\leqslant j\leqslant n,\, x\in X} a_j(x)$, avec $a_j(x) > 0$, $\leqslant j \leqslant n$, valeurs propres de $\dot{R}^L_x$ comme en (0.1.17) *pour tout $x \in X$.*

De plus, si $\mathscr{H}_p \subset L^2(X, E_p)$ est la somme directe des espaces propres de Δ_p associés aux valeurs propres contenues dans $[-\widetilde{C}, \widetilde{C}]$, alors $\mathscr{H}_p \subset \mathscr{C}^\infty(X, E_p)$ pour tout $p \in \mathbb{N}^*$, et pour p suffisament grand, on a

$$\dim \mathscr{H}_p = \int_X \mathrm{Td}(T^{(1,0)}X)\,\mathrm{ch}(E)\exp(p\omega). \tag{0.2.5}$$

Pour tout $p \in \mathbb{N}^*$, on prend comme espace de quantification de X associé à E_p l'espace $\mathscr{H}_p$ des petites valeurs propres de Δ_p défini dans le théorème 0.2.1.

Dans le cas Kählerien et en prenant Φ égal à $-\sqrt{-1}R^E$ contracté avec ω, le Laplacien de Bochner renormalisé est égal à D_p^2 restreint à $\mathscr{C}^\infty(X, E_p)$, où D_p est donné par la partie droite de (0.1.8). Dans ce cas, les petites valeurs propres sont toutes nulles, et on retrouve ainsi les espaces de quantification holomorphes $\mathscr{H}_p = H^0(X, L^p)$ pour p suffisament grand. Dans le cas général, la formule (0.2.5) montre de plus que ces espaces induisent bien une quantification de E au sens de [32, Th.9.1], ce qui en fait une généralisation naturelle de la quantification holomorphe.

Pour tout $p \in \mathbb{N}^*$, on peut considérer la projection orthogonale P_p sur $\mathscr{H}_p$ par rapport au produit hermitien L^2, et on définit le *noyau de Bergman généralisé* $P_p(x, y) \in E_{p,x} \otimes E^*_{p,y}, x, y \in X$, comme le noyau de Schwartz de P_p par rapport à dv_X. La quantification de Berezin-Toeplitz de $f \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ est alors définie comme en (0.1.11) pour tout $p \in \mathbb{N}*$ par la formule

$$T_{f,p} = P_p f P_p : \mathscr{C}^\infty(X, E_p) \to \mathscr{C}^\infty(X, E_p), \tag{0.2.6}$$

et le résultat principal de cette partie est le suivant.

Theorem 0.2.2. *Soient $f, g \in \mathscr{C}^\infty(X, \mathrm{End}(E))$. Le produit $T_{f,p}T_{g,p}$ des quantifications de Berezin-Toeplitz de f et g satisfait l'expansion asymptotique suivante pour tout $k \in \mathbb{N}*$ et lorsque $p \to \infty$ au sens de la norme d'opérateur induite par $\langle \cdot, \cdot \rangle_p$ sur les endomorphismes de $\mathscr{H}_p$,*

$$T_{f,p}T_{g,p} = \sum_{r=0}^{k} p^{-r} T_{C_r(f,g),p} + O(p^{-k}), \tag{0.2.7}$$

où les C_r sont des opérateurs bidifférentiels pour tout $r \in \mathbb{N}$, avec $C_0(f, g) = fg$.

De plus, le théorème 0.1.1 s'étend à ce cadre, où $\nabla^{1,0}$ et $\nabla^{0,1}$ sont les projections sur $T^{(1,0)}X$ et $T^{(0,1)}X$ via (0.1.1) de la connexion sur $\mathrm{End}(E)$ induite par ∇^E, et où $\langle ., . \rangle$ l'accouplement induit par g^{TX} sur $T^*X \otimes \mathrm{End}(E)$ à valeurs dans $\mathrm{End}(E)$.

Theorem 0.2.3. *Supposons que $g^{TX}(\cdot, \cdot) = \omega(\cdot, J\cdot)$. Pour tout $f,\ g \in \mathscr{C}^\infty(X, \mathrm{End}(E))$, on a la formule suivante pour $C_1(f, g)$ dans* (0.2.7)*,*

$$C_1(f, g) = -\frac{1}{2\pi} \langle \nabla^{1,0} f, \nabla^{0,1} g \rangle. \tag{0.2.8}$$

A cause des petites valeurs propres non nulles, la difficulté ici vient du fait que l'expansion (0.1.22) du noyau de Bergman n'est plus vérifiée. D'après Ma et Marinescu [50, Th.1.19], on a néanmoins une expansion proche de la digonale, c'est à dire sur une boule géodésique de rayon $\varepsilon p^{-1/2}$ au lieu de ε pour tout $\varepsilon > 0$ suffisament petit. Une telle expansion n'est pas suffisante pour pouvoir en déduire une expansion pour la composition d'opérateurs comme en Section 0.1.

En fait, une analyse plus fine des techniques de localisation de Bismut-Lebeau et de la théorie spectrale de Δ_p (qui marche aussi pour l'opérateur de Dirac spin^c D_p^c) montre que $P_p(x,y)$ décroit rapidement lorsque $p \to \infty$ dès que $d^X(x,y) > \varepsilon p^{-\theta/2}$, pour tout $\varepsilon > 0$ et $\theta \in]0,1[$, où d^X est la distance Riemannienne sur X induite par g^{TX}. D'autre part, en examinant soigneusement la preuve de l'expansion proche de la diagonale, Liu, Ma et Marinescu [44, Th.2.1] montrent que l'on peut étendre celle ci en une estimée sur une boule géodésique de rayon $\varepsilon p^{-\theta/2}$, pour tout $\theta \in]0,1[$, au lieu de $\varepsilon p^{-1/2}$. On obtient ainsi une estimée globale sur le noyau de Bergman généralisé, ce qui permet de se ramener à des intégrales sur $\mathbb{C}^n$ comme en Section 0.1. Il faut noter ici que la méthode est bien plus délicate à mettre en oeuvre, car les estimées sont moins fines et le rayon des coordonnées dans lesquelles on travaille tend vers 0 lorsque p tend vers l'infini.

La calcul explicite des composition locales (0.1.24) s'étend alors de la même manière, et on peut ainsi calculer explicitement les coefficients de (0.2.7) comme précédemment. Notons que le calcul de (0.2.8) est plus délicat que dans le cas Kählerien, car les dérivées covariantes de la structure complexe ne s'annulent pas. En revanche, il est plus facile que dans le cas spin^c de la Section 0.1, car on est amené à des calculs dans $\mathrm{End}(E)_{x_0}$ plutôt que $\mathrm{End}(\Lambda(T^{*(0,1)}X) \otimes E)_{x_0}$, et les termes de degré strictement positif dans $\Lambda(T^{*(0,1)}X)_{x_0}$ n'apparaissent pas.

0.3 Quantification et sous-variétés isotropes

Cette partie de ma thèse est contenue dans [36].

Soit (X,ω) une variété symplectique compacte de dimension $2n$ et soit (L,h^L) un fibré en droite hermitien au-dessus de X, muni d'une connexion hermitienne ∇^L satisfaisant (0.0.1). Supposons de plus X munie d'une *fibration Lagrangienne régulière*, c'est à dire qu'il existe une submersion propre $\pi : X \to B$ de X vers une variété B avec $\dim B = n$ telle que $\omega(u,v) = 0$ pour tout $x \in X,\ u,v \in \operatorname{Ker} d\pi_x$. Pour tout $b \in B$, la fibre $\pi^{-1}(b) =: \Lambda$ est donc une sous-variété Lagrangienne de X, et on dit qu'elle satisfait la *condition de Bohr-Sommerfeld* s'il existe $s \in \mathscr{C}^\infty(\Lambda, L|_\Lambda)$ non-nulle, *parallèle* pour la connexion induite par ∇^L sur $L|_\Lambda$. Les fibres d'une fibration Lagrangienne régulière sont nécessairement difféomorphes à $\mathbb{T}^n := (S^1)^n$ et la structure symplectique est induite par des *coordonnées actions-angles locales*, ce qui montre en particulier que l'ensemble $B_{BS} \subset B$ des points $b \in B$ tels que $\pi^{-1}(b)$ satisfait la condition de Bohr-Sommerfeld est fini (voir par exemple Śniatycki [56], Duistermaat [23, § 2] et Guillemin et Sternberg [30, § 2]). La *quantification géométrique réelle* de (X,ω) associée à (L,∇^L) et $\pi : X \to B$ est l'espace $\mathscr{H}^{BS} = \mathrm{Span}_{\mathbb{C}} \langle B_{BS} \rangle$.

L'existence d'une fibration Lagrangienne régulière sur X est une condition très restrictive. En général, on considère des *fibrations singulières*, où l'on autorise la dimension des fibres à chuter dans une réunion finie de sous-variétés de X de codimension strictement positive. Ces fibres sont alors seulement *isotropes*, et la condition de Bohr-Sommerfeld a encore un sens. Le cas typique est celui des *variétés toriques*, où $\pi : X \to B$ est l'application moment associée à l'action hamitonienne effective de $\mathbb{T}^n$ sur X, et B est un polytope dans $\mathrm{Lie}(\mathbb{T}^n)^* = \mathbb{R}^n$, appelé *polytope de Delzant* de X. Cette application moment induit une fibration Lagrangienne régulière à l'intérieur du polytope, avec fibre générale un tore de dimension n, et la fibre au dessus d'un point b appartenant à une face de codimension $d \leqslant n$ est un tore de dimension $n - d$.

On considère plus généralement la suite d'espaces de quantification $\mathscr{H}_p^{BS}$ associés à L^p pour tout $p \in \mathbb{N}^*$. On remarque alors que si Λ satisfait la condition de Bohr-Sommerfeld pour L, alors il la satisfait pour L^p pour tout $p \in \mathbb{N}^*$. Inversement, si Λ satisfait la condition de Bohr-Sommerfeld pour L^p pour un certain $p \in \mathbb{N}^*$, alors une section parallèle pour ∇^{L^p} induit une section parallèle pour ∇^L lorsqu'on tire en arrière (L, ∇^L) sur un certain revêtement $\hat{\Lambda}$ d'ordre p de Λ. Dans ce cas, on dira que Λ satisfait la *condition de Bohr-Sommerfeld à l'ordre p*. Pour étudier la limite semi-classique de ces espaces, il est donc plus utile de considérer des immersions propres $\iota : \Lambda \to X$ plutôt que des plongements. On pose ainsi la définition suivante.

Definition 0.3.1. Une immersion propre $\iota : \Lambda \to X$ satisfait la *condition de Bohr-Sommerfeld* s'il existe $0 \neq \zeta \in \mathscr{C}^\infty(\Lambda, \iota^* L)$ tel que $\nabla^{\iota^* L}\zeta = 0$.

Si X est de plus muni d'une structure complexe compatible avec ω, une question naturelle dans ce contexte est de comprendre la relation entre les espaces $\mathscr{H}_p^{BS}$ de quantification géométrique réelle avec les espaces de quantification holomorphe $H^0(X, L^p)$, pour $p \in \mathbb{N}^*$. Dans le cas de la quantification canonique de $X = T^*\mathbb{R}^n \simeq \mathbb{C}^n$, qui n'entre pas dans ce cadre car X n'est pas compacte mais qui sert néanmoins de modèle local, un isomorphisme entre les espaces correspondants est donné par la *transformée de Segal-Bargman* [4], [55].

Pour (X, J, ω) Kählerienne compacte et en faisant l'hypothèse que (X, ω) admet une structure métaplectique, Borthwick, Paul et Uribe [12] associent une section holomorphe de $H^0(X, L^p)$ à toute variété Lagrangienne plongée satisfaisant la condition de Bohr-Sommerfeld d'ordre p, en utilisant le formalisme des demi-formes et l'analyse microlocale de Boutet de Monvel et Guillemin [14]. Ils étudient ensuite les propriétés semi-classiques des ces sections, montrant en particulier qu'elles ne s'annulent pas identiquement pour p suffisament grand, puis donnent une application aux séries de Poincaré relatives en théorie des formes automorphes. Leur définition peut être vue comme une extension de la transformée de Segal-Bargman.

On se place maintenant dans le contexte de la Section 0.2. En particulier, on choisit une structure presque complexe J compatible avec ω et une métrique Riemannienne J-invariante g^{TX} sur TX, et on considère les espaces de quantification $\mathscr{H}_p$ des sections propres de $L^p \otimes E$ pour les petites valeurs propres du Laplacien de Bochner renormalisé (0.2.3). Soit $\iota : \Lambda \to X$ une immersion propre satisfaisant la condition de Bohr-Sommerfeld, soit $\zeta \in \mathscr{C}^\infty(\Lambda, \iota^* L)$ une section associée comme dans la définition 0.3.1, que l'on choisit unitaire, et soit $f \in \mathscr{C}^\infty(\Lambda, \iota^* E)$. On définit l'*état isotrope* (ou *état Lagrangien* si $\dim \Lambda = n$) associé à $\iota : \Lambda \to X$, ζ et f comme la suite $\{s_{f,p} \in \mathscr{H}_p\}_{p \in \mathbb{N}}$, définie pour tout $p \in \mathbb{N}^*$ par

$$s_{f,p} = \int_\Lambda P_p(x, \iota(y))\iota_p.\zeta^p f(y) dv_X(y), \tag{0.3.1}$$

où $\zeta^p f \in \mathscr{C}^\infty(\Lambda, \iota^* E_p)$ est le produit tensoriel de f avec le p-ième produit tensoriel de ζ et $\iota_p : \iota^* E_p \to E_p$ est l'application induite par ι sur l'espace total de $\iota^* E_p$.

L'objectif de cette partie est d'étudier les propriétés semi-classiques de ces états isotropes, en utilisant l'expansion asymptotique du noyau de Bergman généralisé et la quantification de Berezin-Toeplitz pour le Laplacien de Bochner renormalisé développée en Section 0.2. En particulier, la décroissance rapide en $p \to \infty$ du noyau de Bergman en dehors de tout voisinage de la diagonale implique immédiatement la décroissance rapide en $p \to \infty$ de $s_{f,p}$ en dehors de tout voisinage de l'image de ι. Notons $\|.\|_p$ la norme associée au produit hermitien L^2 sur $\mathscr{C}^\infty(X, L^p \otimes E)$, pour tout $p \in \mathbb{N}^*$. Le premier résultat principal de cette partie est le théorème 3.3.6, que nous présentons ici sous sa forme la plus simple.

Theorem 0.3.1. *Soit $\iota : \Lambda \to X$ une immersion propre satisfaisant la condition de Bohr-Sommerfeld, soit $\zeta \in \mathscr{C}^\infty(\Lambda, \iota^* L)$ une section unitaire parallèle pour $\nabla^{\iota^* L}$, et soit $f \in \mathscr{C}^\infty(X, E)$. On pose $d = \dim \Lambda$. Alors il existe $b_r \in \mathbb{R}$, $r \in \mathbb{N}$, tels que pour tout $k \in \mathbb{N}$ et lorsque $p \to \infty$,*

$$\|s_{f,p}\|_p^2 = p^{n-\frac{d}{2}} \sum_{r=0}^{k} p^{-r} b_r + O(p^{n-\frac{d}{2}-(k+1)}), \tag{0.3.2}$$

avec

$$b_0 = 2^{d/2} \int_\Lambda |f|_E^2 dv_\Lambda \tag{0.3.3}$$

si $g^{TX}(\cdot,\cdot) = \omega(\cdot, J\cdot)$.

En particulier, $s_{f,p}$ n'est pas identiquement nul pour p suffisament grand. Sans l'hypothèse $g^{TX}(\cdot,\cdot) = \omega(\cdot, J\cdot)$, je montre de plus que le coefficient b_0 admet la même expression, mais pour une forme volume adaptée à la situation. D'autre part, en considérant un endomorphisme $F \in \mathscr{C}^\infty(X, \mathrm{End}(E))$, je calcule de la même manière une expansion asymptotique pour $\langle T_{F,p} s_{f,p}, s_{f,p}\rangle_p$, où $T_{F,p}$ est la quantification de Berezin-Toeplitz de F comme en (0.2.6), avec une formule analogue pour le premier coefficient.

Dans un second temps, j'étudie le produit hermitien L^2 de deux états isotropes associés à deux variétés de Bohr-Sommerfeld s'intersectant proprement. En particulier, si les deux variétés ne s'intersectent pas, le produit des états isotropes associés décroit rapidement lorsque $p \to \infty$. On voit ainsi que les états associés aux fibres de Bohr-Sommerfeld d'une fibration singulière au sens entendu plus haut ne s'annulent pas et deviennent linéairement indépendants à la limite semi-classique. La définition (0.3.1) nous donne une application linéaire de $\mathscr{H}_p^{BS}$ vers $\mathscr{H}_p$ pour chaque $p \in \mathbb{N}^*$, et ces résultats peuvent être interprétés par le fait que cette suite d'application tend à être injective à la limite semi-classique. Si les dimensions sont égales pour p grand (c'est à dire si $\dim \mathscr{H}_p^{BS}$ est calculée par la formule de Riemann-Roch-Hirzebruch (0.2.5), ce qui est le cas dans tous les exemples naturels), on obtient un isomorphisme à la limite semi-classique. Dans l'exemple des variétés toriques mentionné ci dessus, les états isotropes associés aux éléments de $\mathscr{H}_p^{BS}$ ne s'annulent pas, sont linéairement indépendants et induisent l'isomorphisme attendu, et ce non seulement à la limite semi-classique mais quel que soit $p \in \mathbb{N}^*$.

Maintenant, étant donné deux fibrations Lagrangiennes, on peut s'intéresser à comparer les quantifications associées. La cas typique est celui de $X = T^*\mathbb{R}^n \simeq \mathbb{R}^n \times \mathbb{R}^n$, où l'on peut considérer les deux fibrations induites par les deux projections canoniques associées au produit cartésien. Il est donc naturel dans notre situation de s'intéresser au produit hermitien des états isotropes associés à des variétés de Bohr-Sommerfeld d'intersection non vide. D'autre part, dans le contexte des séries de Poincaré relatives décrit ci-dessous, on s'intéresse aux états isotropes associés aux géodésiques fermées d'une surface de Riemann hyperbolique, qui sont d'intersection non vide en général. Le deuxième résultat principal de cette partie est le théorème 3.4.4, énoncé ici sous sa forme la plus simple.

Theorem 0.3.2. *Soient $\iota_j : \Lambda_j \hookrightarrow X,\ j = 1, 2,$ deux immersions propres satisfaisant la condition de Bohr-Sommerfeld, s'intersectant proprement et d'intersection connexe, soient $\zeta_j \in \mathscr{C}^\infty(\Lambda_j, \iota_j^* L)$ des sections unitaires parallèles pour $\nabla^{\iota_j^* L}, j = 1, 2,$ et soient $f_j \in \mathscr{C}^\infty(X, \iota^* E),\ j = 1, 2.$ Posons $l = \dim \Lambda_1 \cap \Lambda_2$ et $d_j = \dim \Lambda_j,\ j = 1, 2.$ Alors il existe $b_r \in \mathbb{C},\ r \in \mathbb{N},$ tels que pour tout $k \in \mathbb{N}$ et lorsque $p \to \infty$,*

$$\langle s_{f_1,p}, s_{f_2,p} \rangle_p = p^{n - \frac{d_1+d_2}{2} + \frac{l}{2}} \lambda^p \sum_{r=0}^{k} p^{-r} b_r + O(p^{n - \frac{d_1+d_2}{2} + \frac{l}{2} - (k+1)}), \tag{0.3.4}$$

où $\lambda \in \mathbb{C}$ est la valeur de la fonction constante définie pour tout $x \in \Lambda_1 \cap \Lambda_2$ par $\lambda(x) = \langle \zeta_1(x), \zeta_2(x) \rangle_L$. De plus, si $\dim \Lambda_1 = n$ et $g^{TX}(\cdot, \cdot) = \omega(\cdot, J \cdot)$, on a

$$b_0 = 2^{n/2} \int_{\Lambda_1 \cap \Lambda_2} \langle f_1, f_2 \rangle_E \det{}^{-\frac{1}{2}} \Big\{ \sqrt{-1} \sum_{k=1}^{n-l} h^{TX}(e_k, \nu_i) \omega(e_k, \nu_j) \Big\}_{i,j=1}^{d_2 - l} |dv|_{\Lambda_1 \cap \Lambda_2}, \tag{0.3.5}$$

où $\langle e_i \rangle_{i=1}^{n-l}, \langle \nu_j \rangle_{j=1}^{d_2-l}$ sont des bases orthonormées du fibré normal à $\Lambda_1 \cap \Lambda_2$ dans Λ_1, Λ_2, et $|dv|_{\Lambda_1 \cap \Lambda_2}$ est la densité Riemanniennne sur $\Lambda_1 \cap \Lambda_2$ induite par g^{TX}.

Ici, la notion d'intersection utilisée est celle de produit fibré pour les immersions, qui coincide avec l'intersection usuelle dans le cas de plongements. On voit ainsi qu'à la limite semi-classique, le produit hermitien de deux états isotropes est intimement relié à la géométrie de l'intersection des sous-variétés correspondantes. On appelle ce produit le *produit d'intersection* des états isotropes. De la même façon que pour (0.3.3), on a une formule analogue à (0.3.5) pour g^{TX} métrique Riemannienne J-invariante quelconque et pour $\langle T_{F,p} s_{f_1,p}, s_{f_2,p} \rangle_p$ avec $F \in \mathscr{C}^\infty(X, \mathrm{End}(E))$. De plus, le résultat s'étend à des immersions d'intersection non nécessairement connexe. En particulier, si l'intersection est discrète, le premier coefficient (0.3.5) s'écrit comme une somme sur les points d'intersection On note par ailleurs que le théorème 0.3.1 est en fait un cas particulier du théorème 0.3.2.

La preuve du théorème 0.3.2 a pour point de départ la propriété reproduisante suivante de l'état Lagrangien $s_{f,p}$ associé à $\iota : \Lambda \to X$, $s \in \mathscr{C}^\infty(\Lambda, \iota^* L)$ et $f \in \mathscr{C}^\infty(\Lambda, \iota^* E)$ comme en (0.3.1), pour tout $p \in \mathbb{N}^*$ et $s \in \mathscr{H}_p$,

$$\langle s, s_{f,p} \rangle_p = \int_\Lambda \langle s(\iota(x)), \iota_p.\zeta^p f(x) \rangle_{E_p} \, dv_\Lambda(x). \tag{0.3.6}$$

Par (0.3.1) et en reprenant les notations du théorème 0.3.2, cela implique

$$\langle s_{f_1,p}, s_{f_2,p} \rangle_p = \int_{\Lambda_2} \int_{\Lambda_1} \langle P_p(\iota_2(x), \iota_1(y)) \iota_{1,p}.\zeta_1^p f_1(y), \iota_{2,p}.\zeta_2^p f_2(x) \rangle_{E_p} \, dv_{\Lambda_1}(y) dv_{\Lambda_2}(x). \tag{0.3.7}$$

On peut ainsi utiliser la décroissance rapide du noyau de Bergman en dehors d'un voisinage de la diagonale pour localiser le problème autour de l'intersection de Λ_1 et Λ_2. Pour simplifier le raisonnement, supposons dans un premier temps que $\Lambda_1 \cap \Lambda_2$ est discret. Pour tout $x_0 \in \Lambda_1 \cap \Lambda_2$, la stratégie est de ramener le calcul à un modèle local autour de x_0, où les intégrales de long de Λ_1 et Λ_2 dans X sont remplacées par des intégrales le long de leurs espaces tangents respectifs $T_{x_0}\Lambda_1$ et $T_{x_0}\Lambda_2$ dans $T_{x_0}X \simeq \mathbb{R}^{2n}$. Une première difficulté ici est que ces sous-variétés ne sont pas supposées totalement géodésiques, et les coordonnées normales décrites en Section 0.1 ne conviennent plus. Il faut donc adapter les résultats de Dai, Liu et Ma [20] pour l'étude locale du noyau de Bergman à des coordonnées autour de x_0 pour lesquelles les sous-variétés sont envoyées sur des sous-espaces linéaires. La méthode s'étend effectivement pour des coordonnées générales, le point crucial étant que le fibré en droite complexe L soit trivialisé par transport parallèle par rapport à ∇^L le long des rayons passant par x_0. On voit ainsi apparaitre la nécessité de la condition de Bohr-Sommerfeld : les sections s_1 et s_2 étant unitaires et parallèles par rapport à ∇^L, elles deviennent constantes dans cette trivialisation, et on peut alors sortir leur contribution de l'intégrale, qui est notée λ dans (0.3.4). Notons que si celles ci ne sont pas constantes, la puissance de p rend impossible l'évaluation de l'intégrale en général.

On peut alors utiliser l'expansion (0.1.22) dans le cadre décrit en Section 0.2 pour ramener le calcul à des intégrales le long de sous-espaces linéaires de $\mathbb{R}^n$. De la même façon qu'en Section 0.2, on travaille dans des coordonnées dont le rayon tend vers 0 avec p, ce qui demande des précautions analogues à celles rencontrées pour l'établissement de la théorie de Berezin-Toeplitz en Section 0.2. En exploitant l'isotropie de $T_{x_0}\Lambda_1$ et $T_{x_0}\Lambda_2$, l'intégrale associée au r-ième coefficient de l'expansion (0.1.22), pour tout $r \in \mathbb{N}$, se ramène à une intégrale de la forme

$$\int_{T_{x_0}\Lambda_2} G_r(Z) \exp(-\pi \langle Z, CZ \rangle) dv_{\Lambda_2}(Z), \tag{0.3.8}$$

où dv_{Λ_2} est la mesure d'intégration sur $T_{x_0}\Lambda_2$, $G_r(Z)$ est un polynôme en $Z \in \mathbb{R}^n$ de même parité que $r \in \mathbb{N}$ et $C \in \mathrm{GL}_n(\mathbb{C})$ est une matrice symmétrique à partie réelle positive. En particulier, l'intégrale s'annule pour tout r impair pour des raisons de parité, ce qui permet de montrer un développement en puissances entières de p dans (0.3.4). De plus, le polynôme apparaissant dans l'intégrale associée au premier coefficient est constant, ce qui permet de le calculer explicitement. Notons que même dans le cas où g^{TX} est une métrique Riemannienne J-invariante quelconque, il bien plus commode de travailler avec la métrique induite par ω et J pour les calculs locaux, malgré le fait qu'on ne puisse pas se ramener directement à ce cas.

Dans le cas où l'intersection $\Lambda_1 \cap \Lambda_2$ est de dimension strictement positive, on peut travailler dans un voisinage tubulaire de $\Lambda_1 \cap \Lambda_2$ à l'intérieur de Λ_2 pour séparer l'intégrale sur Λ_2 en une intégrale le long de $\Lambda_1 \cap \Lambda_2$ et une intégrale le long des direction orthogonales à $\Lambda_1 \cap \Lambda_2$. En utilisant l'uniformité en x_0 de l'expansion (0.1.22), on est ainsi ramené à la situation précédente dans un voisinage de chaque $x_0 \in \Lambda_1 \cap \Lambda_2$. Pour le calcul explicite du premier terme, on exploite le fait que les direction sont orthogonales et que la dimension de Λ_1 soit maximale, égale à n. Finalement, on remarque que le calcul incluant la quantification de Berezin-Toeplitz $T_{F,p}$ de $F \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ est strictement analogue, en utilisant la propriété reproduisante suivante, pour tout état isotrope $s_{f,p} \in \mathscr{H}_p$,

$$T_{F,p} s_{f,p} = \int_\Lambda T_{F,p}(x, \iota(y)) \iota_p . \zeta^p f(y) dv_\Lambda(y). \tag{0.3.9}$$

On conclut en comparant cette formule avec (0.3.1) et en utilisant l'expansion (0.1.22) pour l'opérateur de Berezin-Toeplitz découlant de la Section 0.2.

Je montre ensuite que les résultats ci-dessus s'étendent pour X orbifold non compact, sous des hypothèses de géométrie bornée et en supposant par simplicité que J est intégrable, avec $(L, h^L), (E, h^E)$ holomorphes munis de leur connexion de Chern. J'étudie deux types d'immersion d'un orbifold Λ dans un orbifold X: les *immersions orbifoldes* d'une part, qui demandent que Λ soit équipée d'une structure d'orbifold compatible avec celle de X, et les *immersions singulières* d'autre part, où l'on autorise Λ à traverser les points singuliers de X sans pour autant qu'ils coincident avec ceux de Λ. Dans ce dernier cas, on a besoin d'une hypothèse de propreté de l'immersion autour des points singuliers, ce qui permet de négliger la contribution des points singuliers à l'intégrale correspondante dans le théorème 0.3.2. On suppose de plus que l'intersection est disjointe des points singuliers. Dans le premier cas, il apparait une multiplicité induite par la structure orbifold dans le calcul correspondant à (0.3.5). J'utilise dans les deux cas l'étude initiée par Ma [45] des techniques de localisation analytique de Bismut-Lebeau dans le cadre orbifold, et l'expansion asymptotique du noyau de Bergman sur un orbifold démontré par Dai, Liu et Ma [20, § 6].

Considérons maintenant l'exemple particulier de $X = \mathbb{H}/\Gamma$, où $\mathbb{H}$ est le demi-plan supérieur du plan complexe et Γ est un sous-groupe discret de $SL_2(\mathbb{Z})$ agissant sur $\mathbb{H}$ par homographies, qui sont des isométries pour la métrique hyperbolique $g^{T\mathbb{H}}$ sur $\mathbb{H}$. Le fibré en droites canonique $K_{\mathbb{H}} = T^{*(1,0)}\mathbb{H}$ satisfait la condition de préquantification (0.0.1) pour la métrique de Kähler $\omega_{\mathbb{H}}$ induite par $g^{T\mathbb{H}}$ sur $\mathbb{H}$ à un facteur 2π près, et toutes ces données passent au quotient $X = \mathbb{H}/\Gamma$. En particulier, le quotient de

$K_{\mathbb{H}}$ par Γ s'identifie au fibré canonique $K_X = T^{*(1,0)}X$. D'autre part, les courbes fermées $\gamma : [0,l] \to X,\ l > 0$, sont automatiquement Lagrangiennes pour des raisons de dimension. On est alors précisément dans la situaton décrite ci-dessus, avec X éventuellement orbifold non compact, et on dit que $\gamma : [0,l] \to X,\ l > 0$, est une *courbe de Bohr-Sommerfeld* si elle est paramétrée par longueur d'arc et si l'immersion induite $\widetilde{\gamma} : S^1 \to X$ est une variété de Bohr-Sommerfeld au sens de la définition 0.3.1. Dans le cas où γ traverse un point singulier, on suppose que γ est une immersion singulière. Le théorème 0.3.1 et le théorème 0.3.2 se traduisent alors en l'énoncé suivant, où on adopte la convention que $\sqrt{-a} = \sqrt{-1}\sqrt{a}$ si $a > 0$.

Theorem 0.3.3. *Soit $X = \mathbb{H}/\Gamma$ avec Γ sous groupe discret de $SL_2(\mathbb{Z})$, soit $\gamma : [0,l] \to X,\ l > 0$, une courbe de Bohr-Sommerfeld paramétrisée par longueur d'arc, et soit $\{s_{\gamma,p}\}_{p\in\mathbb{N}^*}$ un état Lagrangien associé comme dans* (0.3.1)*. Alors*

$$\|s_{\gamma,p}\|_{L^2}^2 = \left(\frac{p}{\pi}\right)^{1/2} l + O(p^{-1/2}). \tag{0.3.10}$$

De plus, si γ_1 et γ_2 sont deux courbes de Bohr-Sommerfeld s'intersectant proprement en dehors d'un voisinage du lieu singulier, alors on a

$$\langle s_{\gamma_1,p}, s_{\gamma_2,p}\rangle = \sqrt{2} \sum_{z\in\gamma_1\cap\gamma_2} \sum_{\substack{t_1,t_2>0,\\ \gamma_1(t_1)=\gamma_2(t_2)=z}} \lambda_{t_1,t_2}^p \frac{e^{\sqrt{-1}(\theta_z/2-\pi/4)}}{\sqrt{\sin(\theta_z)}} + O(p^{-1}), \tag{0.3.11}$$

où $\theta_z \in [0,2\pi[$ est l'angle orienté entre γ_1 et γ_2 en $z \in \gamma_1\cap\gamma_2$ et où pour tout $t_1,t_2 > 0$ tels que $\gamma_1(t_1) = \gamma_2(t_2)$, on définit $\lambda_{t_1,t_2} = \langle \gamma_1^L.s_1(t_1), \gamma_2^L.s_2(t_2)\rangle_{K_X}$.

Le théorème 0.3.3 est particulièrement intéressant dans le cas où $\gamma : [0,l] \to X$ est une courbe associée à l'action d'un élément $g_0 \in \Gamma$, c'est à dire que son relevé à $\mathbb{H}$, que l'on note encore $\gamma : \mathbb{R} \to \mathbb{H}$, satisfait $g_0.\gamma(t) = \gamma(t+l)$ pour tout $t \in \mathbb{R}$. Dans ce cas, on peut utiliser la forme explicite du noyau de Bergman du plan hyperbolique pour calculer $s_{\gamma,p}$ explicitement à partir de (0.3.1) via de l'analyse complexe élémentaire. En particulier, si $\gamma : [0,l] \to X,\ l > 0$, est une courbe géodésique fermée, alors elle est associée à un unique élément *hyperbolique* $g_0 \in \Gamma$ comme ci-dessus, c'est à dire tel que $\mathrm{Tr}(g) > 2$, et l'immersion singulière induite $\widetilde{\gamma} : S^1 \to X$ satisfait la condition de Bohr-Sommerfeld. Par un calcul de Borthwick, Paul et Uribe [12, § 4] via l'identification des sections de K_X au-dessus de X avec les sections Γ-invariantes de $K_{\mathbb{H}}$ au-dessus de $\mathbb{H}$, on obtient pour $g_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ la formule suivante,

$$s_{\gamma,p}(z) = \sum_{[g]\in\Gamma/\Gamma_0} j(g,z)^{-2p}(c(g.z)^2 + (d-a)(g.z) - b)^{-p}, \tag{0.3.12}$$

où la convergence est uniforme pour z dans tout compact de $\mathbb{H}$, avec Γ_0 le sous-goupe de Γ engendré par g_0, et où $j(g,z)$ est la jacobien de l'action de $g \in \Gamma$ en $z \in \mathbb{H}$. On déduit ainsi du théorème 0.3.1 que cette série ne s'annule pas pour p suffisament grand.

Ces séries sont des exemples de *séries de Poincaré relatives*, et ont été utilisées par Katok [40] dans le cadre de la théorie des formes automorphes, c'est à dire pour Γ tel que $\mathrm{Vol}(\mathbb{H}/\Gamma) < +\infty$. Notons que Katok prouve par ailleurs que ces séries engendrent tout l'espace des sections holomorphes L^2 de X.

Chapter 1

On the composition of Berezin-Toeplitz operators on symplectic manifolds

1.1 Introduction

In [51], Ma and Marinescu studied in detail Berezin-Toeplitz quantization for symplectic manifolds, introducing kernel calculus as a method to compute the coefficients of the asymptotic expansion of the associated Toeplitz operators. They considered the following situation: let (X, ω) be a compact symplectic manifold of dimension $2n$, and (E, h^E), (L, h^L) be Hermitian vector bundles on X with $\mathrm{rk}(L) = 1$, endowed with Hermitian connections ∇^E, ∇^L. If R^L denotes the curvature of ∇^L, we assume the following so-called *prequantization condition*:

$$\omega = \frac{\sqrt{-1}}{2\pi} R^L. \tag{1.1.1}$$

Let J be an almost complex structure on TX compatible with ω, and let g^{TX} be the Riemannian metric on TX defined by

$$\omega(u, v) = g^{TX}(Ju, v), \tag{1.1.2}$$

for any $u, v \in TX$. We denote by L^p the p^{th} tensor power of L and D_p the *spin^c Dirac operator* acting on the smooth sections of $\mathbf{E}_p := \Lambda(T^{*(0,1)}X) \otimes L^p \otimes E$. The metrics g^{TX}, h^L and h^E induce the usual L^2-scalar product on the space $L^2(X, \mathbf{E}_p)$ of the square integrable sections of $\mathbf{E}_p$. The orthogonal projection of $L^2(X, \mathbf{E}_p)$ on $\mathrm{Ker}(D_p)$ with respect to this product is denoted by P_p and is called the *Bergman projection*. For $f \in \mathscr{C}^\infty(X, \mathrm{End}(E))$, the *Berezin-Toeplitz quantization* of f is the family $\{T_{f,p}\}_{p\in\mathbb{N}}$ of operators acting on $L^2(X, \mathbf{E}_p)$ by

$$T_{f,p} = P_p f P_p : L^2(X, \mathbf{E}_p) \to L^2(X, \mathbf{E}_p), \tag{1.1.3}$$

where f denotes the operator acting by pointwise multiplication by f.

More generally, a family $\{T_p\}_{p\in\mathbb{N}}$ of bounded operators acting on $L^2(X,\mathbf{E}_p)$ is called a *Toeplitz operator* if $P_pT_pP_p = T_p$ for all $p \in \mathbb{N}$, and if there exists a sequence of sections $g_r \in \mathscr{C}^\infty(X,\mathrm{End}(E))$ for all $r \in \mathbb{N}$ such that

$$T_p = \sum_{r=0}^{\infty} T_{g_r,p}p^{-r} + O(p^{-\infty}), \tag{1.1.4}$$

where, for all $r \in \mathbb{N}$, the family of operators $\{T_{g_r,p}\}_{p\in\mathbb{N}}$ is the Berezin-Toeplitz quantization of g_r in the sense of (1.1.3). Here the notation $O(p^{-\infty})$ means that, for all $k \in \mathbb{N}$, the sum up to order k is a $O(p^{-k})$ of the left member for the operator norm.

In [51, Th.1.1], Ma and Marinescu proved that the set of Toeplitz operators as defined in (1.1.4) forms an algebra. More precisely, given $f, g \in \mathscr{C}^\infty(X,\mathrm{End}(E))$, they established that

$$T_{f,p}T_{g,p} = \sum_{r=0}^{\infty} p^{-r}T_{C_r(f,g),p} + O(p^{-\infty}), \tag{1.1.5}$$

where C_r are bidifferential operators, with $C_0(f,g) = fg$. In particular, we get the following formula:

$$T_{f,p}T_{g,p} = T_{fg,p} + O(p^{-1}), \tag{1.1.6}$$

which shows that the composition of two Toeplitz operators approach the usual pointwise composition of endomorphisms in the semi-classical limit, when p tends to ∞. Moreover, in the case $f, g \in \mathscr{C}^\infty(X)$, they showed that $C_1(f,g) - C_1(g,f) = \sqrt{-1}\{f,g\}$, where $\{.,.\}$ denotes the Poisson bracket associated to the symplectic form $2\pi\omega$. We thus get the following formula:

$$[T_{f,p}, T_{g,p}] = p^{-1}T_{\{f,g\},p} + O(p^{-2}), \tag{1.1.7}$$

which shows that the family $\{T_{f,p}\}_{p\in\mathbb{N}}$ indeed satisfies the expected semi-classical limit for a quantization.

Especially interesting is the case of J coming from a complex structure, making X into a Kähler manifold. In this case, we ask $(E,h^E),(L,h^L)$ to be holomorphic Hermitian vector bundles, and ∇^E, ∇^L to be the associated holomorphic Hermitian connections. The *spin*^{c} Dirac operator D_p is then given by

$$D_p = \sqrt{2}(\overline{\partial}^{L^p\otimes E} + \overline{\partial}^{L^p\otimes E,*}), \tag{1.1.8}$$

where $\overline{\partial}^{L^p\otimes E}$ denotes the holomorphic $\overline{\partial}$-operator on $L^p \otimes E$ acting on the Dolbeault complex $\oplus_q\Omega^{0,q}(X, L^p \otimes E) = \mathscr{C}^\infty(X,\mathbf{E}_p)$, and $\overline{\partial}^{L^p\otimes E,*}$ its formal adjoint for the L^2-scalar product. By Hodge theory, we get

$$\mathrm{Ker}(D_p|_{\Omega^{0,q}(X,L^p\otimes E)}) \simeq H^q(X, L^p \otimes E), \tag{1.1.9}$$

where $H^q(X, L^p \otimes E)$ denotes the q^{th} Dolbeault cohomology group associated to $L^p \otimes E$. The prequantization condition (1.1.1) implying L positive, by the Kodaira-Serre vanishing theorem we get for any $q > 0$,

$$H^q(X, L^p \otimes E) = 0, \tag{1.1.10}$$

whenever p is sufficiently large. Picking such a p, the identification (1.1.9) together with (1.1.10) imply $\mathrm{Ker}(D_p) \simeq H^0(X, L^p \otimes E)$, which gives back the usual setting of geometric quantization on Kähler manifolds, the space $H^0(X, L^p \otimes E)$ being the space of holomorphic sections of $L^p \otimes E$. In the general symplectic setting however, Dolbeault cohomology doesn't exist, and $\mathrm{Ker}(D_p)$ is then a natural generalization of the space $H^0(X, L^p \otimes E)$.

The theory of Berezin-Toeplitz quantization in the Kähler case for $E = \mathbb{C}$ has first been developed by Bordemann, Meinreken and Schlichenmaier in [10] and Schlichenmaier in [54]. Their approach is based on the work of Boutet de Monvel and Sjöstrand on the Szegö kernel in [15], and the theory of Toeplitz structures developed by Boutet de Monvel and Guillemin in [14] (see also [17]).

In the Kähler case, the data given in (1.1.3) can be computed much more explicitly, and in [52, Th.0.3], Ma and Marinescu gave the following formula for the second coefficient $C_1(f, g)$ for $f, g \in \mathscr{C}^\infty(X, \mathrm{End}(E))$:

$$C_1(f, g) = -\frac{1}{2\pi}\langle \nabla^{1,0} f, \nabla^{0,1} g\rangle, \tag{1.1.11}$$

where $\nabla^{1,0}$ and $\nabla^{0,1}$ denote the holomorphic and anti-holomorphic part of the connection on $\mathrm{End}(E)$ induced by ∇^E, and $\langle ., .\rangle$ denotes the pairing induced by g^{TX} on $T^*X \otimes \mathrm{End}(E)$ with values in $\mathrm{End}(E)$. The formula (1.1.11) is compatible with the following description of the Poisson bracket in the case $E = \mathbb{C}$:

$$\sqrt{-1}\{f, g\} = -\frac{1}{2\pi}\left(\langle \nabla^{1,0} f, \nabla^{0,1} g\rangle - \langle \nabla^{1,0} g, \nabla^{0,1} f\rangle\right). \tag{1.1.12}$$

Ma and Marinescu also computed the coefficient $C_2(f, g)$ for $f, g \in \mathscr{C}^\infty(X)$, and gave in [52, Th.0.3] formulas for the first coefficients of the expansion in $p \in \mathbb{N}$ of the kernel of a Berezin-Toeplitz operator on the diagonal. These formulas have been used for the study of canonical metrics via balanced embeddings by Fine in [25, Th.10] for the quantization of the Lichnerowicz operator, and then by Keller, Meyer and Seyyedali in [41, Prop.3.6] for the quantization of the Laplacian operator on vector bundles (see also [22] on this last topic). In [33, Th.1.4, Th.1.5], Hsiao gave a new proof of [52, Th.0.2, Th.0.3] for $E = \mathbb{C}$ using results from microlocal analysis of [15]. For another study of these coefficients, see also Karabegov and Schlichenmaier in [39].

In the context of deformation quantization, the properties (1.1.6) and (1.1.7) in the case $E = \mathbb{C}$ imply that the expansion (1.1.5) defines a star product on $\mathscr{C}^\infty(X)$, called the *Berezin-Toeplitz star product* (see for instance [49, Rem.7.4.2] and [61]).

In this chapter, we use methods developed in [51] as well as results of [48] in order to compute $C_1(f, g)$ for $f, g \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ in the general symplectic case described in the beginning of this section. Analogous to (1.1.11), our result is:

Theorem 1.1.1. *Let (X, ω) be a compact symplectic manifold equiped with a Hermitian line bundle with Hermitian connection (L, h^L, ∇^L) satisfying the prequantization condition* (1.1.1)*, and let (E, h^E, ∇^E) be a Hermitian vector bundle with Hermitian* *connection. Let J be an almost complex structure on TX, and g^{TX} be the Riemannian metric defined by* (1.1.2)*. For any $f, g \in \mathscr{C}^\infty(X, \mathrm{End}(E))$, the second coefficient of the asymptotic expansion of $T_{f,p}T_{g,p}$ as in* (1.1.5) *is*

$$C_1(f,g) = -\frac{1}{2\pi}\langle \nabla^{1,0} f, \nabla^{0,1} g\rangle, \tag{1.1.13}$$

where $\nabla^{1,0}$ and $\nabla^{0,1}$ denote the holomorphic and anti-holomorphic part of the connection on $\mathrm{End}(E)$ *induced by ∇^E, and $\langle .,. \rangle$ denotes the pairing induced by g^{TX} on $T^*X \otimes$* $\mathrm{End}(E)$ *with values in* $\mathrm{End}(E)$*.*

In the case $E = \mathbb{C}$, Charles treated the theory of Berezin-Toeplitz quantization for symplectic manifolds in [18] using microlocal analysis of [14], and computed the coefficient $C_1(f,g)$ for $f, g \in \mathscr{C}^\infty(X, \mathbb{C})$ in [19].

The results of this chapter appear in [35]

1.2 Local model for Toeplitz operators

This section is dedicated to set the context and the notations, and to describe the local model which will be used in the next section for the computations.

1.2.1 Setting

Let (X, ω) be a compact symplectic manifold, endowed with an almost complex structure J compatible with ω on its tangent bundle TX. We denote by g^{TX} the Riemannian metric defined by (1.1.2), and by ∇^{TX} the associated Levi-Civita connection on TX. Writing $TX_{\mathbb{C}} = TX \otimes_{\mathbb{R}} \mathbb{C}$ for the complexification of TX, the almost complex structure J induces a splitting

$$TX_{\mathbb{C}} = T^{(1,0)}X \oplus T^{(0,1)}X \tag{1.2.1}$$

on the complexification of the tangent bundle into the eigenspaces of J correponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$ respectively. This allows us to define the total exterior product bundle $\Lambda(T^{*(0,1)}X)$, which is actually a Clifford bundle: for any $v \in TX$ with decomposition $v = v^{(1,0)} + v^{(0,1)}$ according to (1.2.1), we define the Clifford action of v on $\Lambda(T^{*(0,1)}X)$ by

$$c(v) = \sqrt{2}(v^{(1,0),*} - i_{v^{(0,1)}}), \tag{1.2.2}$$

where $v^{(1,0),*}$ denotes the wedge product by the metric dual of $v^{(1,0)}$ in $T^{*(0,1)}X$, and $i_{v^{(0,1)}}$ denotes the contraction by $v^{(0,1)} \in T^{(0,1)}X$. Let $\nabla^{\det}$ be the connection on $\det(T^{(1,0)}X)$ induced by the natural projection of ∇^{TX} on $T^{(1,0)}X$ via the decomposition (1.2.1). We denote by ∇^{Cl} the Clifford connection on $\Lambda(T^{*(0,1)}X)$ induced by ∇^{TX} and $\nabla^{\det}$ as defined in [49, § 1.3.1].

Recall now that ∇^E and ∇^L are Hermitian connections on the Hermitian vector bundle (E, h^E) and the Hermitian line bundle (L, h^L) respectively. The vector bundle

$$\mathbf{E}_p = L^p \otimes \Lambda(T^{*(0,1)}X) \otimes E \tag{1.2.3}$$

is naturally endowed with the Hermitian product induced by g^{TX}, h^L and h^E. Then there is a natural L^2-scalar product on $\mathscr{C}^\infty(X, \mathbf{E}_p)$ induced by the Hermitian product of $\mathbf{E}_p$ and the Riemannian volume form dv_X on X associated to g^{TX}. We denote by $\nabla^{\mathbf{E}_p}$ the connection on $\mathbf{E}_p$ induced by $\nabla^{\mathrm{Cl}}, \nabla^L$ and ∇^E. We define then the *spin*^{c} Dirac operator D_p locally by

$$D_p = \sum_{j=1}^{2n} c(e_j)\nabla^{\mathbf{E}_p}_{e_j} : \mathscr{C}^\infty(X, \mathbf{E}_p) \to \mathscr{C}^\infty(X, \mathbf{E}_p), \tag{1.2.4}$$

where $\{e_j\}_{j=1}^{2n}$ is any local orthonormal frame of TX with respect to g^{TX}. Then D_p is a formally self-adjoint operator on $\mathscr{C}^\infty(X, \mathbf{E}_p)$ with respect to the L^2-scalar product.

1.2.2 Model operator

Let us fix a point $x_0 \in X$, and let us choose $\varepsilon_0 > 0$ so that the exponential map at x_0 induces a diffeomorphism between the geodesic ball $B^X(x_0, \varepsilon_0) \subset X$ and the open ball $B(0, \varepsilon_0) \subset T_{x_0}X$, where $T_{x_0}X$ is endowed with the Euclidean metric induced by g^{TX}. We trivialize L, E and $\mathbf{E}_p$ over $B(0, \varepsilon_0)$ by identification with their respective fibre at x_0 through parallel transport with respect to their respective connections along geodesics. We then identify L_{x_0} with $\mathbb{C}$ choosing a unit vector. Let us note that as $\mathrm{End}(L^p_{x_0})$ is canonically identified with $\mathbb{C}$, our results will not depend on this choice.

Let $\{w_j\}_{j=1}^n$ be an orthonormal basis of $T^{(1,0)}_{x_0}X$ with respect to the Hermitian product induced by g^{TX}. This induces a basis $\{e_j\}_{j=1}^{2n}$ of $T_{x_0}X$, orthonormal with respect to the Euclidean product induced by g^{TX}, such that

$$e_{2j-1} = \frac{1}{\sqrt{2}}(w_j + \overline{w}_j) \quad \text{and} \quad e_{2j} = \frac{\sqrt{-1}}{\sqrt{2}}(w_j - \overline{w}_j), \tag{1.2.5}$$

for any $1 \leqslant j \leqslant n$. We use this basis to identify $T_{x_0}X$ with $\mathbb{R}^{2n}$. We denote by $Z = (Z_1, \ldots, Z_{2n})$ the induced real coordinates, and by $z = (z_1, \ldots, z_n)$ the complex coordinates on $\mathbb{C}^n$ such that $z_i = Z_{2i-1} + \sqrt{-1}Z_{2i}$ for any $1 \leqslant i \leqslant n$. We thus have $e_j = \partial/\partial Z_j$ for any $1 \leqslant j \leqslant 2n$, and the following equality of vector fields holds:

$$\sum_{i=1}^{2n} Z_i \frac{\partial}{\partial Z_i} = \sum_{j=1}^{2n} \left(z_j \frac{\partial}{\partial z_j} + \bar{z}_j \frac{\partial}{\partial \bar{z}_j} \right), \tag{1.2.6}$$

which is not to be confused with the equality of coordinates $Z = (z + \bar{z})/2$. We write dZ for the canonical Lebesgue measure of $\mathbb{R}^{2n}$ with respect to the variable Z.

Let us now consider the Hilbert space $L^2(\mathbb{R}^{2n}, (\Lambda(T^{*(0,1)}X) \otimes E)_{x_0})$ with the L^2-scalar product induced by the Hermitian product on $(\Lambda(T^{*(0,1)}X) \otimes E)_{x_0}$ and the Lebesgue measure of $\mathbb{R}^{2n}$. By the identification $T_{x_0}X \cong \mathbb{R}^{2n}$ and the trivializations above, we identify sections in $L^2(X, \mathbf{E}_p)$ with sufficiently small compact support around x_0 with functions in $L^2(\mathbb{R}^{2n}, (\Lambda(T^{*(0,1)}X) \otimes E)_{x_0})$.

It is shown in the complex case in [49, Chap.4] and generalized to the symplectic case in [49, Chap.8] how the operator $(D_p)^2$ is approximated for large p, after a convenient rescaling in $\sqrt{p}$, by an operator $\mathscr{L}_0$ acting on $\mathscr{C}^\infty(\mathbb{R}^{2n}, (\Lambda(T^{*(0,1)}X) \otimes E)_{x_0})$ defined by the formulas

$$\begin{aligned} \mathscr{L}_0 = \mathscr{L} + 4\pi \sum_{j=1}^n \overline{w}_j^* i_{\overline{w}_j}, \qquad \mathscr{L} = \sum_{j=1}^n b_j b_j^+, \\ b_i = -2\frac{\partial}{\partial z_i} + \pi \bar{z}_i, \qquad b_i^+ = 2\frac{\partial}{\partial \bar{z}_i} + \pi z_i, \end{aligned} \tag{1.2.7}$$

for any $1 \leqslant i \leqslant n$, where z_i and $\bar{z}_i$ denote the scalar multiplication on $(\Lambda(T^{*(0,1)}X)\otimes E)_{x_0}$ by z_i and $\bar{z}_i$ respectively. The differential operator $\mathscr{L}$ acts on the scalar part of smooth functions with values in $(\Lambda(T^{*(0,1)}X) \otimes E)_{x_0}$, and can thus be seen as a differential operator on $\mathscr{C}^\infty(\mathbb{R}^{2n})$, still denoted by $\mathscr{L}$. We call $\mathscr{L}$ the *model operator*. It is a densely defined self-adjoint operator on $L^2(\mathbb{R}^{2n})$ and has the following spectral properties:

Proposition 1.2.1. *([49, §4.1.20]) The spectrum of $\mathscr{L}$ on $L^2(\mathbb{R}^{2n})$ is given by*

$$\mathrm{Spec}(\mathscr{L}) = \left\{ 4\pi \sum_{i=1}^n \alpha_i \;\middle|\; \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n \right\}, \tag{1.2.8}$$

and an orthogonal basis of the eigenspace indexed by $\alpha \in \mathbb{N}^n$ as in (1.2.8) *is given by*

$$b^\alpha \left(z^\beta \exp\left(-\frac{\pi}{2} \sum_{i=1}^n |z_i|^2 \right) \right) \text{ for any } \beta \in \mathbb{N}^n. \tag{1.2.9}$$

The corresponding orthogonal projection $\mathscr{P} : L^2(\mathbb{R}^{2n}) \to \mathrm{Ker}(\mathscr{L})$ has smooth kernel with respect to dZ, which for all $Z, Z' \in \mathbb{R}^{2n}$ is easily computed to be

$$\mathscr{P}(Z, Z') = \exp\left(-\frac{\pi}{2} \sum_{i=1}^n (|z_i|^2 + |z_i'|^2 - 2 z_i \bar{z}_i') \right). \tag{1.2.10}$$

As the operator $\mathscr{L}$ in (1.2.7) acts only on the scalar part of functions with values in $(\Lambda(T^{*(0,1)}X) \otimes E)_{x_0}$, the kernel of the associated projection

$$\mathscr{P} : L^2(\mathbb{R}^{2n}, (\Lambda(T^{*(0,1)}X) \otimes E)_{x_0})) \to \mathrm{Ker}(\mathscr{L}) \tag{1.2.11}$$

acts on $(\Lambda(T^{*(0,1)}X) \otimes E)_{x_0}$ by scalar multiplication and is still given by (1.2.10).

The space $(\Lambda(T^{*(0,1)}X) \otimes E)_{x_0}$ has a natural $\mathbb{Z}$-graduation given by the one on $\Lambda(T^{*(0,1)}X)$, and we denote by $(\mathbb{C} \otimes E)_{x_0}$ the degree 0 subspace of $(\Lambda(T^{*(0,1)}X) \otimes E)_{x_0}$ for this graduation. We write

$$I_{\mathbb{C}\otimes E} : (\Lambda(T^{*(0,1)}X) \otimes E)_{x_0} \to (\mathbb{C} \otimes E)_{x_0} \tag{1.2.12}$$

for the natural projection, which is orthogonal with respect to the Hermitian product. It induces a projection on the L^2-sections, still denoted by $I_{\mathbb{C}\otimes E}$, which is then orthogonal with respect to the L^2-scalar product.

The two terms defining $\mathscr{L}_0$ in (1.2.7) are positive commuting operators, and the orthogonal projections on their kernels in $L^2(\mathbb{R}^{2n}, (\Lambda(T^{*(0,1)}X) \otimes E)_{x_0})$ are given by $\mathscr{P}$ and $I_{\mathbb{C}\otimes E}$ respectively. Consequently, the orthogonal projection on the kernel of $\mathscr{L}_0$, seen as a densely defined self-adjoint operator acting on $L^2(\mathbb{R}^{2n}, (\Lambda(T^{*(0,1)}X) \otimes E)_{x_0})$, is given by

$$P = \mathscr{P} I_{\mathbb{C}\otimes E} : L^2(\mathbb{R}^{2n}, (\Lambda(T^{*(0,1)}X) \otimes E)_{x_0}) \to \mathrm{Ker}(\mathscr{L}_0). \tag{1.2.13}$$

We denote by $\mathrm{Ker}(\mathscr{L}_0)^\perp$ the orthogonal space of $\mathrm{Ker}(\mathscr{L}_0)$ in $L^2(\mathbb{R}^{2n}, (\Lambda(T^{*(0,1)}X) \otimes E)_{x_0})$, and by $P^\perp$ the associated orthogonal projection. Using Proposition 1.2.1 and (1.2.7), it is easy to compute explicitly the inverse of $\mathscr{L}_0$ on $\mathrm{Ker}(\mathscr{L}_0)^\perp$ by inverting its eigenvalues. It thus makes sense to write

$$\mathscr{L}_0^{-1} P^\perp : L^2(\mathbb{R}^{2n}, (\Lambda(T^{*(0,1)}X) \otimes E)_{x_0}) \to L^2(\mathbb{R}^{2n}, (\Lambda(T^{*(0,1)}X) \otimes E)_{x_0}). \tag{1.2.14}$$

1.2.3 Kernel calculus

We introduce now the kernel calculus on $\mathbb{C}^n$ developed by Ma and Marinescu in [51], which will be the basis for our calculations in the next section.

If T is a bounded operator on $L^2(\mathbb{R}^{2n}, (\Lambda(T^{*(0,1)}X) \otimes E)_{x_0})$ with smooth kernel with respect to dZ, we will denote its evaluation at $Z, Z' \in \mathbb{R}^{2n}$ by

$$T(Z, Z') \in \mathrm{End}(\Lambda(T^{*(0,1)}X) \otimes E)_{x_0}. \tag{1.2.15}$$

If $F(Z, Z') \in \mathrm{End}(\Lambda(T^{*(0,1)}X) \otimes E)_{x_0}$ is a polynomial in $Z, Z' \in \mathbb{R}^{2n}$, we denote by $F\mathscr{P}$ the operator on $L^2(\mathbb{R}^{2n}, (\Lambda(T^{*(0,1)}X) \otimes E)_{x_0})$ defined by the kernel $F(Z, Z')\mathscr{P}(Z, Z')$, so that

$$(F\mathscr{P})(Z, Z') = F(Z, Z')\mathscr{P}(Z, Z'), \tag{1.2.16}$$

for all $Z, Z' \in \mathbb{R}^{2n}$. By the explicit expression of $\mathscr{P}(Z, Z')$ in (1.2.10), the formula (1.2.16) defines in fact a bounded operator on $L^2(\mathbb{R}^{2n}, (\Lambda(T^{*(0,1)}X) \otimes E)_{x_0})$.

Using these notations, we can state the following result, which comes essentially from [51, § 2].

Proposition 1.2.2. *For any $Q(Z, Z')$ and $F(Z, Z') \in \mathrm{End}(\Lambda(T^{*(0,1)}X) \otimes E)_{x_0}$, polynomials in $Z, Z' \in \mathbb{R}^{2n}$, there exists $\mathcal{K}\big[F, Q\big](Z, Z') \in \mathrm{End}(\Lambda(T^{*(0,1)}X) \otimes E)_{x_0}$, polynomial in $Z, Z' \in \mathbb{R}^{2n}$, such that*

$$\mathcal{K}\big[F, Q\big]\mathscr{P} = (F\mathscr{P})(Q\mathscr{P}), \tag{1.2.17}$$

where the left hand side denotes the composition of two operators defined by their kernels as in (1.2.16)*.*

Furthermore, for all $F(Z, Z')$, $G(Z, Z')$ and $H(Z, Z') \in \mathrm{End}(\Lambda(T^{*(0,1)}X) \otimes E)_{x_0}$, polynomials in $Z, Z' \in \mathbb{R}^{2n}$, the following formulas hold:

$$\mathcal{K}\Big[F,\mathcal{K}\Big[G,H\Big]\Big]=\mathcal{K}\Big[\mathcal{K}\Big[F,G\Big],H\Big], \tag{1.2.18}$$

$$\mathcal{K}\Big[I_{\mathbb{C}\otimes E},I_{\mathbb{C}\otimes E}\Big]=I_{\mathbb{C}\otimes E}. \tag{1.2.19}$$

For any $q(Z)$ polynomial in Z with scalar values,

$$\mathcal{K}\Big[F,q(Z)G\Big]=\mathcal{K}\Big[q(Z')F,G\Big]. \tag{1.2.20}$$

For any $Q(Z)$ polynomial in Z with values in $\mathrm{End}(E_{x_0})$*,*

$$\begin{aligned}\mathcal{K}\Big[Q(Z)F,G\Big]&=Q(Z)\mathcal{K}\Big[F,G\Big],\\ \mathcal{K}\Big[F,GQ(Z')\Big]&=\mathcal{K}\Big[F,G\Big]Q(Z').\end{aligned} \tag{1.2.21}$$

Finally, for $A\in\mathrm{End}(E_{x_0})$ and $G(Z,Z')\in\mathrm{End}(\Lambda(T^{*(0,1)}X)\otimes E)_{x_0}$ polynomial in $Z,Z'\in\mathbb{R}^{2n}$ commuting with A, we have:

$$\begin{aligned}A\mathcal{K}\Big[G,F\Big]&=\mathcal{K}\Big[GA,F\Big]=\mathcal{K}\Big[G,AF\Big],\\ \mathcal{K}\Big[FA,G\Big]&=\mathcal{K}\Big[F,AG\Big]=\mathcal{K}\Big[F,G\Big]A.\end{aligned} \tag{1.2.22}$$

Proof. Let us first deal with the case $F=1$. The kernel of the composition $\mathscr{P}(G\mathscr{P})$ is given by $\mathscr{P}(G(Z,Z')\mathscr{P}(Z,Z'))$, where the operator $\mathscr{P}$ acts on the variable $Z\in\mathbb{R}^{2n}$. Thus the variable $Z'\in\mathbb{R}^{2n}$ acts as a parameter in this situation, and we are reduced to the case $G(Z,Z')=G(Z)$ not depending on Z'. Using Proposition 1.2.1, this can be computed by induction on the degree of G in $z,\bar{z}\in\mathbb{C}^n$:

First, by Proposition 1.2.1 and (1.2.10), we get

$$\mathscr{P}(z^\beta\mathscr{P}(Z,Z'))=z^\beta\mathscr{P}(Z,Z'), \tag{1.2.23}$$

for any $\beta\in\mathbb{N}^n$. Next, let us notice that by (1.2.7) and (1.2.13),

$$b_i\mathscr{P}(Z,Z')=2\pi(\bar{z}_i-\bar{z}'_i)\mathscr{P}(Z,Z'). \tag{1.2.24}$$

Here, b_i defined in (1.2.7) acts on the Z variable. As $b_i\mathscr{P}(Z,Z')$ is in the orthogonal of $\mathrm{Ker}(\mathscr{L})$ by Proposition 1.2.1, we get using (1.2.23),

$$\mathscr{P}(\bar{z}_i\mathscr{P}(Z,Z'))=\bar{z}'_i\mathscr{P}(Z,Z'). \tag{1.2.25}$$

Now by the definition of b_i in (1.2.7), we know that

$$[G(Z),b_i]=2\frac{\partial}{\partial z_i}G(Z), \tag{1.2.26}$$

for any $G(Z) \in \mathrm{End}(\Lambda(T^{*(0,1)}X) \otimes E)_{x_0}$ polynomial in $Z \in \mathbb{R}^{2n}$. Using the fact that $b^\alpha z^\beta \mathscr{P}(Z, Z')$ is in $\mathrm{Ker}(\mathscr{L})^\perp$ by Proposition 1.2.1, we can compute the case of general F by induction through repeated applications of (1.2.24) and (1.2.26).

Now for the general case, if $F_1(Z)$ and $F_2(Z') \in \mathrm{End}(\Lambda(T^{*(0,1)}X) \otimes E)_{x_0}$ are two polynomials in Z and $Z' \in \mathbb{R}^{2n}$ respectively, by the definition of operators associated to kernels as in (1.2.16), we have the following two easy facts:

$$\begin{aligned} (F_1(Z)G\mathscr{P})(Z, Z') &= F_1(Z)(G\mathscr{P})(Z, Z'), \\ (GF_2(Z')\mathscr{P})(Z, Z') &= (G\mathscr{P})(Z, Z')F_2(Z'), \end{aligned} \tag{1.2.27}$$

where we use for the second equality the fact that $\mathscr{P}(Z, Z')$ has scalar values by (1.2.10). By (1.2.17) and the usual formula for composition of kernels, recall that

$$\left(\mathcal{K}\Big[F, G\Big]\mathscr{P}\right)(Z, Z') = \int_{\mathbb{R}^{2n}} F(Z, Z'')\mathscr{P}(Z, Z'')G(Z'', Z')\mathscr{P}(Z'', Z')dZ''. \tag{1.2.28}$$

Then from (1.2.27) and (1.2.28), for any $F(Z, Z') \in \mathrm{End}(\Lambda(T^{*(0,1)}X) \otimes E)_{x_0}$ polynomial in $Z, Z' \in \mathbb{R}^{2n}$,

$$\begin{aligned} \left(\mathcal{K}\Big[F_1(Z)F, G\Big]\mathscr{P}\right)(Z, Z') &= F_1(Z)\left(\mathcal{K}\Big[F, G\Big]\mathscr{P}\right)(Z, Z'), \\ \left(\mathcal{K}\Big[FF_2(Z'), G\Big]\mathscr{P}\right)(Z, Z') &= \left(\mathcal{K}\Big[F, F_2(Z)G\Big]\mathscr{P}\right)(Z, Z'), \end{aligned} \tag{1.2.29}$$

so the general case reduces to the previous one. As a byproduct of (1.2.27) and (1.2.29), we get (1.2.20) and (1.2.21) as well.

The associativity (1.2.18) is obvious from (1.2.17). As $\mathscr{P}(Z, Z')$ commutes with $I_{\mathbb{C}\otimes E}$ by (1.2.10), we get (1.2.19). Finally, (1.2.27) and (1.2.29) applied to $A \in \mathrm{End}(E_{x_0})$ constant and commuting with G gives (1.2.22). □

Proposition 1.2.2, together with its proof, is at the basis of the computations in this chapter. As an application, we compute the following special cases of the kernel calculus for any $1 \leqslant i, j \leqslant n$, which will be used constantly in the forthcoming computations:

$$\begin{aligned} &\mathcal{K}\Big[I_{\mathbb{C}\otimes E}, \bar{z}_j I_{\mathbb{C}\otimes E}\Big] = \bar{z}'_j I_{\mathbb{C}\otimes E}, \quad \mathcal{K}\Big[I_{\mathbb{C}\otimes E}, z_j I_{\mathbb{C}\otimes E}\Big] = z_j I_{\mathbb{C}\otimes E}, \\ &\mathcal{K}\Big[z_i I_{\mathbb{C}\otimes E}, \bar{z}_j I_{\mathbb{C}\otimes E}\Big] = z_i \bar{z}'_j I_{\mathbb{C}\otimes E}, \quad \mathcal{K}\Big[\bar{z}_i I_{\mathbb{C}\otimes E}, z_j I_{\mathbb{C}\otimes E}\Big] = \bar{z}_i z_j I_{\mathbb{C}\otimes E}, \\ &\mathcal{K}\Big[z'_i I_{\mathbb{C}\otimes E}, \bar{z}_j I_{\mathbb{C}\otimes E}\Big] = \frac{1}{\pi}\delta_{ij} I_{\mathbb{C}\otimes E} + z_i \bar{z}'_j I_{\mathbb{C}\otimes E}, \\ &\mathcal{K}\Big[\bar{z}'_i I_{\mathbb{C}\otimes E}, z_j I_{\mathbb{C}\otimes E}\Big] = \frac{1}{\pi}\delta_{ij} I_{\mathbb{C}\otimes E} + \bar{z}'_i z_j I_{\mathbb{C}\otimes E}, \\ &\mathcal{K}\Big[I_{\mathbb{C}\otimes E}, \bar{z}_i z_j I_{\mathbb{C}\otimes E}\Big] = \frac{1}{\pi}\delta_{ij} I_{\mathbb{C}\otimes E} + \bar{z}'_i z_j I_{\mathbb{C}\otimes E}, \\ &\mathcal{K}\Big[I_{\mathbb{C}\otimes E}, z'_i I_{\mathbb{C}\otimes E}\Big] = z'_i I_{\mathbb{C}\otimes E}, \quad \mathcal{K}\Big[I_{\mathbb{C}\otimes E}, \bar{z}'_i I_{\mathbb{C}\otimes E}\Big] = \bar{z}'_i I_{\mathbb{C}\otimes E}, \\ &\mathcal{K}\Big[I_{\mathbb{C}\otimes E}, z_i z_j I_{\mathbb{C}\otimes E}\Big] = z_i z_j I_{\mathbb{C}\otimes E}, \quad \mathcal{K}\Big[I_{\mathbb{C}\otimes E}, \bar{z}_i \bar{z}_j I_{\mathbb{C}\otimes E}\Big] = \bar{z}'_i \bar{z}'_j I_{\mathbb{C}\otimes E}. \end{aligned} \tag{1.2.30}$$

1.3 Berezin-Toeplitz quantization and Bergman kernel

In this section, we recall the results of [48], [49] and [51] on the reduction of Berezin-Toeplitz quantization to the local model described in Section 1.2, and then go on to the computation of the second coefficient of the asymptotic expansion. The general reference for the background on the theory is [49].

1.3.1 Asymptotic expansion of Toeplitz operators

For any $p \in \mathbb{N}$, let T_p be an operator acting on $L^2(X, \mathbf{E}_p)$ with smooth kernel $T_p(x, x')$ with respect to dv_X at $x, x' \in X$. For all $x_0 \in X$, it induces

$$T_{p,x_0}(Z, Z') \in \operatorname{End}(\Lambda(T^{*(0,1)}X) \otimes E)_{x_0} \tag{1.3.1}$$

through the trivializations given in Section 1.2, where $Z, Z' \in B(0, \varepsilon_0) \subset T_{x_0}X \simeq \mathbb{R}^{2n}$ are the respective images of $x, x' \in B^X(x_0, \varepsilon_0) \subset X$ in the exponential coordinates.

To estimate the kernels of any family $\{T_p\}_{p\in\mathbb{N}}$ of operators acting on $L^2(X, (\Lambda(T^{*(0,1)}X) \otimes E)_{x_0})$, we will use the following notation given in [51, Not.4.4]: we write

$$p^{-n} T_{p,x_0}(Z, Z') \cong \sum_{r=0}^{\infty} (\mathcal{Q}_{r,x_0} \mathscr{P})(\sqrt{p}Z, \sqrt{p}Z') p^{-r/2} + \mathscr{O}(p^{-\infty}), \tag{1.3.2}$$

where $\{\mathcal{Q}_{r,x_0}(Z, Z')\}_{r\in\mathbb{N}}$ is a family of polynomials in $Z, Z' \in \mathbb{R}^{2n}$ with values in $\operatorname{End}(\Lambda(T^{*(0,1)}X) \otimes E)_{x_0}$ and depending smoothly on $x_0 \in X$, if for any $k \in \mathbb{N}$, there is $\varepsilon > 0, C_0 > 0$ such that for any $l \in \mathbb{N}$, there exist $C > 0, M \in \mathbb{N}$, such that for $|Z|, |Z'| < \varepsilon$, the following estimate holds:

$$\begin{aligned} &\Big| p^{-n} T_{p,x_0}(Z, Z') \kappa_{x_0}^{1/2}(Z) \kappa_{x_0}^{1/2}(Z') - \sum_{r=0}^{k} (\mathcal{Q}_{r,x_0} \mathscr{P})(\sqrt{p}Z, \sqrt{p}Z')\, p^{-r/2} \Big|_{\mathscr{C}^l} \\ &\qquad \leqslant C p^{-\frac{k+1}{2}} (1 + \sqrt{p}|Z| + \sqrt{p}|Z'|)^M \exp(-C_0 \sqrt{p}|Z - Z'|) + O(p^{-\infty}). \end{aligned} \tag{1.3.3}$$

Here $|.|_{\mathscr{C}^l}$ denotes the $\mathscr{C}^l$ norm with respect to $x_0 \in X$ with respect to the induced connection on the pullback bundle $\pi^*(\operatorname{End}(\Lambda(T^{*(0,1)}X) \otimes E))$ over $TX \times_X TX$, where $\pi : TX \times_X TX \to X$ is the fibred product of TX with itself over X. In the same way, when we say that a polynomial in $Z, Z' \in \mathbb{R}^{2n}$ with values in $\operatorname{End}(\Lambda(T^{*(0,1)}X) \otimes E)_{x_0}$ depends smoothly on $x_0 \in X$, it is in that sense.

The function $\kappa_{x_0} \in \mathscr{C}^\infty(B(0, \varepsilon_0))$ is defined for $Z \in B(0, \varepsilon_0) \subset \mathbb{R}^{2n}$ by

$$dv_X(Z) = \kappa_{x_0}(Z) dZ. \tag{1.3.4}$$

Its appearance in the formula (1.3.3) is necessary to make the comparison between kernels consistent. Note that $\kappa_{x_0}(0) = 1$.

We will apply the notation (1.3.3) to estimate the kernel of the Bergman projection P_p on $\operatorname{Ker}(D_p)$ as defined in the introduction, namely through the following *off-diagonal expansion* of the Bergman kernel:

Theorem 1.3.1. *([20, Th.4.18']) There exists a family* $\{J_{r,x_0}\}_{r\in\mathbb{N}}$ *of polynomials in* $Z, Z' \in \mathbb{R}^{2n}$ *with values in* $\mathrm{End}(\Lambda(T^{*(0,1)}X)\otimes E)_{x_0}$ *and depending smoothly on* $x_0 \in X$ *such that the following expansion holds in the sense of* (1.3.3)*:*

$$p^{-n}P_{p,x_0}(Z,Z') \cong \sum_{r=0}^{\infty}(J_{r,x_0}\mathscr{P})(\sqrt{p}Z,\sqrt{p}Z')p^{-r/2} + \mathscr{O}(p^{-\infty}). \tag{1.3.5}$$

Theorem 1.3.1 gives a strong control on the Bergman kernel outside the diagonal, and is used in [51] to prove the following result:

Theorem 1.3.2. *([51, Th.1.1]) Let* f *and* $g \in \mathscr{C}^\infty(X, \mathrm{End}(E))$*. There exist families of polynomials in* $Z, Z' \in \mathbb{R}^{2n}$ *with values in* $\mathrm{End}(\Lambda(T^{*(0,1)}X)\otimes E)_{x_0}$ *and depending smoothly on* $x_0 \in X$*, respectively denoted by* $\{\mathcal{Q}_{r,x_0}(f)\}_{r\in\mathbb{N}}$ *and* $\{\mathcal{Q}_{r,x_0}(f,g)\}_{r\in\mathbb{N}}$*, such that the following expansions hold in the sense of* (1.3.2)*:*

$$p^{-n}T_{f,p,x_0}(Z,Z') \cong \sum_{r=0}^{\infty}(\mathcal{Q}_{r,x_0}(f)\mathscr{P})(\sqrt{p}Z,\sqrt{p}Z')p^{-r/2} + \mathscr{O}(p^{-\infty}), \tag{1.3.6}$$

$$p^{-n}(T_{f,p}T_{g,p})_{x_0}(Z,Z') \cong \sum_{r=0}^{\infty}(\mathcal{Q}_{r,x_0}(f,g)\mathscr{P})(\sqrt{p}Z,\sqrt{p}Z')p^{-r/2} + \mathscr{O}(p^{-\infty}). \tag{1.3.7}$$

Recall that in Section 1.2, we defined an operator $\mathscr{L}_0$ aproximating D_p for large p, after a convenient rescaling in $\sqrt{p}$. We can refine this by the following (see [49, § 4.1.3] for a precise statement): after a convenient rescaling in $\sqrt{p} =: 1/t$, the restriction of D_p on $B^X(x_0,\varepsilon_0)$ is equal, through the trivializations of Section 1.2, to an operator $\mathscr{L}_2^t$ on $B(0,\varepsilon_0)$ satisfying

$$\mathscr{L}_2^t = \mathscr{L}_0 + \sum_{r=1}^{m} t^r\mathcal{O}_r + \mathscr{O}(t^{m+1}), \tag{1.3.8}$$

for any $m \in \mathbb{N}$, where $\{\mathcal{O}_r\}_{r\in\mathbb{N}}$ is a family of differential operators of order equal or less than 2, with coefficients explicitly computable in term of local data, and where the differential operator $\mathscr{O}(t^{m+1})$ has its coefficients and their derivatives up to order k dominated by $C_k t^{m+1}$ for any $k \in \mathbb{N}$.

The family of polynomials $\{J_{r,x_0}\}_{r\in\mathbb{N}}$ defined in (1.3.5) can then be computed explicitly by induction using (1.3.8). In particular, the following lemma, which has been established in [48, § 2.2], gives the first three elements of this family:

Lemma 1.3.3. *For* $\mathcal{O}_1$ *and* $\mathcal{O}_2$ *defined by* (1.3.8)*, the following formulas hold:*

$$J_{0,x_0}\mathscr{P} = P, \quad \textit{i.e.} \quad J_{0,x_0} = I_{\mathbb{C}\otimes E}, \tag{1.3.9}$$

$$J_{1,x_0}\mathscr{P} = -\mathscr{L}_0^{-1}P^\perp\mathcal{O}_1P - P\mathcal{O}_1\mathscr{L}_0^{-1}P^\perp, \tag{1.3.10}$$

$$\begin{aligned} J_{2,x_0}\mathscr{P} = \mathscr{L}_0^{-1}P^{\perp}\mathcal{O}_1\mathscr{L}_0^{-1}P^{\perp}\mathcal{O}_1P - \mathscr{L}_0^{-1}P^{\perp}\mathcal{O}_2P \\ + P\mathcal{O}_1\mathscr{L}_0^{-1}P^{\perp}\mathcal{O}_1\mathscr{L}_0^{-1}P^{\perp} - P\mathcal{O}_2\mathscr{L}_0^{-1}P^{\perp} \\ +(\mathscr{L}_0)^{-1}P^{\perp}\mathcal{O}_1P\mathcal{O}_1\mathscr{L}_0^{-1}P^{\perp} - P\mathcal{O}_1(\mathscr{L}_0)^{-2}P^{\perp}\mathcal{O}_1P. \end{aligned} \tag{1.3.11}$$

Moreover, $\mathcal{O}_1$ commutes with any $A \in \mathrm{End}(E_{x_0})$, and we have the formula

$$P\mathcal{O}_1P = 0. \tag{1.3.12}$$

In particular, J_{0,x_0} and J_{1,x_0} commute with any $A \in \mathrm{End}(E_{x_0})$.

See also [49, § 4.1.7] for a detailed exposition in the complex case. The assertion that $\mathcal{O}_1$ commutes with endomorphisms of E_{x_0} is clear from the explicit description given in [48, § 2.2]. This, together with the fact that P and $\mathscr{L}_0$ act on $\mathrm{End}(E_{x_0})$ by scalar multiplication, implies the last assertion. We point out that due to (1.3.12), we give in (1.3.11) a simpler formula than the one appearing in [49, § 4.1.7].

Following the notations at the beginning of this section, for any $f \in \mathscr{C}^\infty(X, \mathrm{End}(E))$, we write f_{x_0} for the induced function on $B(0, \varepsilon_0) \subset T_{x_0}X$ with values in $\mathrm{End}(E_{x_0})$ through the trivialization described in Section 1.2. Note that in particular, $f_{x_0}(0) = f(x_0)$.

To describe the families $\{\mathcal{Q}_{r,x_0}(f)\}_{r\in\mathbb{N}}$ and $\{\mathcal{Q}_{r,x_0}(f,g)\}_{r\in\mathbb{N}}$ defined in (1.3.6) and (1.3.7) respectively, we will need the introduction of the kernel calculus presented in Section 1.2. This is done in the next lemma, which gives as well a formula for the coefficient $C_1(f,g)$ of (1.1.5).

Lemma 1.3.4. *([51, Lem.4.6, § 4.3]) For f and $g \in \mathscr{C}^\infty(X, \mathrm{End}(E))$, the following formulas hold:*

$$\mathcal{Q}_{r,x_0}(f,g) = \sum_{r_1+r_2=r} \mathcal{K}\Big[\mathcal{Q}_{r_1,x_0}(f), \mathcal{Q}_{r_1,x_0}(g)\Big], \tag{1.3.13}$$

$$\mathcal{Q}_{r,x_0}(f) = \sum_{r_1+r_2+|\alpha|=r} \mathcal{K}\Big[J_{r_1,x_0}, \sum_{|\alpha|=2} \frac{\partial^2 f_{x_0}}{\partial Z^\alpha}(0)\frac{Z^\alpha}{\alpha!}J_{r_2,x_0}\Big], \tag{1.3.14}$$

$$\mathcal{Q}_{1,x_0}(f) = f(x_0)J_{1,x_0} + \mathcal{K}\Big[J_{0,x_0}, \sum_{j=1}^{2n} \frac{\partial f_{x_0}}{\partial Z_j}(0) Z_j J_{0,x_0}\Big], \tag{1.3.15}$$

and for $C_1(f,g) \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ defined by (1.1.5), we have

$$C_1(f,g)(x_0)I_{\mathbb{C}\otimes E} = \mathcal{Q}_{2,x_0}(f,g)(0,0) - \mathcal{Q}_{2,x_0}(fg)(0,0). \tag{1.3.16}$$

Note that the assertion (1.3.13) follows formally from the definitions of $\mathcal{Q}_{r,x_0}(f,g)$ and $\mathcal{Q}_{r,x_0}(f)$ in Theorem 1.3.2 and the definition of $\mathcal{K}\Big[.,.\Big]$ in (1.2.17). In the same way, considering the Taylor expansion of f_{x_0} at 0, equation (1.3.14) follows formally from Theorem 1.3.1, Theorem 1.3.2 and the definition of $T_{f,p}$ in (1.1.3).

As an illustration, let us give the calculation leading to (1.3.15). Notice first that from the identity $T_{f,p} = P_p$ for $f = 1$, the definitions of $J_{r,x_0}, \mathscr{Q}_{r,x_0}$ in (1.3.5), (1.3.6) and (1.3.13), we get

$$J_{1,x_0} = \mathcal{K}\Big[J_{0,x_0}, J_{1,x_0}\Big] + \mathcal{K}\Big[J_{1,x_0}, J_{0,x_0}\Big]. \tag{1.3.17}$$

But J_{0,x_0} and J_{1,x_0} commute with any $A \in \mathrm{End}(E_{x_0})$ by Lemma 1.3.3. Thus

$$\begin{aligned} \mathcal{K}\Big[J_{1,x_0}, f(x_0)J_{0,x_0}\Big] &= f(x_0)\mathcal{K}\Big[J_{1,x_0}, J_{0,x_0}\Big], \\ \mathcal{K}\Big[J_{0,x_0}, f(x_0)J_{1,x_0}\Big] &= f(x_0)\mathcal{K}\Big[J_{0,x_0}, J_{1,x_0}\Big]. \end{aligned} \tag{1.3.18}$$

The assertion (1.3.15) then follows directly from (1.3.17) and (1.3.18).

Finally, (1.3.16) follows from (1.1.6), and can be found in [51, (4.82)]. Notice that (1.3.16) says in particular that the right hand side preserves the degree and vanishes on elements of degree > 0, a fact which is absolutely not obvious from the formulas (1.3.13) and (1.3.14).

Thanks to the known eigenvalues of $\mathscr{L}_0^{-1}P^{\perp}$ coming from Proposition 1.2.1 and the explicit expression of $\mathcal{O}_1$ given in [48, § 2.2], the terms appearing in Lemma 1.3.4 can be computed explicitly. We will only need the following special cases.

Lemma 1.3.5. *([48, (2.33), (2.34)]) Let $\langle .,. \rangle$ be the $\mathbb{C}$-bilinear product on $T_{x_0}X \otimes \mathbb{C}$ induced by g^{TX} and let ∇^X be the connection induced on tensors by the Levi-Civita connection ∇^{TX}. Then the following formulas hold:*

$$(P\mathcal{O}_1\mathscr{L}_0^{-1}P^{\perp})(0,Z') = \frac{2\sqrt{-1}}{3}\sum_{i,l,m=1}^{n} z'_i \left\langle (\nabla^X_{\frac{\partial}{\partial z_i}}J)\frac{\partial}{\partial z_l}, \frac{\partial}{\partial z_m}\right\rangle I_{\mathbb{C}\otimes E}\, i_{\frac{\partial}{\partial \bar{z}_m}} i_{\frac{\partial}{\partial \bar{z}_l}} \mathscr{P}(0,Z'), \tag{1.3.19}$$

$$(P\mathcal{O}_1\mathscr{L}_0^{-1}P^{\perp})(Z,0) = \frac{\sqrt{-1}}{3}\sum_{i,l,m=1}^{n} z_i \left\langle (\nabla^X_{\frac{\partial}{\partial z_i}}J)\frac{\partial}{\partial z_l}, \frac{\partial}{\partial z_m}\right\rangle I_{\mathbb{C}\otimes E}\, i_{\frac{\partial}{\partial \bar{z}_l}} i_{\frac{\partial}{\partial \bar{z}_m}} \mathscr{P}(Z,0), \tag{1.3.20}$$

$$(\mathscr{L}_0^{-1}P^{\perp}\mathcal{O}_1P)(Z,0) = -\frac{\sqrt{-1}}{6}\sum_{i,l,m=1}^{n} \bar{z}_i \left\langle (\nabla^X_{\frac{\partial}{\partial \bar{z}_i}}J)\frac{\partial}{\partial z_l}, \frac{\partial}{\partial z_m}\right\rangle d\bar{z}_l d\bar{z}_m I_{\mathbb{C}\otimes E} \mathscr{P}(Z,0). \tag{1.3.21}$$

In the last formula, $d\bar{z}_l d\bar{z}_m$ denotes the wedge product in $\Lambda(T^{*(0,1)}X)$ by $d\bar{z}_l d\bar{z}_m$.

1.3.2 Calculation of the second coefficient

Let f and $g \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ be fixed in all the sequel. In this last part, we will use the results summarized in the previous sections in order to compute the coefficient $C_1(f,g) \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ defined in (1.1.5), thus giving a proof of Theorem 1.1.1.

Recall that by Proposition 1.2.2, the polynomials J_{1,x_0} and J_{2,x_0} commute with any $A \in \mathrm{End}(E_{x_0})$, thus in particular with $h(x_0) = h_{x_0}(0)$ for any $h \in \mathscr{C}^\infty(X, \mathrm{End}(E))$. Thus from (1.2.19), (1.2.22), (1.3.9) and (1.3.14), we have

$$\mathcal{Q}_{0,x_0}(h) = \mathcal{K}\Big[I_{\mathbb{C}\otimes E}, h(x_0) I_{\mathbb{C}\otimes E}\Big] = h(x_0) I_{\mathbb{C}\otimes E}. \tag{1.3.22}$$

Let us develop the terms in the expression of $C_1(f,g)$ given by (1.3.16). By (1.3.13) and (1.3.22), we get on one hand

$$\begin{aligned}\mathcal{Q}_{2,x_0}(f,g) = \\ \mathcal{K}\Big[f(x_0)J_{0,x_0}, \mathcal{Q}_{2,x_0}(g)\Big] + \mathcal{K}\Big[\mathcal{Q}_{2,x_0}(f), g(x_0)J_{0,x_0}\Big] + \mathcal{K}\Big[Q_{1,x_0}(f), Q_{1,x_0}(g)\Big].\end{aligned} \tag{1.3.23}$$

On another hand, we get from Proposition 1.2.2, Lemma 1.3.3 and (1.3.14) that for any $h \in \mathscr{C}^\infty(X, \mathrm{End}(E))$, thus in particular for $h = fg$,

$$\begin{aligned}\mathcal{Q}_{2,x_0}(h) &= h(x_0)\mathcal{K}\Big[J_{0,x_0}, J_{2,x_0}\Big] + \mathcal{K}\Big[J_{2,x_0}, J_{0,x_0}\Big]h(x_0) \\ &+ h(x_0)\mathcal{K}\Big[J_{1,x_0}, J_{1,x_0}\Big] + \sum_{j=1}^{2n} \frac{\partial h_{x_0}}{\partial Z_j}(0)\mathcal{K}\Big[J_{1,x_0}, Z_j J_{0,x_0}\Big] \\ &+ \sum_{j=1}^{2n} \frac{\partial h_{x_0}}{\partial Z_j}(0)\mathcal{K}\Big[J_{0,x_0}, Z_j J_{1,x_0}\Big] + \sum_{|\alpha|=2} \frac{\partial^2 h_{x_0}}{\partial Z^\alpha}(0)\mathcal{K}\Big[J_{0,x_0}, \frac{Z^\alpha}{\alpha!}J_{0,x_0}\Big].\end{aligned} \tag{1.3.24}$$

The computation of $C_1(f,g)$ will be done in three steps: we will first use Proposition 1.2.2 to simplify these expressions everywhere it's possible, then use Lemma 1.3.3 and formal calculus of operators to cancel most of the terms. Finally, we will use Lemma 1.3.5 and the kernel calculus of the previous section to handle the terms containing Z.

We first develop one by one the terms of (1.3.23). Expanding $\mathcal{Q}_{2,x_0}(g)$ inside the expression $\mathcal{K}\Big[f(x_0)J_{0,x_0}, \mathcal{Q}_{2,x_0}(g)\Big]$ using (1.3.24) and simplifying with Proposition 1.2.2,

$$\begin{aligned}&\mathcal{K}\Big[f(x_0)J_{0,x_0}, \mathcal{Q}_{2,x_0}(g)\Big] = f(x_0)g(x_0)\mathcal{K}\Big[J_{0,x_0}, J_{2,x_0}\Big] \\ &+ f(x_0)\mathcal{K}\Big[J_{0,x_0}, \mathcal{K}\Big[J_{2,x_0}, J_{0,x_0}\Big]\Big]g(x_0) + f(x_0)g(x_0)\mathcal{K}\Big[J_{0,x_0}, \mathcal{K}\Big[J_{1,x_0}, J_{1,x_0}\Big]\Big] \\ &+ f(x_0)\sum_{j=1}^{2n} \frac{\partial g_{x_0}}{\partial Z_j}(0)\mathcal{K}\Big[J_{0,x_0}, \mathcal{K}\Big[J_{1,x_0}, Z_j J_{0,x_0}\Big]\Big] \\ &+ f(x_0)\sum_{j=1}^{2n} \frac{\partial g_{x_0}}{\partial Z_j}(0)\mathcal{K}\Big[J_{0,x_0}, Z_j J_{1,x_0}\Big] + f(x_0)\sum_{|\alpha|=2} \frac{\partial^2 g_{x_0}}{\partial Z^\alpha}(0)\mathcal{K}\Big[J_{0,x_0}, \frac{Z^\alpha}{\alpha!}J_{0,x_0}\Big].\end{aligned} \tag{1.3.25}$$

We expand in the same way $\mathscr{Q}_{2,x_0}(f)$ inside $\mathcal{K}\Big[\mathscr{Q}_{2,x_0}(f), g(x_0)J_{0,x_0}\Big]$ using Proposition 1.2.2 and (1.3.24),

$$\begin{aligned}
&\mathcal{K}\Big[\mathscr{Q}_{2,x_0}(f), g(x_0)J_{0,x_0}\Big] = \mathcal{K}\Big[J_{2,x_0}, J_{0,x_0}\Big]f(x_0)g(x_0)\\
&+ f(x_0)\mathcal{K}\Big[\mathcal{K}\Big[J_{0,x_0}, J_{2,x_0}\Big], J_{0,x_0}\Big]g(x_0) + f(x_0)g(x_0)\mathcal{K}\Big[\mathcal{K}\Big[J_{1,x_0}, J_{1,x_0}\Big], J_{0,x_0}\Big]\\
&+ \sum_{j=1}^{2n}\frac{\partial f_{x_0}}{\partial Z_j}(0)g(x_0)\mathcal{K}\Big[\mathcal{K}\Big[J_{1,x_0}, Z_jJ_{0,x_0}\Big], J_{0,x_0}\Big]\\
&+ \sum_{j=1}^{2n}\frac{\partial f_{x_0}}{\partial Z_j}(0)g(x_0)\mathcal{K}\Big[\mathcal{K}\Big[J_{0,x_0}, Z_jJ_{1,x_0}\Big], J_{0,x_0}\Big]\\
&+ \sum_{|\alpha|=2}\frac{\partial^2 f_{x_0}}{\partial Z^\alpha}(0)g(x_0)\mathcal{K}\Big[\mathcal{K}\Big[J_{0,x_0}, \frac{Z^\alpha}{\alpha!}J_{0,x_0}\Big], J_{0,x_0}\Big].
\end{aligned} \tag{1.3.26}$$

We then use Proposition 1.2.2 and (1.3.15) to expand $\mathscr{Q}_{1,x_0}(f)$ and $\mathscr{Q}_{1,x_0}(g)$ inside the last term of (1.3.23),

$$\begin{aligned}
&\mathcal{K}\Big[\mathscr{Q}_{1,x_0}(f), \mathscr{Q}_{1,x_0}(g)\Big] = f(x_0)g(x_0)\mathcal{K}\Big[J_{1,x_0}, J_{1,x_0}\Big]\\
&+ f(x_0)\sum_{j=1}^{2n}\frac{\partial g_{x_0}}{\partial Z_j}(0)\mathcal{K}\Big[J_{1,x_0}, \mathcal{K}\Big[J_{0,x_0}, Z_jJ_{0,x_0}\Big]\Big]\\
&+ \sum_{j=1}^{2n}\frac{\partial f_{x_0}}{\partial Z_j}(0)g(x_0)\mathcal{K}\Big[\mathcal{K}\Big[J_{0,x_0}, Z_jJ_{0,x_0}\Big], J_{1,x_0}\Big]\\
&+ \sum_{i,j=1}^{2n}\mathcal{K}\Big[\mathcal{K}\Big[J_{0,x_0}, \frac{\partial f_{x_0}}{\partial Z_i}(0)Z_iJ_{0,x_0}\Big], \mathcal{K}\Big[J_{0,x_0}, \frac{\partial g_{x_0}}{\partial Z_j}(0)Z_jJ_{0,x_0}\Big]\Big].
\end{aligned} \tag{1.3.27}$$

Let us now add together (1.3.25), (1.3.26) and (1.3.27) to get (1.3.23). We first point out some useful cancellations: by (1.2.17) and (1.3.10), we get

$$\begin{aligned}
\mathcal{K}\Big[J_{0,x_0}, \mathcal{K}\Big[J_{1,x_0}, J_{1,x_0}\Big]\Big]\mathscr{P} &= P(J_{1,x_0}\mathscr{P})(J_{1,x_0}\mathscr{P})P\\
&= P\mathcal{O}_1(\mathscr{L}_0)^{-2}P^\perp\mathcal{O}_1P,
\end{aligned} \tag{1.3.28}$$

$$\mathcal{K}\Big[\mathcal{K}\Big[J_{1,x_0}, J_{1,x_0}\Big], J_{0,x_0}\Big]\mathscr{P} = P\mathcal{O}_1(\mathscr{L}_0)^{-2}P^\perp\mathcal{O}_1P. \tag{1.3.29}$$

On another hand, by (1.2.17), (1.3.9) and (1.3.11), we get

$$\begin{aligned}
\mathcal{K}\Big[J_{0,x_0}, \mathcal{K}\Big[J_{2,x_0}, J_{0,x_0}\Big]\Big]\mathscr{P} &= \mathcal{K}\Big[\mathcal{K}\Big[J_{0,x_0}, J_{2,x_0}\Big], J_{0,x_0}\Big]\mathscr{P}\\
&= P(J_{2,x_0}\mathscr{P})P\\
&= -P\mathcal{O}_1(\mathscr{L}_0)^{-2}P^\perp\mathcal{O}_1P.
\end{aligned} \tag{1.3.30}$$

As the operator on the right hand side of the equations (1.3.28)-(1.3.30) commutes with constant endomorphisms by Lemma 1.3.3, equations (1.3.28)-(1.3.30) and (1.2.22) show that the second and third terms of (1.3.25) cancel each other, as well as the second and third terms of (1.3.26).

Now, in the difference $\mathcal{Q}_{2,x_0}(f,g)-\mathcal{Q}_{2,x_0}(fg)$, the first three terms of (1.3.24) with $h=fg$ cancel with the first terms of (1.3.25), (1.3.26) and (1.3.27) respectively.

Using (1.3.9) and the cancellations above, we are now ready to describe the terms of $\mathcal{Q}_{2,x_0}(f,g)-\mathcal{Q}_{2,x_0}(fg)$. Let us define I_1, I_2, I_3 and I_4, polynomials in $Z, Z' \in \mathbb{R}^{2n}$ with values in $\mathrm{End}(\Lambda(T^{*(0,1)}X)\otimes E)_{x_0}$, by the following formulas:

$$\begin{aligned}
I_1 = &-\sum_{|\alpha|=2}\frac{\partial^2 (fg)_{x_0}}{\partial Z^\alpha}(0)\mathcal{K}\Big[I_{\mathbb{C}\otimes E}, \frac{Z^\alpha}{\alpha!}I_{\mathbb{C}\otimes E}\Big]\\
&+f(x_0)\sum_{|\alpha|=2}\frac{\partial^2 g_{x_0}}{\partial Z^\alpha}(0)\mathcal{K}\Big[I_{\mathbb{C}\otimes E}, \frac{Z^\alpha}{\alpha!}I_{\mathbb{C}\otimes E}\Big]\\
&+\sum_{|\alpha|=2}\frac{\partial^2 f_{x_0}}{\partial Z^\alpha}(0)g(x_0)\mathcal{K}\Big[\mathcal{K}\Big[I_{\mathbb{C}\otimes E}, \frac{Z^\alpha}{\alpha!}I_{\mathbb{C}\otimes E}\Big], I_{\mathbb{C}\otimes E}\Big]\\
&+\sum_{i,j=1}^{2n}\mathcal{K}\Big[\mathcal{K}\Big[I_{\mathbb{C}\otimes E}, \frac{\partial f_{x_0}}{\partial Z_i}(0)Z_i I_{\mathbb{C}\otimes E}\Big], \mathcal{K}\Big[I_{\mathbb{C}\otimes E}, \frac{\partial g_{x_0}}{\partial Z_j}(0)Z_j I_{\mathbb{C}\otimes E}\Big]\Big],
\end{aligned} \tag{1.3.31}$$

$$\begin{aligned}
I_2 = &f(x_0)\sum_{j=1}^{2n}\frac{\partial g_{x_0}}{\partial Z_j}(0)\mathcal{K}\Big[I_{\mathbb{C}\otimes E}, \mathcal{K}\Big[J_{1,x_0}, Z_j I_{\mathbb{C}\otimes E}\Big]\Big]\\
&+\sum_{j=1}^{2n}\frac{\partial f_{x_0}}{\partial Z_j}(0)g(x_0)\mathcal{K}\Big[\mathcal{K}\Big[I_{\mathbb{C}\otimes E}, Z_j J_{1,x_0}\Big], I_{\mathbb{C}\otimes E}\Big],
\end{aligned} \tag{1.3.32}$$

$$\begin{aligned}
I_3 = &-\sum_{j=1}^{2n}\frac{\partial (fg)_{x_0}}{\partial Z_j}(0)\mathcal{K}\Big[I_{\mathbb{C}\otimes E}, Z_j J_{1,x_0}\Big]\\
&+f(x_0)\sum_{j=1}^{2n}\frac{\partial g_{x_0}}{\partial Z_j}(0)\mathcal{K}\Big[I_{\mathbb{C}\otimes E}, Z_j J_{1,x_0}\Big]\\
&+\sum_{j=1}^{2n}\frac{\partial f_{x_0}}{\partial Z_j}(0)g(x_0)\mathcal{K}\Big[\mathcal{K}\Big[I_{\mathbb{C}\otimes E}, Z_j I_{\mathbb{C}\otimes E}\Big], J_{1,x_0}\Big],
\end{aligned} \tag{1.3.33}$$

$$\begin{aligned}
I_4 = &-\sum_{j=1}^{2n}\frac{\partial (fg)_{x_0}}{\partial Z_j}(0)\mathcal{K}\Big[J_{1,x_0}, Z_j I_{\mathbb{C}\otimes E}\Big]\\
&+\sum_{j=1}^{2n}\frac{\partial f_{x_0}}{\partial Z_j}(0)g(x_0)\mathcal{K}\Big[\mathcal{K}\Big[J_{1,x_0}, Z_j I_{\mathbb{C}\otimes E}\Big], I_{\mathbb{C}\otimes E}\Big]\\
&+f(x_0)\sum_{j=1}^{2n}\frac{\partial g_{x_0}}{\partial Z_j}(0)\mathcal{K}\Big[J_{1,x_0}, \mathcal{K}\Big[I_{\mathbb{C}\otimes E}, Z_j I_{\mathbb{C}\otimes E}\Big]\Big].
\end{aligned} \tag{1.3.34}$$

Then using (1.3.9) and by (1.3.23)-(1.3.34), we get

$$\mathscr{Q}_{2,x_0}(f,g) - \mathscr{Q}_{2,x_0}(fg) = I_1 + I_2 + I_3 + I_4. \tag{1.3.35}$$

Recall that by definition, the terms I_1, I_2, I_3 and I_4 in (1.3.35) are polynomials in $Z, Z' \in \mathbb{R}^{2n}$. Thus by (1.3.16), in order to compute $C_1(f,g)$, it suffices to compute the values of I_1, I_2, I_3 and I_4 at $Z = Z' = 0$. We compute those values one by one in the following propositions, using the kernel calculus described in Section 1.2.

Proposition 1.3.6. *For all f, g and $h \in \mathscr{C}^\infty(\mathbb{R}^{2n}, \mathrm{End}(E_{x_0}))$, the following formulas hold:*

$$\begin{aligned}\sum_{|\alpha|=2} \frac{\partial^2 h}{\partial Z^\alpha}(0) \mathcal{K}\Big[\mathcal{K}\Big[I_{\mathbb{C}\otimes E}, \frac{Z^\alpha}{\alpha!} I_{\mathbb{C}\otimes E}\Big], I_{\mathbb{C}\otimes E}\Big](0,0)& \\ = \sum_{|\alpha|=2} \frac{\partial^2 h}{\partial Z^\alpha}(0) \mathcal{K}\Big[I_{\mathbb{C}\otimes E}, \frac{Z^\alpha}{\alpha!} I_{\mathbb{C}\otimes E}\Big](0,0) &= \frac{1}{\pi}\sum_{i=1}^n \frac{\partial^2 h}{\partial z_i \partial \bar{z}_i}(0) I_{\mathbb{C}\otimes E},\end{aligned} \tag{1.3.36}$$

$$\begin{aligned}\sum_{i,j=1}^{2n} \frac{\partial f}{\partial Z_j}(0)\frac{\partial g}{\partial Z_i}(0)\mathcal{K}\Big[\mathcal{K}\Big[I_{\mathbb{C}\otimes E}, Z_i I_{\mathbb{C}\otimes E}\Big], \mathcal{K}\Big[I_{\mathbb{C}\otimes E}, Z_j I_{\mathbb{C}\otimes E}\Big]\Big](0,0)& \\ = \frac{1}{\pi}\sum_{i=1}^n \frac{\partial f}{\partial \bar{z}_i}(0)\frac{\partial g}{\partial z_i}(0) I_{\mathbb{C}\otimes E},&\end{aligned} \tag{1.3.37}$$

so that the value at $Z = Z' = 0$ of I_1 in (1.3.35) *is given by*

$$I_1(0,0) = -\frac{1}{\pi}\sum_{j=1}^n \frac{\partial f_{x_0}}{\partial z_j}(0)\frac{\partial g_{x_0}}{\partial \bar{z}_j}(0) I_{\mathbb{C}\otimes E}. \tag{1.3.38}$$

Proof. From (1.2.18), (1.2.19) and (1.2.21), we get

$$\begin{aligned}\sum_{|\alpha|=2} \mathcal{K}\Big[\mathcal{K}\Big[I_{\mathbb{C}\otimes E}, Z^\alpha I_{\mathbb{C}\otimes E}\Big], I_{\mathbb{C}\otimes E}\Big] &= \sum_{|\alpha|=2} \mathcal{K}\Big[\mathcal{K}\Big[(Z')^\alpha I_{\mathbb{C}\otimes E}, I_{\mathbb{C}\otimes E}\Big], I_{\mathbb{C}\otimes E}\Big] \\ &= \sum_{|\alpha|=2} \mathcal{K}\Big[(Z')^\alpha I_{\mathbb{C}\otimes E}, I_{\mathbb{C}\otimes E}\Big] \\ &= \sum_{|\alpha|=2} \mathcal{K}\Big[I_{\mathbb{C}\otimes E}, Z^\alpha I_{\mathbb{C}\otimes E}\Big],\end{aligned} \tag{1.3.39}$$

which shows the first equality of (1.3.36). On another hand, by Proposition 1.2.2, (1.2.6) and (1.2.30), we compute for $h \in \mathscr{C}^\infty(\mathbb{R}^{2n}, \mathrm{End}(E_{x_0}))$,

$$\begin{aligned}\sum_{|\alpha|=2} \frac{\partial^2 h}{\partial Z^\alpha}(0)\mathcal{K}\Big[I_{\mathbb{C}\otimes E}, \frac{Z^\alpha}{\alpha!} I_{\mathbb{C}\otimes E}\Big] &= \frac{1}{\pi}\sum_{i=1}^n \frac{\partial^2 h}{\partial z_i \partial \bar{z}_i}(0) I_{\mathbb{C}\otimes E} \\ + \sum_{i,j=1}^n \bigg(\frac{\partial^2 h}{\partial z_i \partial z_j}(0) z_i z_j &+ \frac{\partial^2 h}{\partial \bar{z}_i \partial \bar{z}_j}(0)\bar{z}'_i \bar{z}'_j + \frac{\partial^2 h}{\partial z_i \partial \bar{z}_j}(0) z_i \bar{z}'_j\bigg) I_{\mathbb{C}\otimes E}.\end{aligned} \tag{1.3.40}$$

Evaluating (1.3.40) at $Z = Z' = 0$ then gives (1.3.36).

By (1.2.6) and (1.2.30), we get for any $f \in \mathscr{C}^\infty(X, \mathrm{End}(E))$,

$$\mathcal{K}\Big[I_{\mathbb{C}\otimes E}, \sum_{i=1}^{2n} \frac{\partial f_{x_0}}{\partial Z_i}(0) Z_i I_{\mathbb{C}\otimes E}\Big] = \sum_{i=1}^{n} \left(\frac{\partial f_{x_0}}{\partial z_i}(0) z_i + \frac{\partial f_{x_0}}{\partial \bar{z}_i}(0) \bar{z}'_i \right) I_{\mathbb{C}\otimes E}. \tag{1.3.41}$$

By (1.2.21), (1.2.22), (1.2.30) and (1.3.41), in the same way than in (1.3.40), we get at $Z = Z' = 0$ for all $f, g \in \mathscr{C}^\infty(X, \mathrm{End}(E))$,

$$\begin{aligned} \sum_{i,j=1}^{2n} \mathcal{K}\Big[\mathcal{K}\Big[I_{\mathbb{C}\otimes E}, \frac{\partial f}{\partial Z_i}(0) Z_i I_{\mathbb{C}\otimes E}\Big], \mathcal{K}\Big[I_{\mathbb{C}\otimes E}, \frac{\partial g(0)}{\partial Z_j} Z_j I_{\mathbb{C}\otimes E}\Big]\Big](0,0) \\ = \sum_{i=1}^{n} \frac{\partial f}{\partial \bar{z}_i}(0) \frac{\partial g}{\partial z_i}(0) \mathcal{K}\Big[\bar{z}'_i I_{\mathbb{C}\otimes E}, z_i I_{\mathbb{C}\otimes E}\Big](0,0) \\ = \frac{1}{\pi} \sum_{i=1}^{n} \frac{\partial f}{\partial \bar{z}_i}(0) \frac{\partial g}{\partial z_i}(0) I_{\mathbb{C}\otimes E}. \end{aligned} \tag{1.3.42}$$

As $I_{\mathbb{C}\otimes E}$ commutes with any $A \in \mathrm{End}(E_{x_0})$, from (1.3.42) we get (1.3.37).

Finally, by the formula (1.3.31) for I_1, equations (1.3.36) and (1.3.37) give

$$\begin{aligned} I_1(0,0) = \\ \frac{1}{\pi} \sum_{i=1}^{n} \left(-\frac{\partial^2 (fg)_{x_0}}{\partial z_i \partial \bar{z}_i}(0) + f(x_0) \frac{\partial g_{x_0}}{\partial z_i \partial \bar{z}_i}(0) + \frac{\partial^2 f_{x_0}}{\partial z_i \partial \bar{z}_i}(0) g(x_0) + \frac{\partial f_{x_0}}{\partial \bar{z}_i}(0) \frac{\partial g_{x_0}}{\partial z_i}(0) \right) I_{\mathbb{C}\otimes E}. \end{aligned} \tag{1.3.43}$$

The equality (1.3.38) then follows immediately from (1.3.43) by Leibniz rule. □

Let us point out that all the terms of I_2, I_3 and I_4 in (1.3.31), (1.3.32) and (1.3.33) contain J_{1,x_0}. We already see from its expression in (1.3.10) that the computations will involve the explicit expression of $\mathcal{O}_1$, and we will thus need to use Lemma 1.3.5.

Proposition 1.3.7. *For any $1 \leqslant i \leqslant 2n$, the following formulas hold:*

$$\begin{aligned} \mathcal{K}\Big[I_{\mathbb{C}\otimes E}, \mathcal{K}\Big[J_{1,x_0}, Z_i I_{\mathbb{C}\otimes E}\Big]\Big](0,0) = 0, \\ \mathcal{K}\Big[I_{\mathbb{C}\otimes E}, \mathcal{K}\Big[Z_i J_{1,x_0}, I_{\mathbb{C}\otimes E}\Big]\Big](0,0) = 0. \end{aligned} \tag{1.3.44}$$

so that

$$I_2(0,0) = 0, \tag{1.3.45}$$

i.e. the polynomial I_2 in (1.3.32) *vanishes at $Z = Z' = 0$.*

Proof. First, we've got by (1.2.17) and (1.3.10),

$$\begin{aligned} \mathcal{K}\Big[I_{\mathbb{C}\otimes E}, \mathcal{K}\Big[J_{1,x_0}, Z_i I_{\mathbb{C}\otimes E}\Big]\Big]\mathscr{P} &= P(J_{1,x_0}\mathscr{P})(Z_i I_{\mathbb{C}\otimes E}\mathscr{P}) \\ &= -P\mathcal{O}_1 \mathscr{L}_0^{-1} P^\perp (Z_i I_{\mathbb{C}\otimes E}\mathscr{P}). \end{aligned} \tag{1.3.46}$$

With the convention that operators always act on the Z variable, equation (1.3.46) gives us

$$\begin{aligned}\mathcal{K}\Big[I_{\mathbb{C}\otimes E},\mathcal{K}\Big[J_{1,x_0},Z_iI_{\mathbb{C}\otimes E}\Big]\Big](Z,Z')\mathscr{P}(Z,Z')&\\ =-P\mathcal{O}_1\mathscr{L}_0^{-1}P^{\perp}(Z_iI_{\mathbb{C}\otimes E}\mathscr{P}(Z,Z'))&\\ =-\int_{\mathbb{R}^{2n}}(P\mathcal{O}_1\mathscr{L}_0^{-1}P^{\perp})(Z,Z'')I_{\mathbb{C}\otimes E}Z''_i\mathscr{P}(Z'',Z')dZ''.&\end{aligned}\tag{1.3.47}$$

Recall that by (1.2.10), $\mathscr{P}(Z,Z')$ commutes with $I_{\mathbb{C}\otimes E}$. By the definition of $I_{\mathbb{C}\otimes E}$ in (1.2.12), we have $i_{\frac{\partial}{\partial\bar{z}_m}}i_{\frac{\partial}{\partial\bar{z}_l}}I_{\mathbb{C}\otimes E}=0$, so that (1.3.19) implies

$$(P\mathcal{O}_1\mathscr{L}_0^{-1}P^{\perp})(0,Z'')I_{\mathbb{C}\otimes E}=0.\tag{1.3.48}$$

We thus deduce from (1.3.47) and (1.3.48) that

$$\begin{aligned}\mathcal{K}\Big[I_{\mathbb{C}\otimes E},\mathcal{K}\Big[J_{1,x_0},Z_iI_{\mathbb{C}\otimes E}\Big]\Big](0,0)&\\ =-\int_{\mathbb{R}^{2n}}(P\mathcal{O}_1\mathscr{L}_0^{-1}P^{\perp})(0,Z'')I_{\mathbb{C}\otimes E}Z''_i\mathscr{P}(Z'',0)dZ''=0.&\end{aligned}\tag{1.3.49}$$

Equation (1.3.49) is precisely the first equality of (1.3.44).

On another hand, by (1.2.18) and (1.2.20),

$$\begin{aligned}\mathcal{K}\Big[I_{\mathbb{C}\otimes E},\mathcal{K}\Big[Z_iJ_{1,x_0},I_{\mathbb{C}\otimes E}\Big]\Big]&=\mathcal{K}\Big[I_{\mathbb{C}\otimes E},Z_i\mathcal{K}\Big[J_{1,x_0},I_{\mathbb{C}\otimes E}\Big]\Big]\\ &=\mathcal{K}\Big[Z'_iI_{\mathbb{C}\otimes E},\mathcal{K}\Big[J_{1,x_0},I_{\mathbb{C}\otimes E}\Big]\Big].\end{aligned}\tag{1.3.50}$$

By (1.3.10),

$$\begin{aligned}\mathcal{K}\Big[J_{1,x_0},I_{\mathbb{C}\otimes E}\Big]\mathscr{P}&=\Big(-\mathscr{L}_0^{-1}P^{\perp}\mathcal{O}_1P-P\mathcal{O}_1\mathscr{L}_0^{-1}P^{\perp}\Big)P\\ &=-\mathscr{L}_0^{-1}P^{\perp}\mathcal{O}_1P.\end{aligned}\tag{1.3.51}$$

But $I_{\mathbb{C}\otimes E}d\bar{z}_ld\bar{z}_m=0$ by (1.2.12), so analogous to (1.3.48) and by (1.3.21), we get

$$I_{\mathbb{C}\otimes E}(\mathscr{L}_0^{-1}P^{\perp}\mathcal{O}_1P)(Z'',0)=0,\tag{1.3.52}$$

so that by (1.3.51) and (1.3.52),

$$\begin{aligned}\mathcal{K}\Big[Z'_iI_{\mathbb{C}\otimes E},\mathcal{K}\Big[J_{1,x_0},I_{\mathbb{C}\otimes E}\Big]\Big](0,0)&=-(Z'_iP\mathscr{L}_0^{-1}P^{\perp}\mathcal{O}_1P)(0,0)\\ &=-\int_{\mathbb{R}^{2n}}Z''_i\mathscr{P}(0,Z'')I_{\mathbb{C}\otimes E}(\mathscr{L}_0^{-1}P^{\perp}\mathcal{O}_1P)(Z'',0)dZ''=0.\end{aligned}\tag{1.3.53}$$

From (1.3.50) and (1.3.53), we get the second equality of (1.3.44).

Finally, equation (1.3.45) follows immediately from (1.3.44) and the definition of I_2 in (1.3.32). □

Proposition 1.3.8. *For any $1 \leqslant i \leqslant 2n$, the following formulas hold:*

$$\begin{aligned}\mathcal{K}\Big[I_{\mathbb{C}\otimes E}, Z_i J_{1,x_0}\Big](0,0) &= \mathcal{K}\Big[I_{\mathbb{C}\otimes E}, \mathcal{K}\Big[I_{\mathbb{C}\otimes E}, Z_i J_{1,x_0}\Big]\Big](0,0)\\ &= \mathcal{K}\Big[\mathcal{K}\Big[I_{\mathbb{C}\otimes E}, Z_i I_{\mathbb{C}\otimes E}\Big], J_{1,x_0}\Big](0,0).\end{aligned} \tag{1.3.54}$$

$$\begin{aligned}\mathcal{K}\Big[J_{1,x_0}, Z_i I_{\mathbb{C}\otimes E}\Big](0,0) &= \mathcal{K}\Big[\mathcal{K}\Big[J_{1,x_0}, Z_i I_{\mathbb{C}\otimes E}\Big], I_{\mathbb{C}\otimes E}\Big](0,0)\\ &= \mathcal{K}\Big[J_{1,x_0}, \mathcal{K}\Big[I_{\mathbb{C}\otimes E}, Z_i I_{\mathbb{C}\otimes E}\Big]\Big](0,0).\end{aligned} \tag{1.3.55}$$

so that the values at $Z = Z' = 0$ of I_3 and I_4 in (1.3.33) *and* (1.3.34) *are respectively*

$$I_3(0,0) = I_4(0,0) = 0, \tag{1.3.56}$$

i.e. the polynomials I_3 and I_4 in (1.3.33) *and* (1.3.34) *vanish at $Z = Z' = 0$.*

Proof. From (1.2.18), (1.2.19) and (1.2.20), we immediately get the first lines of (1.3.54) and (1.3.55). Next, we show that

$$\mathcal{K}\Big[\mathcal{K}\Big[I_{\mathbb{C}\otimes E}, Z_i I_{\mathbb{C}\otimes E}\Big], J_{1,x_0}\Big](0,0) = \mathcal{K}\Big[I_{\mathbb{C}\otimes E}, Z_i J_{1,x_0}\Big](0,0). \tag{1.3.57}$$

By (1.2.30), remembering that $Z = (z + \bar{z})/2$, we get on one hand

$$\mathcal{K}\Big[\mathcal{K}\Big[I_{\mathbb{C}\otimes E}, Z_i I_{\mathbb{C}\otimes E}\Big], J_{1,x_0}\Big] = \frac{1}{2}\mathcal{K}\Big[(z_i + \bar{z}'_i) I_{\mathbb{C}\otimes E}, J_{1,x_0}\Big]. \tag{1.3.58}$$

By (1.2.20) and (1.2.30), we get on another hand

$$\mathcal{K}\Big[I_{\mathbb{C}\otimes E}, Z_i J_{1,x_0}\Big] = \mathcal{K}\Big[Z'_i I_{\mathbb{C}\otimes E}, J_{1,x_0}\Big] = \frac{1}{2}\mathcal{K}\Big[(z'_i + \bar{z}'_i) I_{\mathbb{C}\otimes E}, J_{1,x_0}\Big]. \tag{1.3.59}$$

By (1.3.58) and (1.3.59), to get (1.3.57) it suffices to prove the equality

$$\mathcal{K}\Big[z_i I_{\mathbb{C}\otimes E}, J_{1,x_0}\Big](0,0) = \mathcal{K}\Big[z'_i I_{\mathbb{C}\otimes E}, J_{1,x_0}\Big](0,0) = 0. \tag{1.3.60}$$

At first, by (1.2.21) we have

$$\mathcal{K}\Big[z_i I_{\mathbb{C}\otimes E}, J_{1,x_0}\Big] = z_i \mathcal{K}\Big[I_{\mathbb{C}\otimes E}, J_{1,x_0}\Big], \tag{1.3.61}$$

so that

$$\mathcal{K}\Big[z_i I_{\mathbb{C}\otimes E}, J_{1,x_0}\Big](0,0) = 0. \tag{1.3.62}$$

Then, by (1.2.30) and (1.3.10),

$$\begin{aligned}\mathcal{K}\Big[z'_i I_{\mathbb{C}\otimes E}, J_{1,x_0}\Big]\mathscr{P} &= \mathcal{K}\Big[I_{\mathbb{C}\otimes E}, z_i J_{1,x_0}\Big]\mathscr{P}\\ &= -P\left(z_i \mathscr{L}_0^{-1} P^{\perp} \mathcal{O}_1 P\right) - P\left(z_i P \mathcal{O}_1 \mathscr{L}_0^{-1} P^{\perp}\right).\end{aligned} \tag{1.3.63}$$

As $I_{\mathbb{C}\otimes E} d\bar{z}_l d\bar{z}_m = 0$ by (1.2.12), analogous to (1.3.48) and (1.3.52), by (1.3.21),

$$I_{\mathbb{C}\otimes E}(\mathscr{L}_0^{-1} P^{\perp} \mathcal{O}_1 P)(Z,0) = 0. \tag{1.3.64}$$

As $P = \mathscr{P} I_{\mathbb{C}\otimes E}$ by (1.2.13), we deduce that

$$\begin{aligned} P(z_i \mathscr{L}_0^{-1} P^{\perp} \mathcal{O}_1 P)(0,0) &= \int_{\mathbb{R}^{2n}} \mathscr{P}(0, Z'') z_i'' I_{\mathbb{C}\otimes E} (\mathscr{L}_0^{-1} P^{\perp} \mathcal{O}_1 P)(Z'', 0) dZ'' \\ &= 0, \end{aligned} \tag{1.3.65}$$

i.e. the kernel of the first term of the last line of (1.3.63) cancels at $Z = Z' = 0$. On the other hand, by (1.3.20) we can write

$$(z_i P \mathcal{O}_1 \mathscr{L}_0^{-1} P^{\perp})(Z, 0) = H(z) \mathscr{P}(Z, 0), \tag{1.3.66}$$

with $H(z) \in \operatorname{End}(\Lambda(T^{*(0,1)}X) \otimes E)_{x_0}$ polynomial in $z \in \mathbb{C}^n$. Recall from (1.2.23) that in this case, again with the convention that operators act on the Z variable, we get

$$\mathscr{P}(H(z)\mathscr{P})(Z, 0) = H(z) \mathscr{P}(Z, 0), \tag{1.3.67}$$

so that by (1.2.13), (1.2.27), (1.3.66) and (1.3.67),

$$\begin{aligned} P(z_i P \mathcal{O}_1 \mathscr{L}_0^{-1} P^{\perp})(Z, 0) &= (z_i P \mathcal{O}_1 \mathscr{L}_0^{-1} P^{\perp})(Z, 0) \\ &= z_i (P \mathcal{O}_1 \mathscr{L}_0^{-1} P^{\perp})(Z, 0), \end{aligned} \tag{1.3.68}$$

which vanishes at $Z = 0$. By (1.3.63), (1.3.65) and (1.3.68), we thus get

$$\mathcal{K}\Big[z_i' I_{\mathbb{C}\otimes E}, J_{1,x_0}\Big](0,0) = 0. \tag{1.3.69}$$

Equation (1.3.69), together with (1.3.58), (1.3.59) and (1.3.62), proves (1.3.54).

Now concerning (1.3.55), we are left to show that

$$\mathcal{K}\Big[J_{1,x_0}, Z_i I_{\mathbb{C}\otimes E}\Big](0,0) = \mathcal{K}\Big[J_{1,x_0}, \mathcal{K}\Big[I_{\mathbb{C}\otimes E}, Z_i I_{\mathbb{C}\otimes E}\Big]\Big](0,0). \tag{1.3.70}$$

By (1.2.30) we have

$$\mathcal{K}\Big[J_{1,x_0}, \mathcal{K}\Big[I_{\mathbb{C}\otimes E}, Z_i I_{\mathbb{C}\otimes E}\Big]\Big] = \frac{1}{2} \mathcal{K}\Big[J_{1,x_0}, (z_i + \bar{z}_i') I_{\mathbb{C}\otimes E}\Big]. \tag{1.3.71}$$

To get (1.3.70), it suffices thus to show that

$$\mathcal{K}\Big[J_{1,x_0}, \bar{z}_i I_{\mathbb{C}\otimes E}\Big](0,0) = \mathcal{K}\Big[J_{1,x_0}, \bar{z}_i' I_{\mathbb{C}\otimes E}\Big](0,0) = 0. \tag{1.3.72}$$

The equality on the right of (1.3.72) comes from (1.2.21). On another hand, by (1.3.9) and (1.3.10),

$$\mathcal{K}\Big[J_{1,x_0}, \bar{z}_i I_{\mathbb{C}\otimes E}\Big] \mathscr{P} = -P \mathcal{O}_1 \mathscr{L}_0^{-1} P^{\perp} (\bar{z}_i I_{\mathbb{C}\otimes E} \mathscr{P}) - \mathscr{L}_0^{-1} P^{\perp} \mathcal{O}_1 P (\bar{z}_i I_{\mathbb{C}\otimes E} \mathscr{P}). \tag{1.3.73}$$

Now by (1.3.48), once again the kernel of the first term of the left member of (1.3.73) vanishes at $Z = Z' = 0$. On another hand, by (1.2.25),

$$P(\bar{z}_i I_{\mathbb{C}\otimes E} \mathscr{P}) = P(\bar{z}_i' I_{\mathbb{C}\otimes E} \mathscr{P}). \tag{1.3.74}$$

We can thus replace $\bar{z}_i$ by $\bar{z}'_i$ in the second term of the left member of (1.3.73), and by (1.2.27) we get finally

$$(\mathscr{L}_0^{-1}P^{\perp}\mathcal{O}_1 P\bar{z}_i I_{\mathbb{C}\otimes E}\mathscr{P})(Z,Z') = (\mathscr{L}_0^{-1}P^{\perp}\mathcal{O}_1 P I_{\mathbb{C}\otimes E}\mathscr{P})(Z,Z')\bar{z}'_i, \tag{1.3.75}$$

which is 0 at $Z = Z' = 0$. Thus the kernel of the second term of (1.3.73) cancels as well in this case, which means

$$\mathcal{K}\Big[J_{1,x_0}, \bar{z}_i I_{\mathbb{C}\otimes E}\Big](0,0) = 0, \tag{1.3.76}$$

Finally, (1.3.76) implies (1.3.72), which together with (1.3.57) concludes the proof of (1.3.55).

The definitions of I_3 and I_4 in (1.3.33) and (1.3.33) and equation (1.3.54) and (1.3.55) respectively give

$$\begin{aligned} I_3(0,0) &= \sum_{i=1}^{2n}\left(f(x_0)\frac{\partial g_{x_0}}{\partial Z_i}(0) + \frac{\partial f_{x_0}}{\partial Z_i}(0)g(x_0) - \frac{\partial (fg)_{x_0}}{\partial Z_i}(0)\right)\mathcal{K}\Big[I_{\mathbb{C}\otimes E}, Z_i J_{1,x_0}\Big](0,0),\\ I_4(0,0) &= \sum_{i=1}^{2n}\left(f(x_0)\frac{\partial g_{x_0}}{\partial Z_i}(0) + \frac{\partial f_{x_0}}{\partial Z_i}(0)g(x_0) - \frac{\partial (fg)_{x_0}}{\partial Z_i}(0)\right)\mathcal{K}\Big[J_{1,x_0}, Z_i I_{\mathbb{C}\otimes E}\Big](0,0), \end{aligned} \tag{1.3.77}$$

and those two formulas vanish by Leibniz rule. We thus get (1.3.56).

□

Using Proposition 1.3.6, Proposition 1.3.7 and Proposition 1.3.8,the kernel of (1.3.35) at $Z = Z' = 0$ simply is

$$\mathcal{Q}_{2,x_0}(f,g)(0,0) - \mathcal{Q}_{2,x_0}(fg)(0,0) = -\frac{1}{\pi}\sum_{j=1}^{n}\frac{\partial f_{x_0}}{\partial z_j}(0)\frac{\partial g_{x_0}}{\partial \bar{z}_j}(0)I_{\mathbb{C}\otimes E}. \tag{1.3.78}$$

We thus see that $\mathcal{Q}_{2,x_0}(f,g)(0,0) - \mathcal{Q}_{2,x_0}(fg)(0,0)$ is in fact of the form $C_1(f,g)(x_0)I_{\mathbb{C}\otimes E}$ with $C_1(f,g)(x_0) \in \mathrm{End}(E_{x_0})$ given by

$$C_1(f,g)(x_0) = -\frac{1}{\pi}\sum_{j=1}^{n}\frac{\partial f}{\partial z_j}(x_0)\frac{\partial g}{\partial \bar{z}_j}(x_0). \tag{1.3.79}$$

From (1.2.5) and the definition of the pairing $\langle .,.\rangle$ used in (1.1.11), we can take this equality to the manifold through our trivialization and we finally get

$$C_1(f,g) = -\frac{1}{2\pi}\langle \nabla^{1,0}f, \nabla^{0,1}g\rangle. \tag{1.3.80}$$

This proves Theorem 1.1.1.

Chapter 2

Berezin-Toeplitz quantization for eigenstates of the Bochner-Laplacian on symplectic manifolds

2.1 Introduction

Quantization is a recipe in physics for passing from a classical system to a quantum system by obeying certain natural rules. By a classical system we understand a classical phase space (a symplectic manifold) and classical observables (smooth functions). The quantum system consists of a quantum space (a Hilbert space of functions or sections of a bundle) and quantum observables (bounded linear operators on the quantum space). The quantum system should reduce to the classical one as the size of the objects gets large, that is, as the "Planck constant", which, heuristically, corresponds to the magnitude where the quantum phenomena become relevant, tends to zero. This is the so-called semi-classical limit.

The original concept of quantization goes back to Weyl, von Neumann, and Dirac. In the geometric quantization introduced by Kostant [43] and Souriau [57] the quantum space is the Hilbert space of square integrable holomorphic sections of a prequantum line bundle (see also [7, 6, 16, 24]). Berezin-Toeplitz quantization is a particularly efficient version of the geometric quantization theory. Toeplitz operators and more generally Toeplitz structures were introduced in geometric quantization by Berezin [8] and Boutet de Monvel-Guillemin [14]. Using the analysis of Toeplitz structures [14], Bordemann-Meinrenken-Schlichenmaier [10] and Schlichenmaier [54] showed that the Berezin-Toeplitz quantization on a compact Kähler manifold satisfies the correspondence principle asymptotically and introduced the Berezin-Toeplitz star product (cf.(2.1.18) and (2.1.19)) when $E = \mathbb{C}$ and $g^{TX}(\cdot,\cdot) = \omega(\cdot, J\cdot)$.

In order to generalize the Berezin-Toeplitz quantization to arbitrary symplectic manifolds one has to find a substitute for the space of holomorphic sections of tensor powers of the prequantum line bundle. A natural candidate is the kernel of the Dirac operator, since it has similar features to the space of holomorphic sections in the Kähler case, especially the asymptotics of the kernels of the orthogonal projection on both spaces [20]. The Berezin-Toeplitz quantization with quantum space the kernel of the Dirac operator was carried over by Ma-Marinescu [51].

Another appealing candidate is the space of eigenstates of the renormalized Bochner Laplacian [31, 49, 50] corresponding to eigenvalues localized near the origin, cf. (2.1.8), (2.1.9). In this chapter we construct the Berezin-Toeplitz quantization for these spaces and show that it has the correct semiclassical behavior. The difference between this case and the quantization by the kernel of the Dirac operator comes from the possible presence of eigenvalues localized near the origin but different from zero. In this situation the analysis becomes more difficult.

Let us note also that Charles [18] proposed recently another approach to quantization of symplectic manifolds and Hsiao-Marinescu [34] constructed a Berezin-Toeplitz quantization for eigenstates of small eigenvalues in the case of complex manifolds. See also [13].

The readers are referred to the monograph [49] (also [46]) for a comprehensive study of the (generalized) Bergman kernel, Berezin-Toeplitz quantization and its applications.

Let us describe the setting and results in detail. Let (X, ω) be a compact symplectic manifold of real dimension $2n$. Let (L, h^L) be a Hermitian line bundle on X, and let ∇^L be a Hermitian connection on (L, h^L) with the curvature $R^L = (\nabla^L)^2$. Let (E, h^E) be a Hermitian vector bundle with Hermitian connection ∇^E. We will assume throughout the chapter that L is a line bundle satisfying the pre-quantization condition

$$\frac{\sqrt{-1}}{2\pi} R^L = \omega. \tag{2.1.1}$$

We choose an almost complex structure J such that ω is J-invariant. Writing $TX_{\mathbb{C}} = TX \otimes_{\mathbb{R}} \mathbb{C}$, the almost complex structure J induces a splitting $T_{\mathbb{C}} = T^{(1,0)}X \oplus T^{(0,1)}X$, where $T^{(1,0)}X$ and $T^{(0,1)}X$ are the eigenbundles of J corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$, respectively. For any $p \in \mathbb{N}$, set

$$E_p = L^p \otimes E \tag{2.1.2}$$

Let g^{TX} be a J-invariant Riemannian metric on TX. Let dv_X be the Riemannian volume form of (X, g^{TX}). The L^2-Hermitian product on the space $\mathscr{C}^\infty(X, E_p)$ of smooth sections of E_p on X, with $L^p := L^{\otimes p}$, is given by

$$\langle s_1, s_2 \rangle_p = \int_X \langle s_1(x), s_2(x) \rangle_{E^p} dv_X(x), \tag{2.1.3}$$

where $\langle \cdot, \cdot \rangle_{E^p}$ in the integrand is the pointwise Hermitian product on E_p induced by h^L, h^E. Let ∇^{TX} be the Levi-Civita connection on (X, g^{TX}), and let ∇^{E_p} be the connection on E_p induced by ∇^L and ∇^E. Let $\{e_k\}$ be a local orthonormal frame of (TX, g^{TX}).

The Bochner Laplacian acting on $\mathscr{C}^\infty(X, E_p)$ is given by

$$\Delta^{E_p} = -\sum_{j=1}^{2n} \Big[\Big(\nabla^{E_p}_{e_j}\Big)^2 - \nabla^{E_p}_{\nabla^{TX}_{e_j} e_k} \Big]. \tag{2.1.4}$$

Let $\Phi \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ be Hermitian (i.e., self-adjoint with respect to h^E). The renormalized Bochner Laplacian is defined by

$$\Delta_{p,\Phi} = \Delta^{E_p} - \tau p + \Phi, \quad \text{ with } \tau = \frac{\sqrt{-1}}{2} \sum_{j=1}^{2n} R^L(e_j, Je_j). \tag{2.1.5}$$

Write $|\cdot|_{g^{TX}}$ for the Hermitian norm induced by g^{TX} on $T^{(1,0)}X$, and set

$$\mu_0 = \inf_{u \in T_x^{(1,0)}X,\ x \in X} R^L(u, \overline{u})/|u|^2_{g^{TX}}. \tag{2.1.6}$$

By [31], [47, Cor.1.2], [49, Th.8.3.1], there exists $C_L > 0$ independent of p such that

$$\mathrm{Spec}(\Delta_{p,\Phi}) \subset [-C_L, C_L] \cup [2\mu_0 p - C_L, +\infty), \tag{2.1.7}$$

where $\mathrm{Spec}(A)$ denotes the spectrum of the operator A. Since $\Delta_{p,\Phi}$ is an elliptic operator on a compact manifold, it has discrete spectrum and its eigensections are smooth. Let

$$\mathscr{H}_p := \bigoplus_{\lambda \in [-C_L, C_L]} \mathrm{Ker}(\Delta_{p,\Phi} - \lambda) \subset \mathscr{C}^\infty(X, E_p) \tag{2.1.8}$$

be the direct sum of eigenspaces of $\Delta_{p,\Phi}$ corresponding to the eigenvalues lying in $[-C_L, C_L]$.

In mathematical physics terms, the operator $\Delta_{p,\Phi}$ is a semiclassical Schrödinger operator and the space $\mathscr{H}_p$ is the space of its bound states as $p \to \infty$. The space $\mathscr{H}_p$ proves to be an appropriate replacement for the space of holomorphic sections $H^0(X, E_p)$ from the Kähler case. Indeed, if (X, ω) is Kähler, then $\mathscr{H}_p = H^0(X, E_p)$ for p large enough. Moreover, for an arbitrary compact prequantized symplectic manifold (X, ω) as above, the dimension of the space $\mathscr{H}_p$ is given for p large enough as in the Kähler case by the Riemann-Roch-Hirzebruch formula, see [47, Cor.1.2], [49, Th.8.3.1],

$$\dim \mathscr{H}_p = \int_X \mathrm{Td}(T^{(1,0)}X)\, \mathrm{ch}(E) \exp(p\omega) = p^n (\mathrm{rk}\, E) \,\mathrm{Vol}_\omega(X) + O(p^{n-1}), \tag{2.1.9}$$

where $\mathrm{Td}(T^{(1,0)}X)$ represents the Todd class of $T^{(1,0)}X$ and $\mathrm{ch}(E)$ represents the Chern character of E. Another striking similarity is the fact that the kernel of the orthogonal projection on $\mathscr{H}_p$ has an asymptotic expansion analogous to the Bergman kernel expansion for Kähler manifolds, see [49, 50]. We will use the asymptotic expansion of [49, 50] together with the approach of [51] to Berezin-Toeplitz quantization in order to derive the properties of Toeplitz operators modeled on the projection on $\mathscr{H}_p$.

Let P_p be the orthogonal projection from $\mathscr{C}^\infty(X, E_p)$ onto $\mathscr{H}_p$. The kernel $P_p(x, x')$ of P_p with respect to $dv_X(x')$ is called a generalized Bergman kernel [50]. Note that

$P_p(x, x') \in (E_p)_x \otimes (E_p)^*_{x'}$. For a smooth section $f \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ of the bundle $\mathrm{End}(E)$, we define the Berezin-Toeplitz quantization of f by

$$T_{f,p} = P_p f P_p \in \mathrm{End}(L^2(X, E_p)), \tag{2.1.10}$$

where we denote for simplicity by f the endomorphism of $L^2(X, E_p)$ induced by f, namely, $s \mapsto fs$, with $(fs)(x) = f(x)(s(x))$, for $s \in L^2(X, E_p)$ and $x \in X$.

Definition 2.1.1. A Toeplitz operator is a sequence $\{T_p\} = \{T_p\}_{p\in\mathbb{N}}$ of linear operators

$$T_p : L^2(X, E_p) \longrightarrow L^2(X, E_p) \tag{2.1.11}$$

with the following properties:
(i) For any $p \in \mathbb{N}$, we have

$$T_p = P_p T_p P_p; \tag{2.1.12}$$

(ii) There exist a sequence $g_l \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ such that for all $k \geqslant 0$ there exists $C_k > 0$ with

$$\Big\| T_p - P_p\Big(\sum_{l=0}^{k} p^{-l} g_l\Big) P_p \Big\| \leqslant C_k p^{-k-1}, \tag{2.1.13}$$

where $\|\cdot\|$ denotes the operator norm on the space of the bounded operators.
If each T_p is self-adjoint, then $\{T_p\}_{p\in\mathbb{N}}$ is called self-adjoint.

We express (2.1.13) symbolically by

$$T_p = \sum_{l=0}^{k} p^{-l} T_{g_l,p} + O(p^{-k-1}). \tag{2.1.14}$$

If (2.1.13) holds for any $k \in \mathbb{N}$, then we write

$$T_p = \sum_{l=0}^{\infty} p^{-l} T_{g_l,p} + O(p^{-\infty}). \tag{2.1.15}$$

The main result of this chapter is as follows.

Theorem 2.1.2. *Let (X, J, ω) be a compact symplectic manifold with compatible almost complex structure, $(L, h^L, \nabla^L), (E, h^E, \nabla^E)$ be Hermitian vector bundles as above, and g^{TX} be an J-compatible Riemannian metric on TX. Let $f, g \in \mathscr{C}^\infty(X, \mathrm{End}(E))$. Then the product of the Toeplitz operators $T_{f,p}$ and $T_{g,p}$ is a Toeplitz operator, more precisely, it admits the asymptotic expansion in the sense of* (2.1.15)*:*

$$T_{f,p} T_{g,p} = \sum_{r=0}^{\infty} p^{-r} T_{C_r(f,g),p} + O(p^{-\infty}), \tag{2.1.16}$$

where C_r are bidifferential operators, $C_r(f,g) \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ and $C_0(f,g) = fg$.

If $f, g \in \mathscr{C}^\infty(X)$, then we have

$$C_1(f,g) - C_1(g,f) = \sqrt{-1}\{f,g\}\mathrm{Id}_E, \tag{2.1.17}$$

where $\{\cdot,\cdot\}$ is the Poisson bracket on $(X, 2\pi\omega)$, and therefore the correspondence principle holds asymptotically,

$$[T_{f,g}, T_{g,p}] = \frac{\sqrt{-1}}{p} T_{\{f,g\},p} + O(p^{-2}), \ \ p \to \infty. \tag{2.1.18}$$

Corollary 2.1.3. *Let $f, g \in \mathscr{C}^\infty(X, \mathrm{End}(E))$. Set*

$$f * g := \sum_{k=0}^{\infty} C_k(f,g)\hbar^k \in \mathscr{C}^\infty(X, \mathrm{End}(E))[[\hbar]]. \tag{2.1.19}$$

where $C_r(f,g)$ are determined by (2.1.16). Then (2.1.19) defines an associative star-product on $\mathscr{C}^\infty(X, \mathrm{End}(E))$.

Theorem 2.1.4. *For any $f \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ the operator norm of $T_{f,\,p}$ satisfies*

$$\lim_{p\to\infty} \|T_{f,\,p}\| = \|f\|_\infty := \sup_{0\neq u \in E_x, x\in X} |f(x)(u)|_{h^E} / |u|_{h^E}\,. \tag{2.1.20}$$

In the special case when the Riemannian metric g^{TX} is associated with ω we can even calculate $C_1(f,g)$, not only the difference $C_1(f,g) - C_1(g,f)$ from (2.1.17). To state the result, let

$$\begin{aligned} (\nabla^E)^{1,0} &: \mathscr{C}^\infty(X, \mathrm{End}(E)) \to \mathscr{C}^\infty(X, T^{*(1,0)}X \otimes \mathrm{End}(E)), \\ (\nabla^E)^{0,1} &: \mathscr{C}^\infty(X, \mathrm{End}(E)) \to \mathscr{C}^\infty(X, T^{*(0,1)}X \otimes \mathrm{End}(E)). \end{aligned} \tag{2.1.21}$$

be the $(1,0)$-component and $(0,1)$-component respectively of the connection ∇^E, and let $\langle\cdot,\cdot\rangle$ denote the pairing induced by g^{TX} on $T^*X \otimes \mathrm{End}(E)$ with values in $\mathrm{End}(E)$.

Following an argument of [36], we get the last result of this chapter.

Theorem 2.1.5. *If $g^{TX}(\cdot,\cdot) = \omega(\cdot, J\cdot)$, then for any $f, g \in \mathscr{C}^\infty(X, \mathrm{End}(E))$, the coefficient $C_1(f,g) \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ defined in (2.1.16) is given by*

$$C_1(f,g) = -\frac{1}{2\pi}\langle(\nabla^E)^{1,0} f, (\nabla^E)^{0,1} g\rangle. \tag{2.1.22}$$

Note that this formula is clearly compatible with the formula (2.1.17) for the Poisson bracket in the case $f, g \in \mathscr{C}^\infty(X)$. Note also that (2.1.22) is a direct generalization of the formula [52, (0.20)] for Kähler manifolds.

Recently Kordyukov informed us about his preprint [42] in which the Berezin-Toeplitz quantization by eigenstates of the Bochner-Laplacian is reconsidered.

The organization of the chapter is as follows. In Section 2.2, we recall the asymptotic expansion of the generalized Bergman kernel obtained in [44]. In Section 2.3, we obtain the asymptotic expansion of the kernel of a Toeplitz operator. In Section 2.4, we show that the asymptotic expansion is also a sufficient condition for a family to be Toeplitz. In Section 2.5, we conclude that the set of Toeplitz operators forms an algebra. In Section 2.6 we prove Theorem 2.1.5.

The results of this chapter appear in [37].

2.2 The asymptotic expansion of the generalized Bergman kernel

Let a^X be the injectivity radius of (X, g^{TX}). Let $d(x, y)$ denote the Riemannian distance from x to y on (X, g^{TX}). By [49, Prop.8.3.5] (cf. also [44, (1.11)]), we have the following far off-diagonal behavior of the generalized Bergman kernel.

Proposition 2.2.1. *For any $b > 0$ and any $k, l \in \mathbb{N}$ and $0 < \theta < 1$, there exists $C_{b,k,l,\theta} > 0$ such that*

$$\Big|P_p(x,x')\Big|_{\mathscr{C}^k(X\times X)} \leqslant C_{b,k,l,\theta}\, p^{-l}, \quad \text{for } d(x,x') > bp^{-\frac{\theta}{2}}, \tag{2.2.1}$$

here the $\mathscr{C}^k$-norm is induced by $\nabla^L, \nabla^E, h^L, h^E$ and g^{TX}.

Let $\varepsilon \in (0, a^X/4)$ be fixed. We denote by $B^X(x,\varepsilon)$ and $B^{T_xX}(0,\varepsilon)$ the open balls in X and T_xX with center x and radius ε, respectively. We identify $B^{T_xX}(0,\varepsilon)$ with $B^X(x,\varepsilon)$ by using the exponential map of (X, g^{TX}).

Let $x_0 \in X$. For $Z \in B^{T_{x_0}X}(0,\varepsilon)$ we identify (L_Z, h^L_Z), (E_Z, h^E_Z) and $(E_p)_Z$ to $(L_{x_0}, h^L_{x_0})$, $(E_{x_0}, h^E_{x_0})$ and $(E_p)_{x_0}$ by parallel transport with respect to the connections ∇^L, ∇^E and ∇^{E_p} along the curve $\gamma_Z : [0,1] \ni u \to \exp^X_{x_0}(uZ)$. This is the basic trivialization we use in this chapter.

Using this trivialization we identify $f \in \mathscr{C}^\infty(X, \operatorname{End}(E))$ to a family $\{f_{x_0}\}_{x_0\in X}$ where f_{x_0} is the function f in normal coordinates near x_0, i. e., $f_{x_0} : B^{T_{x_0}X}(0,\varepsilon) \to \operatorname{End}(E_{x_0})$, $f_{x_0}(Z) = f \circ \exp^X_{x_0}(Z)$. In general, for functions expressed in normal coordinates centered at $x_0 \in X$, we will add a subscript x_0 to indicate the base point x_0.

Similarly, $P_p(x,x')$ induces in terms of the basic trivialization a smooth section

$$(Z, Z') \longmapsto P_{p,x_0}(Z, Z')$$

of $\pi^* \operatorname{End}(E)$ over $\{(Z,Z') \in TX \times_X TX : |Z|, |Z'| < \varepsilon\}$, which depends smoothly on x_0. Here $\pi : TX \times_X TX \to X$ is the natural projection from the fibered product $TX \times_X TX$ on X and we identify a section $S \in \mathscr{C}^\infty\Big(TX \times_X TX, \pi^* \operatorname{End}(E)\Big)$ with the family $(S_x)_{x\in X}$, where $S_x = S|_{\pi^{-1}(x)}$.

Let dv_{TX} be the Riemannian volume form on $(T_{x_0}X, g^{T_{x_0}X})$. Let $\kappa_{x_0}(Z)$ be the smooth positive function defined by the equation

$$dv_X(Z) = \kappa_{x_0}(Z) dv_{TX}(Z), \quad \kappa_{x_0}(0) = 1, \tag{2.2.2}$$

where the subscript x_0 of $\kappa_{x_0}(Z)$ indicates the base point $x_0 \in X$.

Writing $\langle \cdot, \cdot \rangle$ for the $\mathbb{C}$-bilinear product induced by g^{TX} on $T^{(1,0)}X$, we identify the 2-form R^L with the Hermitian matrix $\dot{R}^L \in \operatorname{End}(T^{(1,0)}X)$ such that for any $u, v \in T^{(1,0)}X$,

$$R^L(u, \overline{v}) = \langle \dot{R}^L u, \overline{v} \rangle. \tag{2.2.3}$$

Choose $\{w_j\}_{j=1}^n$ an orthonormal basis of $T_{x_0}^{(1,0)}X$ such that

$$\dot{R}^L_{x_0} = \operatorname{diag}(a_1, \dots, a_n) \in \operatorname{End}(T_{x_0}^{(1,0)}X). \tag{2.2.4}$$

Then $a_j > 0$, for all $1 \leqslant j \leqslant n$. We fix an orthonormal basis of $T_{x_0}X$ given by $e_{2j-1} = \frac{1}{\sqrt{2}}(w_j + \overline{w}_j)$ and $e_{2j} = \frac{\sqrt{-1}}{\sqrt{2}}(w_j - \overline{w}_j)$. Then $z_j = Z_{2j-1} + \sqrt{-1} Z_{2j}$ is a complex coordinate of $Z \in \mathbb{R}^{2n} \simeq (T_{x_0}X, J)$.

By [44, Th.2.1] and [51, Th.1.18], we obtain the following version of the off diagonal expansion of the generalized Bergman kernel.

Theorem 2.2.2. *For any $x_0 \in X$ and $r \in \mathbb{N}$, there exist polynomials $J_{r,x_0}(Z, Z') \in \operatorname{End}(E_{x_0})$ in Z, Z' with the same parity as r and with $\deg J_{r,x_0} \leqslant 3r$ such that by setting*

$$\mathscr{F}_{r,x_0}(Z, Z') = J_{r,x_0}(Z, Z') \mathscr{P}_{x_0}(Z, Z'), \quad J_{0,x_0}(Z, Z') \equiv \operatorname{Id}_{E_{x_0}}, \tag{2.2.5}$$

with

$$\mathscr{P}_{x_0}(Z, Z') = \prod_{i=1}^n \frac{a_i}{2\pi} \exp\Big[-\frac{1}{4} \sum_{j=1}^n a_j \Big(|z_j|^2 + |z'_j|^2 - 2 z_j \bar{z}'_j \Big) \Big], \tag{2.2.6}$$

the following statement holds: for any $b > 0$ and $k_0, m, m' \in \mathbb{N}$, there exists $C_{b,k_0,m,m'} > 0$ such that for $|\alpha| + |\alpha'| \leqslant m'$ and any $|Z|, |Z'| < b p^{-\frac{1}{2}+\theta}$ with

$$\theta = \frac{1}{2\Big(2n + 8 + 2k_0 + 3m' + 2m\Big)}\,, \tag{2.2.7}$$

we have

$$\begin{aligned} &\left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \Big(p^{-n} P_{p,x_0}(Z, Z') - \sum_{r=0}^k \mathscr{F}_{r,x_0}(\sqrt{p} Z, \sqrt{p} Z') \kappa_{x_0}^{-1/2}(Z) \kappa_{x_0}^{-1/2}(Z') p^{-\frac{r}{2}} \Big) \right|_{\mathscr{C}^m(X)} \\ &\qquad \leqslant C_{b,k_0,m,m'}\, p^{-\frac{k_0}{2}-1}, \end{aligned} \tag{2.2.8}$$

where $k = k_0 + m' + 2$ and $\mathscr{C}^m(X)$ is the $\mathscr{C}^m$ norm for the parameter $x_0 \in X$.

In particular when $m' = 0$, the following statement holds: for any $b > 0$ and $k, m \in \mathbb{N}$, there exists $C > 0$ such that for any $|Z|, |Z'| < b p^{-\frac{1}{2}+\theta_2}$ with

$$\theta_2 = \frac{1}{4\Big(n + k + m + 2\Big)}\,, \tag{2.2.9}$$

we have

$$\left|p^{-n}P_{p,x_0}(Z,Z') - \sum_{r=0}^{k}\mathscr{F}_{r,x_0}(\sqrt{p}Z,\sqrt{p}Z')\kappa_{x_0}^{-1/2}(Z)\kappa_{x_0}^{-1/2}(Z')p^{-\frac{r}{2}}\right|_{\mathscr{C}^m(X)} \leqslant Cp^{-k/2}. \tag{2.2.10}$$

Note that the more expansion term in (2.2.10), the smaller of the expansion domain for the variables Z and Z'. This serves as the main ingredient for the generalized Bergman kernel case.

By [51, Lem.2.2], for any polynomials $F,G \in \mathbb{C}[Z,Z']$ there exist $\mathscr{K}[F,G] \in \mathbb{C}[Z,Z']$ such that

$$\Big((F\mathscr{P})\circ(G\mathscr{P})\Big)(Z,Z') = \mathscr{K}[F,G](Z,Z')\mathscr{P}(Z,Z'). \tag{2.2.11}$$

2.3 Asymptotic expansion of Toeplitz operators

For $f \in \mathscr{C}^\infty(X,\mathrm{End}(E))$ we define the Toeplitz operator $T_{f,p}$ on $L^2(X,E_p)$ by (2.1.10). The Schwartz kernel of $T_{f,p}$ is given by

$$T_{f,p}(x,x') = \int_X P_p(x,x'')f(x'')P_p(x'',x')dv_X(x''). \tag{2.3.1}$$

Note that if $f \in \mathscr{C}^\infty(X,\mathrm{End}(E))$ is self-adjoint, i.e., $f(x) = f(x)^*$ for all $x \in X$, then the operator $T_{f,p}$ is self-adjoint.

We examine now the asymptotic expansion of the kernel of Toeplitz operators $T_{f,p}$. Outside the diagonal of $X \times X$, we have the following analogue of [51, Lem.4.2].

Lemma 2.3.1. *Let $\theta \in\,]0,1[$ and $f \in \mathscr{C}^\infty(X,\mathrm{End}(E))$ fixed. For any $b > 0$ and $k,l \in \mathbb{N}$, there exists $C_{b,k,l} > 0$ such that*

$$\left|T_{f,p}(x,x')\right|_{\mathscr{C}^k(X\times X)} \leqslant C_{b,k,l}p^{-l}, \tag{2.3.2}$$

for all $p \geqslant 1$ and all $(x,x') \in X\times X$ with $d(x,x') > bp^{-\theta}$, where the $\mathscr{C}^k$-norm is induced by $\nabla^L, \nabla^E, h^L, h^E$ and g^{TX}.

Proof. From Proposition 2.2.1 and (2.2.10), we know that for any $k \in \mathbb{N}$ there exist $C_k > 0$ and $M_k > 0$ such that for all $(x,x') \in X \times X$,

$$\left|P_p(x,x')\right|_{\mathscr{C}^k(X\times X)} \leqslant C_k p^{M_k}. \tag{2.3.3}$$

We split the integral in (2.3.1) in a sum of two integrals as follows:

$$T_{f,p}(x,x') = \Big(\int_{B^X(x,\frac{b}{2}p^{-\theta})} + \int_{X\setminus B^X(x,\frac{b}{2}p^{-\theta})}\Big)P_p(x,x'')f(x'')P_p(x'',x')dv_X(x''). \tag{2.3.4}$$

Assume that $d(x,x') > bp^{-\theta}$. Then

$$\begin{aligned} d(x'',x') &> \frac{b}{2}p^{-\theta} \quad \text{for } x'' \in B^X(x, \frac{b}{2}p^{-\theta}), \\ d(x,x'') &\geqslant \frac{b}{2}p^{-\theta} \quad \text{for } x'' \in X \setminus B^X(x, \frac{b}{2}p^{-\theta}). \end{aligned} \tag{2.3.5}$$

Now from (2.2.1) and (2.3.3)-(2.3.5), we get (2.3.2). The proof of Lemma 2.3.1 is complete. □

We concentrate next on a neighborhood of the diagonal of $X \times X$ in order to obtain the asymptotic expansion of the kernel $T_{f,p}(x,x')$.

Let $\{\Xi_p\}_{p\in\mathbb{N}}$ be a sequence of linear operators $\Xi_p : L^2(X,E_p) \to L^2(X,E_p)$ with smooth kernel $\Xi_p(x,y)$ with respect to $dv_X(y)$. Recall that $\pi : TX \times_X TX \to X$ is the natural projection. Under our trivialization, $\Xi_p(x,y)$ induces a smooth section $\Xi_{p,x_0}(Z,Z')$ of $\pi^*\Big(\mathrm{End}(E)\Big)$ over $TX \times_X TX$ with $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| < a^X$. Recall also that $\mathscr{P}_{x_0}$ was defined by (2.2.6).

Consider the following condition for $\{\Xi_p\}_{p\in\mathbb{N}}$.

Condition A. There exists a family $\{Q_{r,x_0}\}_{r\in\mathbb{N},x_0\in X}$ such that
(a) $Q_{r,x_0} \in \mathrm{End}(E_{x_0})[Z,Z']$;
(b) $\{Q_{r,x_0}\}_{r\in\mathbb{N},x_0\in X}$ is smooth with respect to the parameter $x_0 \in X$ and there exist $b_1, b_0 \in \mathbb{N}$ such that $\deg Q_r \leqslant b_1 r + b_0$;
(c) for any $k,m \in \mathbb{N}$, there exists $\theta_{k,m} \in\,]0,1/2[$ such that for any $b > 0$, there exist $C_{b,k,m} > 0$ such that for every $x_0 \in X$, $Z, Z' \in T_{x_0}X$ with $|Z|, |Z'| < bp^{-\frac{1}{2}+\theta_{k,m}}$ and $p \in \mathbb{N}^*$, the following estimate holds:

$$\left| p^{-n}\Xi_{p,x_0}(Z,Z')\kappa_{x_0}^{1/2}(Z)\kappa_{x_0}^{1/2}(Z') - \sum_{r=0}^{k}(Q_{r,x_0}\mathscr{P}_{x_0})(\sqrt{p}Z,\sqrt{p}Z')p^{-r/2} \right|_{\mathscr{C}^m(X)} \leqslant C_{b,k,m}p^{-k/2}. \tag{2.3.6}$$

(d) For any $\theta \in (0,1), b > 0, k,m \in \mathbb{N}$, there exists $C > 0$ such that for any $p \in \mathbb{N}^*, d(x,x') > bp^{-\theta/2}$, we have

$$\Big|\Xi_p(x,x')\Big|_{\mathscr{C}^m(X\times X)} \leqslant Cp^{-k}.$$

Notation A. For any $k,m \in \mathbb{N}$ we write the equation (2.3.6) for $|Z|, |Z'| < bp^{-\frac{1}{2}+\theta_{k,m}}$ as

$$p^{-n}\Xi_{p,x_0}(Z,Z') \cong \sum_{r=0}^{k}(Q_{r,x_0}\mathscr{P}_{x_0})(\sqrt{p}Z,\sqrt{p}Z')p^{-r/2} + \mathscr{O}_m(p^{-k/2}). \tag{2.3.7}$$

Remark 2.3.2. By Theorem 2.2.2, (2.2.9) and (2.2.10), we have

$$p^{-n}P_{p,x_0}(Z,Z') \cong \sum_{r=0}^{k}(J_{r,x_0}\mathscr{P}_{x_0})(\sqrt{p}Z,\sqrt{p}Z')p^{-r/2} + \mathscr{O}_m(p^{-k/2}), \tag{2.3.8}$$

in the sense of Notation A with

$$\theta_{k,m} = \frac{1}{4(n+k+m+2)} \quad \text{for } k, m \in \mathbb{N}, \tag{2.3.9}$$

where $J_{r,x_0}(Z, Z') \in \operatorname{End}(E_{x_0})$ are the polynomials in Z, Z' defined in (2.2.5). Note that $J_{r,x_0}(Z, Z')$ has the same parity as r and $\deg J_{r,x_0} \leqslant 3r$, $J_{0,x_0} = \operatorname{Id}_{E_{x_0}}$.

The following result is about the near diagonal asymptotic expansion of the kernel $T_{f,p}(x, x')$. It is a version of [51, Lem.4.6] in our situation.

Lemma 2.3.3. *Let $f \in \mathscr{C}^\infty(X, \operatorname{End}(E))$. There exists a family $\{Q_{r,x_0}(f)\}_{r\in\mathbb{N}, x_0\in X}$ such that*
(a) $Q_{r,x_0}(f) \in \operatorname{End}(E_{x_0})[Z, Z']$ *are polynomials with the same parity as r;*
(b) $\{Q_{r,x_0}(f)\}_{r\in\mathbb{N}, x_0\in X}$ *is smooth with respect to $x_0 \in X$, and $\deg Q_{r,x_0} \leqslant 3r$;*
(c) *for any $k_0, m \in \mathbb{N}$, we have*

$$p^{-n} T_{f,p,x_0}(Z, Z') \cong \sum_{r=0}^{k_0} \Big(Q_{r,x_0}(f)\mathscr{P}_{x_0}\Big)(\sqrt{p}Z, \sqrt{p}Z') p^{-r/2} + \mathscr{O}_m(p^{-k_0/2}), \tag{2.3.10}$$

in the sense of Notation A with

$$\theta_{k_0,m} = \frac{1}{4(n+k+m+2)} \text{ for some } k \geqslant k_0, \tag{2.3.11}$$

and $Q_{r,x_0}(f)$ are expressed by

$$Q_{r,x_0}(f) = \sum_{r_1+r_2+|\alpha|=r} \mathscr{K}\Big[J_{r_1,x_0}, \frac{\partial^{|\alpha|} f_{x_0}}{\partial Z^\alpha}(0)\frac{Z^\alpha}{\alpha!} J_{r_2,x_0}\Big]. \tag{2.3.12}$$

Especially,

$$Q_{0,x_0}(f) \equiv f(x_0). \tag{2.3.13}$$

Proof. For $k_0, m \in \mathbb{N}$ fixed, let $k \geqslant k_0$ to be determined. Set

$$\theta_2 = \frac{1}{4(n+k+m+2)}, \quad \theta_1 = 1 - 2\theta_2. \tag{2.3.14}$$

By (2.2.10), we have for any $|Z|, |Z'| < 2bp^{-\frac{1}{2}+\theta_2} = 2bp^{-\theta_1/2}$,

$$\begin{aligned} \Big| p^{-n} P_{p,x_0}(Z, Z') - \sum_{r=0}^{k} \mathscr{F}_{r,x_0}(\sqrt{p}Z, \sqrt{p}Z') \kappa_{x_0}^{-1/2}(Z) \kappa_{x_0}^{-1/2}(Z') p^{-r/2} \Big|_{\mathscr{C}^m(X)} \\ \leqslant C_{k,l} p^{-k/2}. \end{aligned} \tag{2.3.15}$$

For $|Z|,|Z'| < \frac{b}{2}p^{-\frac{1}{2}+\theta_2} = \frac{b}{2}p^{-\theta_1/2}$, we get from (2.3.1) that

$$T_{f,p,x_0}(Z,Z') = \int_X P_{p,x_0}(Z,y)f(y)P_{p,x_0}(y,Z')dv_X(y). \tag{2.3.16}$$

We split the integral into integrals over $B^X(x,bp^{-\theta_1/2})$ and $X\setminus B^X(x,bp^{-\theta_1/2})$. We have

$$d(y,\exp_{x_0} Z) \geqslant d(y,x_0) - |Z| > \frac{b}{2}p^{-\theta_1/2} \text{ on } X\setminus B^X(x,bp^{-\theta_1/2}), \tag{2.3.17}$$

since on this set $d(y,x_0) > bp^{-\theta_1/2}$ holds. By Proposition 2.2.1 for θ_1 in (2.3.14), (2.3.3) and (2.3.17) we have for $|Z|,|Z'| < \frac{b}{2}p^{-\theta_1/2}$,

$$\begin{aligned} &T_{f,p,x_0}(Z,Z') \\ &= \int_{|Z''|<bp^{-\frac{\theta_1}{2}}} P_{p,x_0}(Z,Z'')f_{x_0}(Z'')P_{p,x_0}(Z'',Z')\kappa_{x_0}(Z'')dv_{TX}(Z'') + \mathscr{O}_m(p^{-\infty}). \end{aligned} \tag{2.3.18}$$

Then

$$\begin{aligned} &p^{-n}T_{f,p,x_0}(Z,Z')\kappa_{x_0}^{1/2}(Z)\kappa_{x_0}^{1/2}(Z') \\ &= p^{-n}\int_{|Z''|<bp^{-\frac{\theta_1}{2}}} P_{p,x_0}(Z,Z'')\kappa_{x_0}^{\frac{1}{2}}(Z)\kappa_{x_0}^{\frac{1}{2}}(Z'')f_{x_0}(Z'')P_{p,x_0}(Z'',Z')\kappa_{x_0}^{\frac{1}{2}}(Z'')\kappa_{x_0}^{\frac{1}{2}}(Z')dv_{TX}(Z'') \\ &\qquad + \mathscr{O}_m(p^{-\infty}). \end{aligned} \tag{2.3.19}$$

We consider the Taylor expansion of f_{x_0}:

$$\begin{aligned} f_{x_0}(Z) &= \sum_{|\alpha|\leqslant k} \frac{\partial^{|\alpha|}f_{x_0}}{\partial Z^\alpha}(0)\frac{Z^\alpha}{\alpha!} + O(|Z|^{k+1}) \\ &= \sum_{|\alpha|\leqslant k} p^{-|\alpha|/2}\frac{\partial^{|\alpha|}f_{x_0}}{\partial Z^\alpha}(0)\frac{(\sqrt{p}Z)^\alpha}{\alpha!} + p^{-\frac{k+1}{2}}O(|\sqrt{p}Z|^{k+1}). \end{aligned} \tag{2.3.20}$$

Combining the asymptotic expansion (2.3.15) and (2.3.20), to obtain the asymptotic expansion of (2.3.19), we need to consider $I_{r_1,|\alpha|,r_2}(T_{x_0}X)(Z,Z')$ defined by

$$\begin{aligned} &p^{-n+\frac{r_1+|\alpha|+r_2}{2}}I_{r_1,|\alpha|,r_2}(T_{x_0}X)(Z,Z') := \\ &\int_{T_{x_0}X}(J_{r_1,x_0}\mathscr{P}_{x_0})(\sqrt{p}Z,\sqrt{p}Z'')\frac{\partial^{|\alpha|}f_{x_0}}{\partial Z^\alpha}(0)\frac{(\sqrt{p}Z'')^\alpha}{\alpha!}(J_{r_2,x_0}\mathscr{P}_{x_0})(\sqrt{p}Z'',\sqrt{p}Z')dv_{TX}(Z''). \end{aligned} \tag{2.3.21}$$

Clearly, we can define $I_{r_1,|\alpha|,r_2}\Big(B^{T_{x_0}X}(0,a)\Big)$ and $I_{r_1,|\alpha|,r_2}\Big(T_{x_0}X\setminus B^{T_{x_0}X}(0,a)\Big)$ for $a>0$ in the same manner. Then by (2.3.19),

$$\begin{aligned} &p^{-n}T_{f,p,x_0}(Z,Z')\kappa^{1/2}(Z)\kappa^{-1/2}(Z') \\ &= \sum_{r_1,|\alpha|,r_2\leqslant k} I_{r_1,|\alpha|,r_2}\Big(B^{T_{x_0}X}(0,bp^{-\theta_1/2})\Big)(Z,Z') \\ &\quad + I_1(Z,Z') + I_2(Z,Z') + I_3(Z,Z') + \mathscr{O}_m(p^{-\infty}), \end{aligned} \tag{2.3.22}$$

with

$$
\begin{aligned}
&I_1(Z,Z') \qquad (2.3.23)\\
&=\int_{|Z''|<bp^{-\theta_1/2}}\Big[p^{-n}P_p(Z,Z'')\kappa^{1/2}(Z)\kappa^{1/2}(Z'')-\sum_{r\leqslant k}(J_{r,x_0}\mathscr{P}_{x_0})(\sqrt{p}Z,\sqrt{p}Z'')p^{-r/2}\Big]\\
&\quad\times f_{x_0}(Z'')P_p(Z'',Z')\kappa^{1/2}(Z'')\kappa^{1/2}(Z')dv_{TX}(Z''),
\end{aligned}
$$

and

$$
\begin{aligned}
&I_2(Z,Z') \qquad (2.3.24)\\
&=\int_{|Z''|<bp^{-\theta_1/2}}\sum_{r_1\leqslant k}(J_{r_1,x_0}\mathscr{P}_{x_0})(\sqrt{p}Z,\sqrt{p}Z'')p^{-r_1/2}\\
&\times\Big[f_{x_0}(Z'')-\sum_{|\alpha|\leqslant k}\frac{\partial^{|\alpha|}f_{x_0}}{\partial Z^\alpha}(0)\frac{(\sqrt{p}Z'')^\alpha}{\alpha!}p^{-|\alpha|/2}\Big]P_p(Z'',Z')\kappa^{1/2}(Z'')\kappa^{1/2}(Z')dv_{TX}(Z''),\\
&I_3(Z,Z')\\
&=p^n\int_{|Z''|<bp^{-\theta_1/2}}\sum_{r_1\leqslant k}(J_{r_1,x_0}\mathscr{P}_{x_0})(\sqrt{p}Z,\sqrt{p}Z'')p^{-r_1/2}\sum_{|\alpha|\leqslant k}\frac{\partial^{|\alpha|}f_{x_0}}{\partial Z^\alpha}(0)\frac{(\sqrt{p}Z'')^\alpha}{\alpha!}p^{-|\alpha|/2}\\
&\times\Big[p^{-n}P_p(Z'',Z')\kappa^{1/2}(Z'')\kappa^{1/2}(Z')-\sum_{r_2\leqslant k}(J_{r_2}\mathscr{P}_{x_0})(\sqrt{p}Z'',\sqrt{p}Z')p^{-r_2/2}\Big]dv_{TX}(Z'').
\end{aligned}
$$

We claim that for k large,

$$
\Big|I_j(Z,Z')\Big|_{\mathscr{C}^m(X)}\leqslant Cp^{-k_0/2}\quad\text{for } j=1,2,3. \qquad (2.3.25)
$$

In fact, by (2.3.3), there exists $C_0>0$ and $M_0>0$ such that for all $(x,x')\in X\times X$,

$$
\Big|P_p(x,x')\Big|_{\mathscr{C}^0(X\times X)}\leqslant C_0p^{M_0}. \qquad (2.3.26)
$$

Combining (2.3.15), (2.3.23) and (2.3.26) yields

$$
|I_1(Z,Z')|_{\mathscr{C}^m(X)}\leqslant Cp^{-\frac{k}{2}+M_0}. \qquad (2.3.27)
$$

By (2.3.20), (2.3.26) and the fact that $\deg J_r\leqslant 3r$,

$$
\begin{aligned}
\Big|I_2(Z,Z')\Big|_{\mathscr{C}^m(X)}&\leqslant C(1+\sqrt{p}|Z|)^{3k}\cdot p^{-\frac{k+1}{2}}\cdot p^{M_0}\\
&\leqslant Cp^{-\frac{k+1}{2}+3k\theta_2+M_0}.
\end{aligned} \qquad (2.3.28)
$$

By (2.3.15) and the fact that $\deg J_r\leqslant 3r$, we have for $|Z|,|Z'|<\frac{b}{2}p^{-\theta_1/2}$,

$$
\Big|I_3(Z,Z')\Big|_{\mathscr{C}^m(X)}\leqslant C(1+\sqrt{p}|Z|)^{3k}p^{-\frac{k}{2}}. \qquad (2.3.29)
$$

From (2.3.27)–(2.3.29), choose $k > k_0$ big enough such that

$$k + 1 - 6k\theta_2 - 2M_0 = k\Big(1 - \frac{3}{2(n+k+m+2)}\Big) - 2M_0 + 1 > k_0. \tag{2.3.30}$$

Then the claim (2.3.25) holds. By (2.3.22) and (2.3.25),

$$\begin{aligned} &\Big|p^{-n}T_{f,p,x_0}(Z,Z')\kappa^{1/2}(Z)\kappa^{-1/2}(Z') \\ &\quad - \sum_{r_1,\alpha,r_2\leqslant k} I_{r_1,|\alpha|,r_2}(B^{T_{x_0}X}(0,bp^{-\theta_1/2}))(Z,Z')\Big|_{\mathscr{C}^m(X)} \leqslant Cp^{-k_0/2}. \end{aligned} \tag{2.3.31}$$

Note by (2.1.6) and (2.2.6),

$$\Big|\mathscr{P}(\sqrt{p}Z,\sqrt{p}Z')\Big| = \prod_{i=1}^{n}\frac{a_i}{2\pi}e^{-\frac{p}{4}\sum_{j=1}^{n}a_j|z_j-z_j'|^2} \leqslant Ce^{-\frac{p}{4}\mu_0|Z-Z'|^2}. \tag{2.3.32}$$

By (2.3.32) and the fact that $\deg J_r \leqslant 3r$, we obtain

$$\begin{aligned} &\Big|I_{r_1,|\alpha|,r_2}\Big(T_{x_0}X\backslash B^{T_{x_0}X}(0,bp^{-\theta_1/2})\Big)(Z,Z')\Big|_{\mathscr{C}^m(X)} \\ &\leqslant Cp^n\int_{|Z''|>bp^{-\theta_1/2}}(1+\sqrt{p}|Z|+\sqrt{p}|Z''|)^{3r_1}(1+\sqrt{p}|Z'|+\sqrt{p}|Z''|)^{3r_2} \\ &\qquad \times(\sqrt{p}|Z''|)^{|\alpha|}\ \exp\Big(-\frac{\mu_0}{2}\sqrt{p}|Z-Z''|-\frac{\mu_0}{2}\sqrt{p}|Z''-Z'|\Big)dv_{TX}(Z''). \end{aligned} \tag{2.3.33}$$

Note that for any $|Z|,|Z'| < \frac{b}{2}p^{-\theta_1/2}$ and $|Z''| > bp^{-\theta_1/2}$, we have

$$|Z| < |Z''|,\quad |Z'| < |Z''|,\quad |Z-Z''| \geqslant \frac{1}{2}|Z''|,\quad |Z'-Z''| \geqslant \frac{1}{2}|Z''|. \tag{2.3.34}$$

Substituting (2.3.34) into (2.3.33) yields for any $|Z|,|Z'| < \frac{b}{2}p^{-\theta_1/2}$,

$$\begin{aligned} &\Big|I_{r_1,|\alpha|,r_2}\Big(T_{x_0}X\backslash B^{T_{x_0}X}(0,bp^{-\theta_1/2})\Big)(Z,Z')\Big|_{\mathscr{C}^m(X)} \\ &\leqslant Cp^n\int_{|Z''|>bp^{-\theta_1/2}}\Big(1+\sqrt{p}|Z''|\Big)^{3(r_1+r_2)}(\sqrt{p}|Z''|)^{|\alpha|}e^{-\frac{\mu_0}{2}\sqrt{p}|Z''|}dv_{TX}(Z'') \\ &\leqslant Cp^n\exp\left(-\frac{b}{4}\mu_0p^{\theta_2}\right)\int_{|Z''|>bp^{-\theta_1/2}}\Big(1+\sqrt{p}|Z''|\Big)^{3(r_1+r_2)+|\alpha|}e^{-\frac{\mu_0}{4}\sqrt{p}|Z''|}dv_{TX}(Z'') \\ &\leqslant C\exp\left(-\frac{b}{4}\mu_0p^{\theta_2}\right)\int_{|Z''|>bp^{\theta_2}}\Big(1+|Z''|\Big)^{3(r_1+r_2)+|\alpha|}e^{-\frac{\mu_0}{4}|Z''|}dv_{TX}(Z'') \\ &\leqslant C\exp\left(-\frac{b}{4}\mu_0p^{\theta_2}\right). \end{aligned} \tag{2.3.35}$$

Combining (2.3.31) and (2.3.35), we obtain

$$\Big|p^{-n}T_{f,p,x_0}(Z,Z')\kappa^{1/2}(Z)\kappa^{1/2}(Z') - \sum_{r_1,|\alpha|,r_2\leqslant k} I_{r_1,|\alpha|,r_2}(T_{x_0}X)(Z,Z')\Big|_{\mathscr{C}^m(X)} \leqslant Cp^{-\frac{k_0}{2}}. \tag{2.3.36}$$

Clearly,

$$
\begin{aligned}
&\sum_{r_1,|\alpha|,r_2\leqslant k} I_{r_1,|\alpha|,r_2}(T_{x_0}X)(Z,Z') \\
&= \Big(\sum_{r_1+|\alpha|+r_2\leqslant k_0} + \sum_{r_1+|\alpha|+r_2=k_0+1}^{3k}\Big) I_{r_1,|\alpha|,r_2}(T_{x_0}X)(Z,Z').
\end{aligned} \tag{2.3.37}
$$

By (2.2.11) and (2.3.21),

$$
\begin{aligned}
&I_{r_1,|\alpha|,r_2}(T_{x_0}X)(Z,Z') \\
&= p^{-(r_1+|\alpha|+r_2)/2}\Big(\mathscr{K}\Big[J_{r_1,x_0}, \frac{\partial^{|\alpha|} f_{x_0}}{\partial Z^\alpha}(0)\frac{Z^\alpha}{\alpha!}J_{r_2,x_0}\Big]\mathscr{P}\Big)(\sqrt{p}Z,\sqrt{p}Z').
\end{aligned} \tag{2.3.38}
$$

In view of (2.3.36)-(2.3.38), to finish the proof of Lemma 2.3.3, it suffices to prove that the $\mathscr{C}^m$ norm with respect to the parameter $x_0\in X$ of the term

$$
\sum_{r_1+|\alpha|+r_2=k_0+1}^{3k} I_{r_1,|\alpha|,r_2}(T_{x_0}X)(Z,Z'), \text{ for } |Z|,|Z'|<\frac{b}{2}p^{-\theta_1/2}, \tag{2.3.39}
$$

is controlled by $Cp^{-k_0/2}$ for large k.

Estimating $I_{r_1,|\alpha|,r_2}(T_{x_0}X)(Z,Z')$ for $|Z|,|Z'|<\frac{b}{2}p^{-\theta_1/2}$, using (2.3.32), (2.3.38) and the fact that $\deg J_r\leqslant 3r$, we obtain

$$
\begin{aligned}
&\Big|I_{r_1,|\alpha|,r_2}(T_{x_0}X)(Z,Z')\Big|_{\mathscr{C}^m(X)} \\
&\leqslant Cp^{-s/2}\Big(1+\sqrt{p}|Z|+\sqrt{p}|Z'|\Big)^{3s}\exp\Big(-\frac{p}{4}\mu_0|Z-Z'|^2\Big) \\
&\leqslant Cp^{-\frac{s}{2}}p^{\frac{1-\theta_1}{2}\cdot 3s} = Cp^{-s(1-6\theta_2)/2},
\end{aligned} \tag{2.3.40}
$$

with $s=r_1+|\alpha|+r_2$. If $s>k_0$, then

$$
s(1-6\theta_2)\geqslant (k_0+1)\left(1-\frac{6}{4(n+k+m+2)}\right). \tag{2.3.41}
$$

Choose k big enough such that

$$
(k_0+1)\left(1-\frac{6}{4(n+k+m+2)}\right)>k_0. \tag{2.3.42}
$$

Then

$$
\Big|I_{r_1,|\alpha|,r_2}(T_{x_0}X)(Z,Z')\Big|_{\mathscr{C}^m(X)}\leqslant Cp^{-k_0/2} \quad \text{for } r_1+|\alpha|+r_2=s>k_0. \tag{2.3.43}
$$

To sum up, we have proved the following statement: for fixed k_0, choose $k>k_0$ such that (2.3.30) and (2.3.42) hold. Set

$$
\theta_2=\frac{1}{4(n+k+m+2)},\quad \theta_1=1-2\theta_2, \tag{2.3.44}
$$

then for any $|Z|, |Z'| < \frac{b}{2}p^{-\theta_1/2}$, we have

$$\begin{aligned} &\Big|p^{-n}T_{f,p,x_0}(Z,Z')\kappa_{x_0}^{1/2}(Z)\kappa_{x_0}^{1/2}(Z') - \sum_{r=0}^{k_0}\Big(Q_{r,x_0}(f)\mathscr{P}_{x_0}\Big)(\sqrt{p}Z,\sqrt{p}Z')p^{-r/2}\Big|_{\mathscr{C}^m(X)} \\ &\leqslant Cp^{-k_0/2}, \end{aligned} \tag{2.3.45}$$

where $Q_{r,x_0}(f)$ is given by (2.3.12). This completes the proof of Lemma 2.3.3. □

Remark 2.3.4. Let Ξ_p be a sequence of operators satisfying Condition A and assume that $\Xi_p = P_p\Xi_pP_p$ for all $p \in \mathbb{N}$. Applying the proof of Lemma 2.3.3, by splitting integrals and studying different integration regions, we deduce by Theorem 2.2.2 and (2.3.6):

For any $k, m, m' \in \mathbb{N}$, there exist $\theta_{k,m,m'} \in]0, 1/2[$ such that for any $b > 0$, there exist $C > 0$ such that for every $x_0 \in X$, $Z, Z' \in T_{x_0}X$ with $|Z|, |Z'| < bp^{-\frac{1}{2}+\theta_{k,m,m'}}$ and $p \in \mathbb{N}^*$, $|\alpha| + |\alpha'| \leqslant m'$, we have

$$\begin{aligned} \Bigg|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}}\Bigg(p^{-n}\Xi_{p,x_0}(Z,Z')\kappa_{x_0}^{1/2}(Z)\kappa_{x_0}^{1/2}(Z') - \sum_{r=0}^{k}(Q_{r,x_0}\mathscr{P}_{x_0})(\sqrt{p}Z,\sqrt{p}Z')p^{-r/2}\Bigg)\Bigg|_{\mathscr{C}^m(X)} \\ \leqslant Cp^{-(k-m')/2}. \end{aligned} \tag{2.3.46}$$

In fact, by $\Xi_p = P_p\Xi_pP_p$, for $|Z|, |Z'| < bp^{-\frac{1}{2}+\theta_{k,m,m'}}$, we have the analogue of (2.3.16):

$$\Xi_{p,x_0}(Z,Z') = \int_X P_{p,x_0}(Z,y)\Xi_p(y,Z')P_{p,x_0}(y,Z')dv_X(y). \tag{2.3.47}$$

Then the estimate (2.3.46) follows from Theorem 2.2.2, (2.3.6), (2.3.15) and (2.3.47) in the same manner as (2.3.45) follows from (2.3.15), (2.3.16) and (2.3.20).

2.4 Criterion for Toeplitz operators

In this section we prove a useful criterion which ensures that a given family of bounded linear operators is a Toeplitz operator.

Theorem 2.4.1. *Let $\{T_p : L^2(X,E_p) \to L^2(X,E_p)\}$ be a family of bounded linear operators which satisfies the following three conditions:*
(i) *For any $p \in \mathbb{N}$, $P_pT_pP_p = T_p$.*
(ii) *For any $b > 0$, $l \in \mathbb{N}$ and $0 < \theta < 1$, there exists $C_{b,l,\theta} > 0$ such that for all $p \geqslant 1$ and all $(x,x') \in X \times X$ with $d(x,x') > bp^{-\theta/2}$,*

$$\Big|T_p(x,x')\Big| \leqslant C_{b,l,\theta}p^{-l}. \tag{2.4.1}$$

(iii) *There exists a family of polynomials* $\left\{\mathscr{Q}_{r,x_0} \in \operatorname{End}(E_{x_0})[Z,Z']\right\}_{x_0 \in X}$ *such that*
(a) *each* $\mathscr{Q}_{r,x_0}$ *has the same parity as* r,
(b) *there exist* $b_1, b_0 \in \mathbb{N}$ *such that* $\deg \mathscr{Q}_r \leqslant b_1 r + b_0$,
(c) *the family is smooth in* $x_0 \in X$,
(d) *for any* $k_0, m \in \mathbb{N}$, *there exists* $\theta_{k_0,m} \in \,]0,1/2[$ *such that for any* $b > 0$, $p \in \mathbb{N}^*$, $x_0 \in X$ *and every* $Z, Z' \in T_{x_0}X$ *with* $|Z|, |Z'| < bp^{-\frac{1}{2}+\theta_{k_0,m}}$, *we have*

$$p^{-n} T_{p,x_0}(Z,Z') \cong \sum_{r=0}^{k_0} \Big(\mathscr{Q}_{r,x_0}\mathscr{P}_{x_0}\Big)(\sqrt{p}Z, \sqrt{p}Z') p^{-r/2} + \mathscr{O}_m(p^{-k_0/2}), \tag{2.4.2}$$

in the sense of Notation A for $k_0, m, \theta_{k_0,m}$.
Then $\{T_p\}$ *is a Toeplitz operator.*

Remark 2.4.2. By Lemmas 2.3.1 and 2.3.3, by (2.1.12), (2.1.13) and the Sobolev inequality (cf. [20, (4.14)], it follows that every Toeplitz operator in the sense of Definition 2.1.1 verifies the Conditions (i), (ii) and (iii) of Theorem 2.4.1.

The rest of this sections is dedicated to the proof of Theorem 2.4.1. We will define inductively the sequence $(g_l)_{l \geqslant 0}$, $g_l \in \mathscr{C}^\infty(X, \operatorname{End}(E))$ such that in the sense of Definition 2.1.1,

$$T_p = \sum_{l=0}^{q} P_p g_l p^{-l} P_p + O(p^{-q-1}) \quad \text{for all } q \geqslant 0. \tag{2.4.3}$$

Let us start with the case $q = 0$ of (2.4.3). For any $x_0 \in X$, we set

$$g_0(x_0) = \mathscr{Q}_{0,x_0}(0,0) \in \operatorname{End}(E_{x_0}). \tag{2.4.4}$$

We will show that

$$T_p = P_p g_0 P_p + O(p^{-1}). \tag{2.4.5}$$

The proof of (2.4.5) is the result of Proposition 2.4.3 and Proposition 2.4.9.

Proposition 2.4.3. *In the conditions of* Theorem 2.4.1, *we have*

$$\mathscr{Q}_{0,x_0}(Z,Z') = \mathscr{Q}_{0,x_0}(0,0) \in \operatorname{End}(E_{x_0}) \tag{2.4.6}$$

for all $x_0 \in X$ *and all* $Z, Z' \in T_{x_0}X$.

Proof. The proof is divided in the series of Lemmas 2.4.4– 2.4.8. Our first observation is as follows.

Lemma 2.4.4. $\mathscr{Q}_{0,x_0} \in \operatorname{End}(E_{x_0})[Z,Z']$ *only depends on* $z, \bar{z}'$, *so that there is* $\mathscr{Q}_{x_0} \in \operatorname{End}(E_{x_0})[z, \bar{z}']$ *such that*

$$\mathscr{Q}_{x_0}(z, \bar{z}') = \mathscr{Q}_{0,x_0}(Z,Z'). \tag{2.4.7}$$

Proof. By (2.4.2), for $k_0 = 2$ there exists $\theta_3 \in]0, 1/2[$ such that for any $b > 0$ and every $Z, Z' \in T_{x_0}X$ with $|Z|, |Z'| < bp^{-\frac{1}{2}+\theta_3}$, we have

$$p^{-n} T_{p,x_0}(Z, Z') \cong \sum_{r=0}^{2} (\mathcal{Q}_{r,x_0} \mathscr{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z') p^{-r/2} + \mathscr{O}_m(p^{-1}). \tag{2.4.8}$$

By (2.3.8),

$$p^{-n} P_{p,x_0}(Z, Z') \cong \sum_{r=0}^{2} (J_{r,x_0} \mathscr{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z') p^{-r/2} + \mathscr{O}_m(p^{-1}), \tag{2.4.9}$$

in the sense of Notation A with $\theta_4 = 1/4(n + m + 4)$. Combining (2.4.8) and (2.4.9), as we got (2.3.45) from (2.3.15) and (2.3.20), we obtain that for every $Z, Z' \in T_{x_0}X$ with $|Z|, |Z'| < bp^{-\frac{1}{2}+\theta_2}$ and θ_2 given in (2.3.44) for some large k,

$$\begin{aligned} &p^{-n}(P_p T_p P_p)_{x_0}(Z, Z') \\ \cong &\sum_{r=0}^{2} \sum_{r_1+r_2+r_3=r} \Big[(J_{r_1,x_0} \mathscr{P}_{x_0}) \circ (\mathcal{Q}_{r_2,x_0} \mathscr{P}_{x_0}) \circ (J_{r_3,x_0} \mathscr{P}_{x_0}) \Big] (\sqrt{p}Z, \sqrt{p}Z') p^{-r/2} \\ &+ \mathscr{O}_m(p^{-1}). \end{aligned} \tag{2.4.10}$$

Since $P_p T_p P_p = T_p$, we deduce from (2.4.8) and (2.4.10) that

$$\mathcal{Q}_{0,x_0} \mathscr{P}_{x_0} = \mathscr{P}_{x_0} \circ (\mathcal{Q}_{0,x_0} \mathscr{P}_{x_0}) \circ \mathscr{P}_{x_0}. \tag{2.4.11}$$

By [51, (2.8)] and (2.4.11), we obtain that $\mathcal{Q}_{0,x_0}$ only depends on $z, \bar{z}'$. The proof of Lemma 2.4.4 is complete. □

For any $x_0 \in X$, let $\mathcal{Q}_{0,x_0} = \sum_{i \geqslant 0} \mathcal{Q}_{x_0}^{(i)}$ be the decomposition of the polynomial $\mathcal{Q}_{x_0}$ in (2.4.7) in homogeneous polynomials $\mathcal{Q}_{x_0}^{(i)}$ of degree i. We will show that $\mathcal{Q}_{x_0}^{(i)}$ vanishes identically for $i > 0$, that is

$$\mathcal{Q}_{x_0}^{(i)}(z, \bar{z}') = 0 \quad \text{for all } x_0 \in X,\ i > 0 \text{ and } z, \bar{z}' \in \mathbb{C}. \tag{2.4.12}$$

The first step is to prove

$$\mathcal{Q}_{x_0}^{(i)}(0, \bar{z}') = 0 \quad \text{for all } x_0 \in X,\ i > 0 \text{ and } z' \in \mathbb{C}. \tag{2.4.13}$$

Consider $0 < \theta_{k_0,m} < 1$ as in hypothesis (iii)(c) of Theorem 2.4.1. For $x \in X$, $Z' \in \mathbb{R}^{2n} \simeq T_xX$ with $|Z'| < a^X$ and $y = \exp_x^X(Z')$, set

$$\begin{aligned} F^{(i)}(x, y) = &\mathcal{Q}_x^{(i)}(0, \bar{z}') \in \operatorname{End}(E_x), \\ \widetilde{F}^{(i)}(x, y) = &\Big(F^{(i)}(y, x) \Big)^* \in \operatorname{End}(E_y). \end{aligned} \tag{2.4.14}$$

Then F^i and $\widetilde{F}^{(i)}$ define smooth sections on a neighborhood of the diagonal of $X \times X$. Clearly, the $\widetilde{F}^{(i)}(x, y)$'s need not be polynomials in z and $\bar{z}'$.

Since we wish to define global operators induced by these kernels, we use a cut-off function in the neighborhood of the diagonal. Pick a smooth function $\eta \in \mathscr{C}^\infty(\mathbb{R})$ such that

$$\eta(u) = 1 \text{ for } |u| \leqslant \varepsilon/2 \quad \text{and} \quad \eta(u) = 0 \text{ for } |u| \geqslant \varepsilon. \tag{2.4.15}$$

We denote by $F^{(i)}P_p$ and $P_p\widetilde{F}^{(i)}$ the operators defined by the kernels

$$\eta(d(x,y))F^{(i)}(x,y)P_p(x,y) \ \text{ and } \ \eta(d(x,y))P_p(x,y)\widetilde{F}^{(i)}(x,y) \tag{2.4.16}$$

with respect to $dv_X(y)$. Set

$$\mathscr{T}_p = T_p - \sum_{i\geqslant 0}(F^{(i)}P_p)p^{i/2}. \tag{2.4.17}$$

The operators $\mathscr{T}_p$ extend naturally to bounded operators on $L^2(X, E_p)$.

From (2.4.2) and (2.4.17), we deduce that for any $k_0, m \in \mathbb{N}$, there exists $\theta_{k_0,m} \in]0, 1/2[$ such that for any $|Z'| < \varepsilon p^{-\frac{1}{2}+\theta_{k_0,m}}$, we have the following expansion in the normal coordinates around $x_0 \in X$,

$$p^{-n}\mathscr{T}_{p,x_0}(0,Z') \cong \sum_{r=1}^{k_0}(R_{r,x_0}\mathscr{P}_{x_0})(0,\sqrt{p}Z')p^{-r/2} + \mathscr{O}_m(p^{-k_0/2}), \tag{2.4.18}$$

for some polynomials R_{r,x_0} of the same parity as r. For simplicity we denote by $R_{r,p}$ the operator defined as in (2.4.14) by the kernel

$$R_{r,p}(x,y) = p^n(R_{r,x}\mathscr{P}_x)(0,\sqrt{p}Z')\kappa_x^{-1/2}(Z')\eta(d(x,y)), \tag{2.4.19}$$

where $y = \exp_x^X(Z')$.

Lemma 2.4.5. *For $k_0 \geqslant 2(n+1)$, there exists $C > 0$ such that for every $p \geqslant 1$ and $s \in L^2(X, E_p)$, we have*

$$\begin{aligned} \|\mathscr{T}_p s\|_{L^2} &\leqslant Cp^{-1/2}\|s\|_{L^2}, \\ \|\mathscr{T}_p^* s\|_{L^2} &\leqslant Cp^{-1/2}\|s\|_{L^2}. \end{aligned} \tag{2.4.20}$$

Proof. In order to use (2.4.18) we write

$$\|\mathscr{T}_p s\|_{L^2} \leqslant \Big\|\Big(\mathscr{T}_p - \sum_{r=1}^{k_0}p^{-r/2}R_{r,p}\Big)s\Big\|_{L^2} + \Big\|\sum_{r=1}^{k_0}p^{-r/2}R_{r,p}s\Big\|_{L^2}. \tag{2.4.21}$$

By the Cauchy-Schwarz inequality we have

$$\begin{aligned} &\Big\|\Big(\mathscr{T}_p - \sum_{r=1}^{k_0}p^{-r/2}R_{r,p}\Big)s\Big\|_{L^2}^2 \\ &\leqslant \int_X \Big(\int_X \Big|\Big(\mathscr{T}_p - \sum_{r=1}^{k_0}p^{-r/2}R_{r,p}\Big)(x,y)\Big|dv_X(y)\Big) \\ &\qquad \times \Big(\int_X \Big|\Big(\mathscr{T}_p - \sum_{r=1}^{k_0}p^{-r/2}R_{r,p}\Big)(x,y)\Big||s(y)|^2dv_X(y)\Big)dv_X(x). \end{aligned} \tag{2.4.22}$$

By (2.2.1), (2.4.1), (2.4.16), (2.4.17) and (2.4.19), we obtain uniformly in $x \in X$,

$$\begin{aligned}
&\int_X \Big|\Big(\mathscr{T}_p - \sum_{r=1}^{k_0} p^{-r/2} R_{r,p}\Big)(x,y)\Big| |s(y)|^2 dv_X(y) \\
&\leqslant \int_{B^X(x,\frac{\varepsilon}{2}p^{-\frac{1}{2}+\theta_{k_0,m}})} \Big|\Big(\mathscr{T}_p - \sum_{r=1}^{k_0} p^{-r/2} R_{r,p}\Big)(x,y)\Big| |s(y)|^2 dv_X(y) \\
&\qquad\qquad + O(p^{-\infty}) \int_{X\setminus B^X(x,\frac{\varepsilon}{2}p^{-\frac{1}{2}+\theta_{k_0,m}})} |s(y)|^2 dv_X(y),
\end{aligned} \tag{2.4.23}$$

where $\theta_{k_0,m}$ is given by (2.4.18). By (2.3.6) and (2.4.18) we obtain

$$\begin{aligned}
&\int_{B^X(x,\frac{\varepsilon}{2}p^{-\frac{1}{2}+\theta_{k_0,m}})} \Big|\Big(\mathscr{T}_p - \sum_{r=1}^{k_0} p^{-r/2} R_{r,p}\Big)(x,y)\Big| |s(y)|^2 dv_X(y) \\
&= O(p^{-1}) \int_{B^X(x,\frac{\varepsilon}{2}p^{-\frac{1}{2}+\theta_{k_0,m}})} |s(y)|^2 dv_X(y).
\end{aligned} \tag{2.4.24}$$

In the same vein (by splitting the integral region as above) we obtain

$$\int_X \Big|\Big(\mathscr{T}_p - \sum_{r=1}^{k_0} p^{-r/2} R_{r,p}\Big)(x,y)\Big| dv_X(y) = O(p^{-1}) + O(p^{-\infty}). \tag{2.4.25}$$

Combining (2.4.22)–(2.4.25) yields

$$\Big\|\Big(\mathscr{T}_p - \sum_{r=1}^{k_0} p^{-r/2} R_{r,p}\Big)s\Big\|_{L^2} \leqslant Cp^{-1}\|s\|_{L^2}. \tag{2.4.26}$$

A similar proof as for (2.4.26) delivers for $s \in L^2(X, E_p)$,

$$\|R_{r,p}s\|_{L^2} \leqslant C\|s\|_{L^2}, \tag{2.4.27}$$

which implies

$$\Big\|\sum_{r=1}^{k_0} p^{-r/2} R_{r,p}s\Big\|_{L^2} \leqslant Cp^{-1/2}\|s\|_{L^2} \quad \text{for } s \in L^2(X, E_p), \tag{2.4.28}$$

for some constant $C > 0$. Relations (2.4.26) and (2.4.28) entail the first inequality of (2.4.20), which is equivalent to the second of (2.4.20), by taking the adjoint. This completes the proof of Lemma 2.4.5. □

For any $x_0 \in X$ and $Z, Z' \in \mathbb{R}^{2n} \simeq T_{x_0}X$ such that $|Z|, |Z'| < a^X$, recall that $\widetilde{F}^{(i)}_{x_0}(Z, Z') \in \mathrm{End}(E_{x_0})$ denotes the image of $\widetilde{F}^{(i)}(x, y) \in \mathrm{End}(E_x)$, with $x = \exp_{x_0}(Z)$, $y = \exp_{x_0}(Z')$, in the trivialization around $x_0 \in X$ defined in 2.2. Let us consider the following Taylor expansion, for any $k \in \mathbb{N}$,

$$\widetilde{F}^{(i)}_{x_0}(0, Z') = \sum_{|\alpha|\leqslant k} \frac{\partial^{|\alpha|}\widetilde{F}^{(i)}_{x_0}}{\partial Z'^{\alpha}}(0,0)\frac{(\sqrt{p}Z')^{\alpha}}{\alpha!}p^{-|\alpha|/2} + O(|Z'|^{k+1}). \tag{2.4.29}$$

The next step of the proof of Proposition 2.4.3 is the following.

Lemma 2.4.6. *For any $x_0 \in X$, we have*

$$\frac{\partial^{|\alpha|}\widetilde{F}^{(i)}_{x_0}}{\partial Z'^{\alpha}}(0,0) = 0 \quad \text{for } i - |\alpha| > 0. \tag{2.4.30}$$

Proof. The definition (2.4.17) of $\mathscr{T}_p$ shows that

$$\mathscr{T}_p^* = T_p^* - \sum_{i \geqslant 0} p^{i/2}(P_p \widetilde{F}^{(i)}). \tag{2.4.31}$$

Pick $x_0 \in X$, and let us develop the sum on the right-hand side. Combining the Taylor expansion (2.4.29) with the expansion (2.3.8) of the Bergman kernel, for any $k \in \mathbb{N}$ we obtain

$$\begin{aligned} p^{-n}\Big(P_p\widetilde{F}^{(i)}\Big)_{x_0}&(0,Z')\kappa^{1/2}(Z') \\ &- \sum_{r,|\alpha|\leqslant k}\Big(J_{r,x_0}\mathscr{P}_{x_0}\Big)(0,\sqrt{p}Z')\frac{\partial^{|\alpha|}\widetilde{F}^{(i)}_{x_0}}{\partial Z'^{\alpha}}(0,0)\frac{(\sqrt{p}Z')^{\alpha}}{\alpha!}p^{-\frac{|\alpha|+r}{2}} \\ =& \Big[p^{-n}P_{p,x_0}(0,Z')\kappa^{1/2}(Z') - \sum_{r\leqslant k}\Big(J_{r,x_0}\mathscr{P}_{x_0}\Big)(0,\sqrt{p}Z')p^{-r/2}\Big]\widetilde{F}^{(i)}_{x_0}(0,Z') \\ +\sum_{r\leqslant k}\Big(J_{r,x_0}\mathscr{P}_{x_0}\Big)&(0,\sqrt{p}Z')p^{-r/2}\Big[\widetilde{F}^{(i)}_{x_0}(0,Z') - \sum_{|\alpha|\leqslant k}\frac{\partial^{|\alpha|}\widetilde{F}^{(i)}_{x_0}}{\partial Z'^{\alpha}}(0,0)\frac{(\sqrt{p}Z')^{\alpha}}{\alpha!}p^{-\frac{|\alpha|}{2}}\Big]. \end{aligned} \tag{2.4.32}$$

By (2.3.8), (2.3.32), (2.4.29) and $\deg J_{r,x_0} \leqslant 3r$, we obtain that for $k \geqslant \deg \mathcal{Q}_{x_0} + 1$ and $m \in \mathbb{N}$, there exists $\theta_{k,m} \in]0,1[$ such that for any $Z' \in T_{x_0}X$ with $|Z'| \leqslant bp^{-\frac{1}{2}+\theta_{k,m}}$, we have

$$\begin{aligned} p^{-n}\sum_{i\geqslant 0}\Big(P_p\widetilde{F}^{(i)}\Big)_{x_0}&(0,Z')p^{i/2} \\ \cong \sum_{i\geqslant 0}\sum_{|\alpha|,r\leqslant k}&\Big(J_{r,x_0}\mathscr{P}_{x_0}\Big)(0,\sqrt{p}Z')\frac{\partial^{|\alpha|}\widetilde{F}^{(i)}_{x_0}}{\partial Z'^{\alpha}}(0,0)\frac{(\sqrt{p}Z')^{\alpha}}{\alpha!}p^{(i-|\alpha|-r)/2} \\ &+ \mathscr{O}_m\Big(p^{(\deg \mathcal{Q}_{x_0}-k-1)/2}\Big). \end{aligned} \tag{2.4.33}$$

Having in mind the second inequality of (2.4.20), this is only possible if for every $j > 0$, the coefficients of $p^{j/2}$ in the right-hand side of (2.4.33) vanish. Thus, we have for every $j > 0$,

$$\sum_{i=j}^{\deg \mathcal{Q}_{x_0}}\sum_{j+r=i-|\alpha|}J_{r,x_0}(0,\sqrt{p}Z')\frac{\partial^{|\alpha|}\widetilde{F}^{(i)}_{x_0}}{\partial Z'^{\alpha}}(0,0)\frac{(\sqrt{p}Z')^{\alpha}}{\alpha!} = 0. \tag{2.4.34}$$

This implies immediately that (2.4.30) holds for $i > \deg \mathcal{Q}_{x_0}$. From (2.4.34), we will prove by a descending recurrence on $j > 0$ that (2.4.30) holds for $i - |\alpha| > j$. As the first step of the recurrence, let us take $j = \deg \mathcal{Q}_{x_0}$ in (2.4.34). Since $J_{0,x_0} = \mathrm{Id}_{E_{x_0}}$, we get immediately $\widetilde{F}^{(j)}_{x_0}(0,0) = 0$ in that case. Hence (2.4.30) holds for $i - |\alpha| \geqslant \deg \mathcal{Q}_{x_0}$. Assume that (2.4.30) holds for $i - |\alpha| > j_0 > 0$. Then for $j = j_0$, the coefficient with $r > 0$ in (2.4.34) is zero. Since $J_{0,x_0} = \mathrm{Id}_{E_{x_0}}$, (2.4.34) reads

$$\sum_{\alpha \in \mathbb{N}^{2n}} \frac{\partial^{|\alpha|} \widetilde{F}^{(j_0+|\alpha|)}_{x_0}}{\partial Z'^{\alpha}}(x_0, 0) \frac{(\sqrt{p} Z')^{\alpha}}{\alpha!} = 0, \tag{2.4.35}$$

which entails (2.4.30) for $i - |\alpha| \geqslant j_0$. The proof of (2.4.30) is complete. □

Lemma 2.4.7. *For any $x_0 \in X$, we have*

$$\frac{\partial^{|\alpha|} \mathcal{Q}^{(i)}_{x_0}}{\partial \bar{z}'^{\alpha}}(0,0) = 0, \;\; |\alpha| \leqslant i. \tag{2.4.36}$$

Therefore, $\mathcal{Q}^{(i)}_{x_0}(0, \bar{z}') = 0$ for all $x_0 \in X$, $i > 0$ and $z' \in \mathbb{C}$, i.e., (2.4.13) holds true. Moreover,

$$\mathcal{Q}^{(i)}_{x_0}(z, 0) = 0 \text{ for all } x_0 \in X,\ i > 0 \text{ and all } z \in \mathbb{C}. \tag{2.4.37}$$

Proof. Let us start with some preliminary observations. From (2.4.30) and (2.4.33), we get for any $x_0 \in X$,

$$p^{-n} \sum_{i \geqslant 0} \left(P_p \widetilde{F}^{(i)} \right)_{x_0} (0, Z') p^{i/2} \cong \sum_{|\alpha| = i} \mathscr{P}_{x_0}(0, \sqrt{p} Z') \frac{\partial^{|\alpha|} \widetilde{F}^{(i)}_{x_0}}{\partial Z'^{\alpha}}(0,0) \frac{(\sqrt{p} Z')^{\alpha}}{\alpha!} + \mathscr{O}_m\left(p^{-1/2}\right). \tag{2.4.38}$$

On the other hand, taking the adjoint of (2.4.2) we get

$$p^{-n} T^*_{p,x_0}(0, Z') \cong \mathscr{P}_{x_0}(0, \sqrt{p} Z') \left(\mathcal{Q}_{0,x_0}(\sqrt{p} Z', 0) \right)^* + \mathscr{O}_m(p^{-1/2}). \tag{2.4.39}$$

In view of (2.4.7), (2.4.20), (2.4.29) and (2.4.30), comparing (2.4.31) with (2.4.38) and (2.4.39) for any $x_0 \in X$ gives

$$\widetilde{F}^{(i)}_{x_0}(0, Z') = \left(\mathcal{Q}^{(i)}_{x_0}(z', 0) \right)^* + O(|Z'|^{i+1}). \tag{2.4.40}$$

Expressing (2.4.14) in our trivialization around $x_0 \in X$ and taking the adjoint of (2.4.40), we get

$$F^{(i)}_{x_0}(Z, 0) = \mathcal{Q}^{(i)}_{x_0}(z, 0) + O(|Z|^{i+1}). \tag{2.4.41}$$

Now recall that in our trivialization around $x_0 \in X$, (2.4.14) writes

$$F^{(i)}_{x_0}(0, Z') = \mathcal{Q}_{x_0}(0, \bar{z}'). \tag{2.4.42}$$

Thus to get (2.4.36) we need to show that

$$\frac{\partial^{|\alpha|} F^{(i)}_{x_0}}{\partial \bar{z}'^{\alpha}}(0,0) = 0, \quad \text{for } |\alpha| \leqslant i. \tag{2.4.43}$$

We prove this by induction over $|\alpha|$. For $|\alpha| = 0$, we have $F^{(i)}_{x_0}(0,0) = 0$ for all $x_0 \in X$, since $\mathcal{Q}^{(i)}_{x_0}(z, \bar{z}')$ is a homogeneous polynomial of degree $i > 0$. For the induction step, take $\alpha \in \mathbb{N}^n$ and assume that (2.4.43) holds for $|\alpha| - 1$ at any $x_0 \in X$. Together with (2.4.42), this precisely means that all derivatives in y up to order $|\alpha| - 1$ of $F^{(i)}(x,y)$ vanish for $x = y$. In particular, consider j with $\alpha_j > 0$ and set

$$\beta = (\alpha_1, \ldots, \alpha_j - 1, \ldots, \alpha_n). \tag{2.4.44}$$

Then at any $x_0 \in X$ and for all $Z \in \mathbb{R}^{2n}$ with $|Z| < a^X$, the induction hypothesis implies

$$\frac{\partial^{|\beta|} F^{(i)}_{x_0}}{\partial \bar{z}'^{\beta}}(Z,Z) = 0. \tag{2.4.45}$$

Let $j_\Delta : \mathbb{R}^{2n} \to \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ be the diagonal injection. By (2.4.41) and (2.4.58), we deduce from the induction hypothesis that

$$\frac{\partial^{|\alpha|} F^{(i)}_{x_0}}{\partial \bar{z}'^{\alpha}}(0,0) = \left(\frac{\partial}{\partial \bar{z}_j} j_\Delta^* \frac{\partial^{|\beta|} F^{(i)}_{x_0}}{\partial \bar{z}'^{\beta}}\right)(0) - \left(\frac{\partial^\beta}{\partial \bar{z}'^\beta} \frac{\partial F^{(i)}_{x_0}}{\partial \bar{z}_j}\right)(0,0) = 0. \tag{2.4.46}$$

Thus, (2.4.36) is proved, and this is equivalent to (2.4.13). Then (2.4.37) follows from (2.4.13), (2.4.14) and (2.4.40), taking the adjoint of (2.4.42). This finishes the proof of Lemma 2.4.7. □

Lemma 2.4.8. *We have* $\mathcal{Q}^{(i)}_{x_0}(z, \bar{z}') = 0$ *for all* $x_0 \in X$, $i > 0$ *and* $z, z' \in \mathbb{C}^n$.

Proof. For any $x_0 \in X$, let us consider the operator

$$\frac{1}{\sqrt{p}} P_p \Big(\nabla^{E_p}_v T_p\Big) P_p \quad \text{with } v \in \mathscr{C}^\infty(X, TX_{\mathbb{C}}),\ v_{x_0} = \frac{\partial}{\partial z_j}. \tag{2.4.47}$$

By Remark 2.3.4, the operator (2.4.47) admits an expansion as in (2.4.2), with leading term at $x_0 \in X$ equal to

$$\left(\frac{\partial \mathcal{Q}_{x_0}}{\partial z_j}\right)(\sqrt{p}z, \sqrt{p}\bar{z}')\mathscr{P}_{x_0}(\sqrt{p}Z, \sqrt{p}Z'). \tag{2.4.48}$$

On the other hand, note that the proofs of Lemmas 2.4.5 to 2.4.7 did not use the condition (a) in Theorem 2.4.1, so that Lemma 2.4.7 holds for the operator (2.4.47). Following the notations above, we thus get for $i > 0$,

$$\frac{\partial \mathcal{Q}^{(i+1)}_{x_0}}{\partial z_j}(0, \bar{z}') = \left(\frac{\partial \mathcal{Q}_{x_0}}{\partial z_j}\right)^{(i)}(0, \bar{z}') = 0. \tag{2.4.49}$$

Now (2.4.37) tells us that the constant term of $(\frac{\partial}{\partial z_j}\mathscr{Q}_{x_0})(z,\bar{z}')$ vanishes, so that (2.4.49) holds as well for $i=0$. Then by (2.4.14), (2.4.39), (2.4.40) and (2.4.41), taking the adjoint of (2.4.49) we further get for any $i>0$,

$$\frac{\partial \mathscr{Q}^{(i)}_{x_0}}{\partial \bar{z}'_j}(z,0)=0. \tag{2.4.50}$$

By continuing this process, we show that for all $x_0\in X$, $i>0$, $\alpha\in\mathbb{N}^n$, $z,z'\in\mathbb{C}^n$,

$$\frac{\partial^{|\alpha|}\mathscr{Q}^{(i)}_{x_0}}{\partial z^\alpha}(0,\bar{z}')=\frac{\partial^{|\alpha|}\mathscr{Q}^{(i)}_{x_0}}{\partial \bar{z}'^\alpha}(z,0)=0. \tag{2.4.51}$$

This proves Lemma 2.4.8 and (2.4.12) holds true. □

Lemma 2.4.8 finishes the proof of Proposition 2.4.3. □

We come now to the proof of the first induction step leading to (2.4.3).

Proposition 2.4.9. *We have*

$$p^{-n}(T_p-T_{g_0,p})_{x_0}(Z,Z')\cong\mathscr{O}_m(p^{-1}) \tag{2.4.52}$$

in the sense of Notation A for some $\theta_m\in\,]0,1/2[$. *Consequently,*

$$T_p=P_pg_0P_p+O(p^{-1}), \tag{2.4.53}$$

i.e., relation (2.4.5) *holds true in the sense of* (2.1.14).

Proof. Let us compare the asymptotic expansion of T_p and $T_{g_0,p}=P_pg_0P_p$. Using the Notation A, the expansion (2.3.10) (for $k_0=2$) reads for $\theta_m=1/4(n+k+m+2)$,

$$\begin{aligned}&p^{-n}T_{g_0,p,x_0}(Z,Z') \\ &\cong\Big(g_0(x_0)\mathscr{P}_{x_0}+Q_{1,x_0}(g_0)\mathscr{P}_{x_0}p^{-1/2}+Q_{2,x_0}(g_0)\mathscr{P}_{x_0}p^{-1}\Big)(\sqrt{p}Z,\sqrt{p}Z')+\mathscr{O}_m(p^{-1}),\end{aligned} \tag{2.4.54}$$

since $Q_{0,x_0}(g_0)=g_0(x_0)$ by (2.3.13). The expansion (2.4.2) (also for $k_0=2$) takes the form for θ_m in (2.4.2),

$$\begin{aligned}&p^{-n}T_{p,x_0}(Z,Z') \\ &\cong\Big(g_0(x_0)\mathscr{P}_{x_0}+\mathscr{Q}_{1,x_0}\mathscr{P}_{x_0}p^{-1/2}+\mathscr{Q}_{2,x_0}\mathscr{P}_{x_0}p^{-1}\Big)(\sqrt{p}Z,\sqrt{p}Z')+\mathscr{O}_m(p^{-1}),\end{aligned} \tag{2.4.55}$$

where we have used Proposition 2.4.3 and the definition (2.4.4) of g_0. Thus subtracting (2.4.54) from (2.4.55) we obtain for some $\theta_m\in\,]0,1/2[$,

$$\begin{aligned}&p^{-n}(T_p-T_{g_0,p})_{x_0}(Z,Z') \\ &\cong\Big((\mathscr{Q}_{1,x_0}-Q_{1,x_0}(g_0))\mathscr{P}_{x_0}\Big)(\sqrt{p}Z,\sqrt{p}Z')p^{-1/2}+\mathscr{O}_m(p^{-1}).\end{aligned} \tag{2.4.56}$$

Thus, it suffices to prove the following result.

Lemma 2.4.10. *For any $x_0 \in X$,*

$$\mathscr{Q}_{1,x_0} - Q_{1,x_0}(g_0) \equiv 0. \tag{2.4.57}$$

Proof. From (2.4.56), we see that $\mathscr{Q}_{1,x_0} - Q_{1,x_0}(g_0)$ is the polynomial associated with the first coefficient of the expansion as in (2.4.2) of

$$\sqrt{p}(T_p - P_p g_0 P_p). \tag{2.4.58}$$

As before, we see that this operator satisfies the hypotheses of Lemmas 2.4.5 to 2.4.8, so that all the homogeneous components of degree $i > 0$ of $\mathscr{Q}_{1,x_0} - Q_{1,x_0}(g_0)$ vanish. Furthermore, $\mathscr{Q}_{1,x_0} - Q_{1,x_0}(g_0)$ is an odd polynomial, so that in particular its constant term vanishes as well. This shows (2.4.57). □

Lemma 2.4.10 and the expansion (2.4.56) imply immediately Proposition 2.4.9. □

Now Proposition 2.4.9 shows that the asymptotic expansion (2.4.3) of T_p holds for $q = 0$. To prove (2.4.3) for $q = 1$, note that by (2.4.52), the operator $p(T_p - P_p g_0 P_p)$ satisfies the hypotheses of Theorem 2.4.1, so that Proposition 2.4.3 and Proposition 2.4.9 applied to $p(T_p - P_p g_0 P_p)$ yield $g_1 \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ such that (2.4.3) holds true for $q = 1$. We can then continue this process to get (2.4.3) for any $q = 1$. This completes the proof of Theorem 2.4.1.

2.5 Algebra of Toeplitz operators

The Poisson bracket $\{\cdot,\cdot\}$ on $(X, 2\pi\omega)$ is defined as follows. For $f, g \in \mathscr{C}^\infty(X)$, let ξ_f be the Hamiltonian vector field generated by f, which is defined by $2\pi i_{\xi_f}\omega = df$. Then

$$\{f, g\} = i_{\xi_f} dg. \tag{2.5.1}$$

Proof of Theorem 2.1.2. First, it is obvious that $P_p T_{f,p} T_{g,p} P_p = T_{f,p} T_{g,p}$. To prove (2.4.1), note that from Lemma 2.3.1 and (2.3.10), we know that for any $k \in \mathbb{N}$ there exist $C_k > 0$ and $M_k > 0$ such that for all $(x, x') \in X \times X$,

$$\Big| T_{f,p}(x, x') \Big|_{\mathscr{C}^k(X\times X)} \leqslant C_k p^{M_k}. \tag{2.5.2}$$

For any $b > 0$ and $0 < \theta < 1$, if $d(x, x') > bp^{-\theta/2}$, then

$$T_{f,p}T_{g,p}(x, x') = \Big(\int_{B^X(x, \frac{b}{2}p^{-\theta/2})} + \int_{X \setminus B^X(x, \frac{b}{2}p^{-\theta/2})} \Big) T_{f,p}(x, x'') T_{g,p}(x'', x') dv_X(x''). \tag{2.5.3}$$

Then (2.4.1) follows from (2.3.2), (2.3.5), (2.5.2) and (2.5.3). Like (2.3.18), for $|Z|, |Z'| < \frac{b}{2}p^{-\theta/2}$, we have

$$\begin{aligned} &(T_{f,p}T_{g,p})_{x_0}(Z, Z') \\ &= \int_{|Z''| < bp^{-\theta_1/2}} T_{f,p,x_0}(z, z'') T_{g,p,x_0}(Z'', Z') \kappa_{x_0}(Z'') dv_{TX}(Z'') + \mathscr{O}_m(p^{-\infty}). \end{aligned} \tag{2.5.4}$$

By Lemma 2.3.1 and Lemma 2.3.3 and (2.5.4), we deduce as we obtain (2.3.45) from Proposition 2.2.1, (2.3.15) and (2.3.18) in the proof of Lemma 2.3.3 that for $|Z|, |Z'| < \frac{b}{2}p^{-\theta/2}$, we have

$$p^{-n}(T_{f,p}T_{g,p})_{x_0}(Z,Z') \cong \sum_{r=0}^{k_0} \Big(Q_{r,x_0}(f,g)\mathscr{P}_{x_0}\Big)(\sqrt{p}Z, \sqrt{p}Z')p^{-r/2} + \mathscr{O}_m(p^{-k_0/2}), \quad (2.5.5)$$

with

$$Q_{r,x_0}(f,g) = \sum_{r_1+r_2=r} \mathscr{K}[Q_{r_1,x_0}(f), Q_{r_2,x_0}(g)]. \quad (2.5.6)$$

Thus, $T_{f,p}T_{g,p}$ is a Toeplitz operator by Theorem 2.4.1. Moreover, it follows form the proofs of Lemma 2.3.3 and Theorem 2.4.1 that $g_l = C_l(f,g)$, where C_l are bidifferential operators.

The rest of the proof of Theorem 2.1.2 is exactly the same as that of [51, Th.1.1] and we omit it here. This finishes the proof of Theorem 2.1.2. □

Proof of Theorem 2.1.4. Take a point $x_0 \in X$ and $u_0 \in E_{x_0}$ with $|u_0|_{h^E} = 1$ such that $|f(x_0)(u_0)| = \|f\|_\infty$. Recall that we trivialized the bundles L, E in normal coordinates near x_0, and e_L is the unit frame of L which trivializes L. Moreover, in this normal coordinates, u_0 is a trivial section of E. Considering the sequence of sections $S^p_{x_0} = p^{-n/2}P_p(e_L^{\otimes p} \otimes u_0)$, we have by (2.3.8),

$$\left\|T_{f,p}\, S^p_{x_0} - f(x_0)S^p_{x_0}\right\|_{L^2} \leqslant \frac{C}{\sqrt{p}} \left\|S^p_{x_0}\right\|_{L^2}, \quad (2.5.7)$$

which immediately implies (2.1.20). □

2.6 Proof of Theorem Theorem 2.1.5

In this section, we show how to adapt the results of [36] in order to give a proof to Theorem 2.1.5, that is the computation of the coefficient $C_1(f,g)$ of Theorem 2.1.2.

Fix $x_0 \in X$ and $\varepsilon \in (0, a^X/4)$. It is shown in [50, Th.1.4] that the restriction on $B^X(x_0,\varepsilon)$ of the operator $\Delta_{p,\Phi}$ defined in (2.1.5) is equal, through the trivializations given in Section 2.2 and after a convenient rescaling in $\sqrt{p} := 1/t$, to an operator $\mathscr{L}_t$ on $B^{T_{x_0}X}(0,\varepsilon/t)$ satisfying

$$\mathscr{L}_t = \mathscr{L}_0 + \sum_{r=1}^{m} t^r \mathcal{O}_r + \mathscr{O}(t^{m+1}), \quad (2.6.1)$$

for any $m \in \mathbb{N}$, where $\{\mathcal{O}_r\}_{r\in\mathbb{N}}$ is a family of differential operators of order equal or less than 2, with coefficients explicitly computable in terms of local data, and where the differential operator $\mathscr{O}(t^{m+1})$ has its coefficients and their derivatives up to order k dominated by $C_k t^{m+1}$ for any $k \in \mathbb{N}$.

Moreover, as explained in [50, § 1.4], the differential operator $\mathscr{L}_0$ acts on the scalar part of smooth functions on $\mathbb{R}^{2n}$ with values in E_{x_0}, and the spectrum of its restriction to $L^2(\mathbb{R}^{2n})$ is given by $\{4\pi n \mid n \in \mathbb{N}\}$. Furthermore, the kernel of the orthogonal projection P from $L^2(\mathbb{R}^{2n}, E_{x_0})$ to $\operatorname{Ker}(\mathscr{L}_0)$ is given by $\mathscr{F}_{0,x_0}(Z,Z') = \mathscr{P}(Z,Z')\operatorname{Id}_{E_{x_0}}$ as in (2.2.5). We write $P^\perp = \operatorname{Id}_{E_{x_0}} - P$, and define the operator $\mathscr{L}_0^{-1}P^\perp$ by inverting the positive eigenvalues of $\mathscr{L}_0|_{L^2(\mathbb{R}^{2n})}$.

As shown in [50], there is a direct method to compute the family $\{\mathscr{F}_{r,x_0}(Z,Z')\}_{r\in\mathbb{N}}$ defined in Theorem 2.2.2, using (2.6.1). The following lemma, which has been established in [50, Th.1.16, (1.30), (1.111)], gives the first three elements of this family.

Lemma 2.6.1. *For any $r \in \mathbb{N}$, let $\mathscr{F}_{r,x_0}$ be the operator associated to the kernel $\mathscr{F}_{r,x_0}(Z,Z')$, and let the differential operators $\mathcal{O}_1$ and $\mathcal{O}_2$ be as in* (2.6.1)*. Then the following formulas hold:*

$$\begin{aligned}
\mathscr{F}_{1,x_0} =& -\mathscr{L}_0^{-1}P^\perp\mathcal{O}_1P - P\mathcal{O}_1\mathscr{L}_0^{-1}P^\perp,\\
\mathscr{F}_{2,x_0} =& \mathscr{L}_0^{-1}P^\perp\mathcal{O}_1\mathscr{L}_0^{-1}P^\perp\mathcal{O}_1P - \mathscr{L}_0^{-1}P^\perp\mathcal{O}_2P\\
&+ P\mathcal{O}_1\mathscr{L}_0^{-1}P^\perp\mathcal{O}_1\mathscr{L}_0^{-1}P^\perp - P\mathcal{O}_2\mathscr{L}_0^{-1}P^\perp\\
&+ (\mathscr{L}_0)^{-1}P^\perp\mathcal{O}_1P\mathcal{O}_1\mathscr{L}_0^{-1}P^\perp - P\mathcal{O}_1(\mathscr{L}_0)^{-2}P^\perp\mathcal{O}_1P.
\end{aligned} \tag{2.6.2}$$

Moreover, $\mathcal{O}_1$ commutes with any $A \in \operatorname{End}(E_{x_0})$, and we have the formula

$$P\mathcal{O}_1P = 0. \tag{2.6.3}$$

In particular, $\mathscr{F}_{0,x_0}$ and $\mathscr{F}_{1,x_0}$ commute with any $A \in \operatorname{End}(E_{x_0})$.

Lemma 2.6.1 corresponds to [36, Lem.3.3], and the following technical Lemma corresponds to [36, Lem.3.5]. It was essentially proved in [50, (2.25)].

Lemma 2.6.2. *The following formulas hold:*

$$\begin{aligned}
(P\mathcal{O}_1\mathscr{L}_0^{-1}P^\perp)(0,Z') &= 0,\\
(P\mathcal{O}_1\mathscr{L}_0^{-1}P^\perp)(Z,0) &= 0,\\
(\mathscr{L}_0^{-1}P^\perp\mathcal{O}_1P)(Z,0) &= 0.
\end{aligned} \tag{2.6.4}$$

The result of Lemma 2.6.2 is a simple computation from the first line of [50, (2.25)], using [50, (1.98), (1.99)] and recalling the formula $T^*(Z,Z') = T(Z',Z)^*$ for the kernel of the dual T^* of an operator T. In fact, all the kernels associated to the situation in this chapter are the degree-0 part of the kernels of the corresponding situation in [36]. Lemma 2.6.2 is then an expression of the fact that the corresponding formulas in [36, Lem.3.5] have vanishing degree-0 part.

Now, from the proof of Theorem 2.4.1, the following formula holds,

$$C_1(f,g)(x_0) = Q_{2,x_0}(f,g)(0,0) - Q_{2,x_0}(fg)(0,0), \tag{2.6.5}$$

where the coefficients $Q_{2,x_0}(f,g)$ and $Q_{2,x_0}(fg)$ have been defined in Lemma 2.3.3 and (2.5.6) respectively. Note that the formula (2.6.5) is actually simpler than the one given in [51, (4.82)], due to the fact that we only need to consider the degree-0 part. The following Proposition corresponds to [36, (3.19)], and is easily seen to imply Theorem 2.1.5 in the trivialization described in Section 2.2.

Proposition 2.6.3. *Assume that $g^{TX}(\cdot,\cdot) = \omega(\cdot, J\cdot)$. Then in the complex coordinates of $Z \in \mathbb{R}^{2n} \simeq (T_{x_0}X, J)$ as in Section 2.2, the following formula holds,*

$$Q_{2,x_0}(f,g)(0,0) - Q_{2,x_0}(fg)(0,0) = -\frac{1}{\pi}\sum_{j=1}^{n} \frac{\partial f_{x_0}}{\partial z_j}(0)\frac{\partial g_{x_0}}{\partial \bar{z}_j}(0). \tag{2.6.6}$$

Proof. By (2.2.11) and following [51, Lem.2.2, Ex.2.3], the kernel calculus of [49, § 7.1] as described in [36, § 2.3] is still valid. Note that the assumption $g^{TX}(\cdot,\cdot) = \omega(\cdot, J\cdot)$ is equivalent to $a_j = 2\pi$ in (2.2.4), for all $1 \leqslant j \leqslant n$.

Recall the formulas (2.3.12) and (2.5.6) with $r = 2$ for the second and the first term of (2.6.5) respectively. Furthermore, by (2.6.2), by (2.3.12) with $r = 1$ and as in the proof of [36, Lem.3], the following formula still holds,

$$Q_{1,x_0}(f) = f(x_0)J_{1,x_0} + \mathscr{K}\Big[J_{1,x_0}, \sum_{j=1}^{2n} \frac{\partial f_{x_0}}{\partial Z_j}(0) Z_j J_{0,x_0}\Big]. \tag{2.6.7}$$

Then the computations of [36, § 3.2] go through, and even simplify due to Lemma 2.6.2. In particular, writing Z_j for the operator of scalar multiplication by Z_j in $\mathrm{End}(E_{x_0}[Z,Z'])$ for all $1 \leqslant j \leqslant 2n$, by (2.2.5), (2.2.11), (1.3.11) and the first line of (1.3.19), we have as in [36, (3.46), (3.49)],

$$\begin{aligned} \mathscr{K}\Big[J_{0,x_0}, \mathscr{K}\Big[J_{1,x_0}, Z_j J_{0,x_0}\Big]\Big](0,0) &= (\mathscr{F}_{0,x_0}\mathscr{F}_{1,x_0} Z_j \mathscr{F}_{0,x_0})(0,0) \\ &= -(P\mathcal{O}_1\mathscr{L}_0^{-1}P^{\perp}Z_jP)(0,0) \\ &= -\int_{\mathbb{R}^{2n}} (P\mathcal{O}_1\mathscr{L}_0^{-1}P^{\perp})(0,Z) Z_j \mathscr{P}(Z,0)dZ = 0. \end{aligned} \tag{2.6.8}$$

In the same way, writing Z'_j for the operator of multiplication by Z'_j in $\mathrm{End}(E_{x_0}[Z,Z'])$ for all $1 \leqslant j \leqslant 2n$, by (2.6.2) and the second line of (2.6.4), we have as in [36, (3.53)],

$$\begin{aligned} \mathscr{K}\Big[Z'_jJ_{0,x_0}, \mathscr{K}\Big[J_{1,x_0}, J_{0,x_0}\Big]\Big](0,0) &= (Z'_j\mathscr{F}_{0,x_0}\mathscr{F}_{1,x_0}\mathscr{F}_{0,x_0})(0,0) \\ &= -(Z'_jP\mathscr{L}_0^{-1}P^{\perp}\mathcal{O}_1P)(0,0) \\ &= -\int_{\mathbb{R}^{2n}} Z'_j\mathscr{P}(0,Z')(\mathscr{L}_0^{-1}P^{\perp}\mathcal{O}_1P)(Z',0)dZ' = 0. \end{aligned} \tag{2.6.9}$$

Finally, writing z_j for the operator of multiplication by z_j in $\mathrm{End}(E_{x_0}[Z,Z'])$ for all $1 \leqslant j \leqslant n$, by (2.6.2) and the last line of (2.6.4), we have as in [36, (3.65)],

$$\begin{aligned} P(z_j\mathscr{L}_0^{-1}P^{\perp}\mathcal{O}_1P)(0,0) &= \int_{\mathbb{R}^{2n}} \mathscr{P}(0,Z)z_j(\mathscr{L}_0^{-1}P^{\perp}\mathcal{O}_1P)(Z,0)dZ \\ &= 0. \end{aligned} \tag{2.6.10}$$

Then (2.6.6) is precisely [36, (3.78)]. □

Chapter 3

Quantization and isotropic submanifolds

3.1 Introduction

Let (X, ω) be a compact symplectic manifold of dimension $2n$, and let (L, h^L) be a Hermitian line bundle over X, endowed with a Hermitian connection ∇^L such that its curvature R^L satisfies the following *prequantization condition*,

$$\omega = \frac{\sqrt{-1}}{2\pi} R^L. \tag{3.1.1}$$

Let J be an almost complex structure on TX compatible with ω, and let g^{TX} be the Riemannian metric on TX induced by ω and J. For any $p \in \mathbb{N}^*$, we denote by L^p the p-th tensor power of L. Then following [31, (1.7)], we consider the *renormalized Bochner Laplacian* acting on $\mathscr{C}^\infty(X, L^p)$, given for any $p \in \mathbb{N}^*$ by the formula

$$\Delta^{L^p} - 2\pi n p, \tag{3.1.2}$$

where Δ^{L^p} denotes the usual Bochner Laplacian. By analogy with the complex case, we define the finite dimensional space $\mathscr{H}_p \subset \mathscr{C}^\infty(X, L^p)$ of *almost holomorphic sections* of L^p for any $p \in \mathbb{N}^*$ as the direct sum of the eigenspaces associated with the small eigenvalues of (3.1.2) (see Section 3.2.1).

In fact, consider the special case of J being integrable, making (X, J, ω) into a *Kähler manifold*, together with a holomorphic Hermitian line bundle (L, h^L) such that its *Chern connection* ∇^L (that is its unique Hermitian connection compatible with the holomorphic structure) satisfies (3.1.1). For any $p \in \mathbb{N}^*$, writing $\overline{\partial}_p$ for the holomorphic $\overline{\partial}$-operator on forms with values in L^p and $\overline{\partial}_p^*$ for its formal adjoint with respect to the L^2-Hermitian product, the *Bochner-Kodaira* formula tells us that the operator (3.1.2) is equal to $2\overline{\partial}_p^*\overline{\partial}_p$. Then by a result of [9, Th.1.1], this operator shows a *spectral gap*, so that the small eigenvalues are all equal to 0. The space $\mathscr{H}_p$ of almost holomorphic sections considered above reduces then to the space $H^0(X, L^p)$ of holomorphic sections of L^p in the Kähler case.

Given (L, h^L, ∇^L) over (X, J, ω) satisfying (3.1.1), the family $\{\mathscr{H}_p\}_{p\in\mathbb{N}^*}$ defined above is a natural generalization of its *holomorphic quantization*, where $p \in \mathbb{N}^*$ can be thought of as the inverse of the Planck constant and $\mathscr{H}_p$ is the associated space of quantum states. In this context, asymptotic results when p tends to infinity are supposed to describe the so-called *semi-classical limit*, when the scale gets so large that we recover the laws of classical mechanics as an approximation of the laws of quantum mechanics.

On the other hand, in the framework of geometric quantization associated with a regular Lagrangian fibration on X, the quantum states of X are represented by immersed Lagrangian submanifolds $\iota : \Lambda \hookrightarrow X$ satisfying a property called the *Bohr-Sommerfeld condition*, which asks for the existence of a non-vanishing section $\zeta \in \mathscr{C}^\infty(\Lambda, \iota^* L)$ parallel with respect to $\nabla^{\iota^* L}$ (see for example [56]). We call the data of (Λ, ι, ζ) a *Bohr-Sommerfeld Lagrangian*. The existence of a regular Lagrangian fibration on X being very restrictive, we consider in general singular Lagrangian fibrations, in which we allow the dimension of the fibres to drop on a finite union of submanifolds of positive codimension in X. Removing the condition $\dim \Lambda = n$, the immersed submanifold $\iota : \Lambda \hookrightarrow X$ is only isotropic, and we call the data of (Λ, ι, ζ) a *Bohr-Sommerfeld submanifold*. The typical case of a singular fibration is the case of *toric manifolds*, where X is endowed with an effective Hamiltonian action of $\mathbb{T}^n = (S^1)^n$ and the fibres are given by the orbits of this action. For a comparison between holomorphic and real quantization in this context, see for example [3].

In this chapter, we use the theory of the generalized Bergman kernel of Ma and Marinescu in [50] to study semi-classical properties of Bohr-Sommerfeld submanifolds in the context of the *almost holomorphic quantization* described above. Here, the quantization of a Bohr-Sommerfeld submanifold is represented by a sequence $\{s_p \in \mathscr{H}_p\}_{p\in\mathbb{N}^*}$, called an *isotropic state*, defined for any $p \in \mathbb{N}^*$ by the formula

$$s_p = \int_\Lambda P_p(x, \iota(y))\zeta^p(y) dv_\Lambda(y), \tag{3.1.3}$$

where dv_Λ is the Riemannian volume form of $(\Lambda, \iota^* g^{T\Lambda})$, $\zeta^p \in \mathscr{C}^\infty(\Lambda, \iota^* L^p)$ is the p-th tensor power of ζ and $P_p(\cdot,\cdot)$ is the *generalized Bergman kernel*, that is the Schwartz kernel with respect to dv_X of the orthogonal projection P_p from $\mathscr{C}^\infty(X, L^p)$ to $\mathscr{H}_p$ with respect to the natural L^2-Hermitian product. The expected behaviour of a quantum state in the semi-classical limit is to rapidly localize around the corresponding classical object, and we show in Proposition 3.3.5 that isotropic states indeed concentrate around the associated Bohr-Sommerfeld submanifold when p tends to infinity. Furthermore, we establish in Theorem 3.3.6 the following estimate on the norm of these sections, which is the first main result of this chapter and which we state here in its simplest form.

Theorem 3.1.1. *Let (Λ, ζ, ι) be a Bohr-Sommerfeld submanifold of dimension $d = \dim \Lambda$. Then there exist $b_r \in \mathbb{R}$, $r \in \mathbb{N}$, such that for any $k \in \mathbb{N}$ and as $p \to +\infty$,*

$$\|s_p\|_p^2 = p^{n-\frac{d}{2}} \sum_{r=0}^{k} p^{-r} b_r + O(p^{n-\frac{d}{2}-(k+1)}). \tag{3.1.4}$$

Furthermore, we have $b_0 = 2^{d/2}\operatorname{Vol}(\Lambda)$.

In Section 3.4, we study the L^2-Hermitian product $\langle\cdot,\cdot\rangle_p$ of two such sections for any $p \in \mathbb{N}^*$. We show that this product tends to 0 rapidly as p tends to infinity whenever the two associated submanifolds do not intersect, and we establish Theorem 3.4.4, which is the second main result of this chapter and which we state here in its simplest form.

Theorem 3.1.2. *Let* $(\Lambda_1, \iota_1, \zeta_1)$ *and* $(\Lambda_2, \iota_2, \zeta_2)$ *be two Bohr-Sommerfeld submanifolds with clean and connected intersection, and let* $\{s_{j,p}\}_{p\in\mathbb{N}^*}, j = 1, 2$, *denote the associated isotropic states. Set* $l = \dim \Lambda_1 \cap \Lambda_2$ *and* $d_j = \dim \Lambda_j,\ j = 1, 2$. *Then there exist* $b_r \in \mathbb{C},\ r \in \mathbb{N}$, *such that for any* $k \in \mathbb{N}$ *and as* $p \to +\infty$,

$$\langle s_{1,p}, s_{2,p}\rangle_p = p^{n-\frac{d_1+d_2}{2}+\frac{l}{2}}\lambda^p \sum_{r=0}^{k} p^{-r} b_r + O(p^{n-\frac{d_1+d_2}{2}+\frac{l}{2}-(k+1)}), \tag{3.1.5}$$

where $\lambda \in \mathbb{C}$ *is the value of the constant function on* $\Lambda_1\cap\Lambda_2$ *defined for any* $x \in \Lambda_1\cap\Lambda_2$ *by* $\lambda(x) = \langle\zeta_1(x), \zeta_2(x)\rangle_L$. *Furthermore, if* $\dim \Lambda_1 = n$, *the following formula holds,*

$$b_0 = 2^{n/2} \int_{\Lambda_1\cap\Lambda_2} \det{}^{-\frac{1}{2}} \Big\{ \sqrt{-1} \sum_{k=1}^{n-l} h^{TX}(e_k, \nu_i)\omega(e_k, \nu_j) \Big\}_{i,j=1}^{d_2-l} |dv|_{\Lambda_1\cap\Lambda_2}, \tag{3.1.6}$$

where $\langle e_i\rangle_{i=1}^{n-l}, \langle\nu_j\rangle_{j=1}^{d_2-l}$ *are local orthonormal frames of the normal bundles of* $\Lambda_1\cap\Lambda_2$ *in* Λ_1, Λ_2 *respectively, and* $|dv|_{\Lambda_1\cap\Lambda_2}$ *is the Riemannian density on* $\Lambda_1\cap\Lambda_2$ *induced by* g^{TX}.

Here the intersection of two immersed submanifolds is taken to be the fibred product over X of the immersions. We thus see that in the semi-classical limit, the Hermitian product of two isotropic states is closely related to the geometry of the intersection of the corresponding submanifolds. In the case of non-connected intersection, the expansion (3.1.5) takes the form of a sum over the connected components. The left hand side of (3.1.5) is called the *intersection product* of $s_{1,p}$ and $s_{2,p}$, and can be thought as the cup product of some Lagrangian intersection theory (see [59] for a discussion on this idea).

To give the most general formulation of Theorem 3.1.1 and Theorem 3.1.2, we use the theory of Berezin-Toeplitz operators for the generalized Bergman kernel on symplectic manifolds of [37], we consider any J-invariant Riemannian metric g^{TX} on TX and isotropic states taking values in an auxiliary Hermitian vector bundle (E, h^E) with Hermitian connection ∇^E. This is Theorem 3.3.6 and Theorem 3.4.4 respectively, in the case X smooth and compact.

In Section 3.5, we explain how the results of Section 3.3 extend to the case of (X, g^{TX}) complete non-compact orbifold, when the immersed isotropic submanifold Λ is compact and (X, J, ω) is Kähler. As an application to the case where X is the quotient of the Poincaré upper-half plane $\mathbb{H}$ by a discrete subgroup Γ of $SL_2(\mathbb{R})$, we derive in Section 3.6 asymptotic results on relative Poincaré series in the theory of automorphic forms.

In the case (X, J, ω) compact Kähler manifold with $c_1(TX)$ even, $E = \mathbb{C}$, $g^{TX}(\cdot,\cdot) = \omega(\cdot, J\cdot)$ and $\dim \Lambda_1 = \dim \Lambda_2 = n$, Theorem 3.4.4 is the main result of Borthwick, Paul and Uribe in [11, Th.3.2], with the expansion (3.1.5) given with half-integer powers of p instead of integer powers as in [11, (85)]. This is explained in Remark 3.4.5, where we translate their use of the formalism of half-forms by taking for E a square root of the canonical bundle of X. In the case where Γ acts freely on $\mathbb{H}$ and where $X = \mathbb{H}/\Gamma$ is compact, the application to relative Poincaré series in Section 3.6 is the result of [11, § 4]. In the case where (X, J, ω, g^{TX}) is additionally equipped with an Hamiltonian action of a compact Lie group lifting to (L, h^L, ∇^L), an equivariant version of the results of [11] has been obtained by Debernardi and Paoletti [21]. Semi-classical asymptotics on Lagrangian states have also been obtained by Charles in [17] in the case of discrete intersections and in the same particular context than in [11].

The theory of Berezin-Toeplitz operators was first developed by Bordemann, Meinreken and Schlichenmaier in [10] and Schlichenmaier in [54] for the Kähler case, $E = \mathbb{C}$ and $g^{TX}(\cdot, \cdot) = \omega(\cdot, J\cdot)$. The approach of both [10], [11], [17] and [21] is based on the work of Boutet de Monvel and Sjöstrand on the Szegö kernel in [15], and the theory of Toeplitz structures developed by Boutet de Monvel and Guillemin in [14]. Note that the definitions of Section 3.3.1 extend in a straightforward way to the case of the spin^c quantization considered in Chapter 1, and the results of Section 3.3 and Section 3.4 certainly hold in this case. If (X, J, ω, g^{TX}) is further endowed with an Hamiltonian action of a compact Lie group G lifting to $(L, h^L, \nabla^L), (E, h^E, \nabla^E)$ such that $0 \in \mathrm{Lie}(G)^*$ is a regular point of the associated moment map $\mu : X \to \mathrm{Lie}(G)^*$, and if $\iota : \Lambda \to X$ intersects $\mu^{-1}(0)$ cleanly in the sense of Definition 3.4.1, then one can use the full off-diagonal expansion of the G-invariant Bergman kernel of Ma and Zhang in [53, Th.0.2, Rem.0.3] to prove a result analogous to Theorem 3.3.6 for the G-invariant part of the associated isotropic state.

About relative Poincaré series, Barron (previously Foth) studied in [26] the case of Bohr-Sommerfeld tori in higher dimensional symmetric spaces. The results of Section 3.5 could be used to generalize [26, § 1.3] to the case of non-compact or orbifold symmetric spaces. In another direction, the results of Section 3.3.2 and Section 3.4.2 could be applied to study relative Poincaré series associated with isotropic submanifolds in higher dimensional symmetric spaces. The case of geodesics on some specific compact quotient of the ball has been studied by Barron in [5]. On the other hand, Alluhaibi and Barron in [1] studied the case of relative Poincaré series associated to submanifolds of the ball which are not necessarily isotropic.

A final motivation for this work is towards the program initiated by Witten in [60] in holomorphic quantization of Chern-Simons theory, showing an asymptotic expansion for Lagrangian states associated to some special Bohr-Sommerfeld Lagrangians inside the moduli space of flat connections on a Riemann surface, defined in [38, Prop.7.2] and [27, Prop.3.27]. Bohr-Sommerfeld Lagrangians in this context have also been studied by Tyurin in [59], and in the more general context of the Abelian Lagrangian Algebraic Geometry program of Gorodentsev and Tyurin [29]. In both cases, it is of particular importance to be able to consider orbifolds.

The results of this chapter appear in [36].

3.2 Generalized Bergman kernels on Symplectic Manifolds

In this section, we set the context and notations, and recall the results of [47], [50] and [37] we will need throughout the chapter. We refer to the book [49, Chap.4-8] as a basic reference for the theory.

3.2.1 Setting

Let (X, ω) be a compact symplectic manifold of dimension $2n$ with tangent bundle TX, and let J be an almost complex structure on TX compatible with ω. Take g^{TX} to be any J-invariant Riemannian metric on TX, and let ∇^{TX} be the associated Levi-Civita connection.

For any Euclidean vector bundle $(\mathscr{E}, g^{\mathscr{E}})$, we write $\mathscr{E}_{\mathbb{C}}$ for its complexification and still write $g^{\mathscr{E}}$ for the induced $\mathbb{C}$-bilinear product on $\mathscr{E}_{\mathbb{C}}$. Let us then write

$$TX_{\mathbb{C}} = T^{(1,0)}X \oplus T^{(0,1)}X \tag{3.2.1}$$

for the splitting of $TX_{\mathbb{C}}$ into the eigenspaces of J corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$ respectively. Then for any $x \in X$, $v, w \in T_x^{(1,0)}X$, we define the positive Hermitian endomorphism $\dot{R}_x^L \in \mathrm{End}(T_x^{(1,0)}X)$ by the formula

$$g^{TX}(\dot{R}_x^L v, \overline{w}) = R^L(v, \overline{w}). \tag{3.2.2}$$

We denote by $K_X = \det(T^{*(1,0)}X)$ the canonical line bundle of (X, J), endowed with the Hermitian structure and connection h^{K_X}, ∇^{K_X} induced by g^{TX}, ∇^{TX} via (3.2.1). We will consider as well the Riemannian metric g_ω^{TX} on TX defined by the formula

$$g_\omega^{TX}(\cdot,\cdot) = \omega(\cdot, J\cdot), \tag{3.2.3}$$

and the Hermitian metric h_ω^{TX} on (TX, J) defined by

$$h_\omega^{TX} = g_\omega^{TX} - \sqrt{-1}\omega. \tag{3.2.4}$$

Note that if $g^{TX} = g_\omega^{TX}$, then $\dot{R}^L = 2\pi \mathrm{Id}_{T^{(1,0)}X}$. For any submanifold $Y \subset X$, we will write g^{TY}, g_ω^{TY} for the Riemannian metrics on Y induced by g^{TX}, g_ω^{TX} and $dv_Y, dv_{Y,\omega}$ for the induced Riemannian volume forms. In particular, we have

$$dv_{X,\omega} = \det\left(\dot{R}^L/2\pi\right) dv_X. \tag{3.2.5}$$

Consider a Hermitian line bundle (L, h^L) over X, together with a Hermitian connection ∇^L satisfying (3.1.1), and let (E, h^E) be an auxiliary Hermitian vector bundle over X with Hermitian connection ∇^E and curvature R^E. For any $p \in \mathbb{N}^*$, we write

$$E_p = L^p \otimes E, \tag{3.2.6}$$

endowed with the Hermitan metric and connection h^{E_p}, ∇^{E_p} induced by h^L, h^E, ∇^L, ∇^E.

Definition 3.2.1. The *Bochner Laplacian* Δ^{E_p} is the second order differential operator acting on $\mathscr{C}^\infty(X, E_p)$ by the formula

$$\Delta^{E_p} = -\sum_{j=1}^{2n}\left[(\nabla^{E_p}_{e_j})^2 - \nabla^{E_p}_{\nabla^{TX}_{e_j}e_j}\right], \tag{3.2.7}$$

where $\{e_j\}_{j=1}^{2n}$ is any local orthonormal frame of TX with respect to g^{TX}.

For any $p \in \mathbb{N}^*$ and any Hermitian smooth section $\Phi \in \mathscr{C}^\infty(X, \mathrm{End}(E))$, the *renormalized Bochner Laplacian* $\Delta_{p,\Phi}$ is the second order differential operator acting on $\mathscr{C}^\infty(X, E_p)$ by the formula

$$\Delta_{p,\Phi} = \Delta^{E_p} - p\,\mathrm{Tr}[\dot{R}^L] + \Phi. \tag{3.2.8}$$

From now on, we fix $\Phi \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ and simply write Δ_p for the associated renormalized Bochner Laplacian. In the Kähler case, if $g^{TX} = g^{TX}_\omega$ and if Φ is equal to $-\sqrt{-1}R^E$ contracted with ω, we recover twice the Kodaira Laplacian of E_p. On the other hand, if $g^{TX} = g^{TX}_\omega$ and $E = \mathbb{C}$, we recover (3.1.2).

Let $\langle\cdot,\cdot\rangle_{E_p}$ denote the Hermitian product on E_p induced by h^L and h^E. The L^2-*Hermitian product* $\langle\cdot,\cdot\rangle_p$ on $\mathscr{C}^\infty(X, E_p)$ is given for any $s_1, s_2 \in \mathscr{C}^\infty(X, E_p)$ by the formula

$$\langle s_1, s_2\rangle_p = \int_X \langle s_1(x), s_2(x)\rangle_{E_p} dv_X(x). \tag{3.2.9}$$

Let $L^2(X, E_p)$ be the completion of $\mathscr{C}^\infty(X, E_p)$ with respect to $\langle\cdot,\cdot\rangle_p$. Then Δ_p is a self-adjoint second order differential operator on $L^2(X, E_p)$, and has discrete spectrum contained in $\mathbb{R}$. Furthermore, we have the following refinement of [31, Th.2.a].

Theorem 3.2.2. *[47, Cor.1.2] There exist $\widetilde{C},\ C > 0$ such that for all $p \in \mathbb{N}^*$,*

$$\mathrm{Spec}(\Delta_p) \subset\ [-\widetilde{C}, \widetilde{C}] \cup\ ]2\mu_0 p - C, +\infty[, \tag{3.2.10}$$

where $\mu_0 = \inf\limits_{x\in X, v\in T^{(1,0)}_x X} R^L_x(v, \overline{v})/g^{TX}_x(v, \overline{v})$.

For any $p \in \mathbb{N}^*$, we define the *space of almost holomorphic sections* $\mathscr{H}_p \subset L^2(X, E_p)$ of E_p as the direct sum of the eigenspaces of Δ_p associated with the eigenvalues in $[-\widetilde{C}, \widetilde{C}]$. Then by standard elliptic theory, we have $\mathscr{H}_p \subset \mathscr{C}^\infty(X, E_p)$ and $\dim \mathscr{H}_p < +\infty$. Actually, by [47, Cor.1.2], the dimension of $\mathscr{H}_p$ is computed by the Riemann-Roch-Hirzebruch formula, and is in particular a polynomial of degree n in p.

We write $\pi_j : X \times X \to X,\ j = 1, 2$, for the first and second projections on X. For any vector bundles E' and E'' over X, we define a vector bundle over $X \times X$ by the formula

$$E' \boxtimes E'' = \pi_1^* E' \otimes \pi_2^* E''. \tag{3.2.11}$$

The orthogonal projection $P_p : \mathscr{C}^\infty(X, E_p) \to \mathscr{H}_p$ with respect to (3.2.9) has smooth Schwartz kernel $P_p(\cdot,\cdot) \in \mathscr{C}^\infty(X \times X, E_p \boxtimes E_p^*)$ with respect to dv_X, defined for any $\zeta \in \mathscr{C}^\infty(X, E_p)$ and $x \in X$ by

$$(P_p s)(x) = \int_X P_p(x, y)s(y)dv_X(y). \tag{3.2.12}$$

For any $F \in \mathscr{C}^\infty(X, \mathrm{End}(E))$, we define the *Berezin-Toeplitz quantization* of F as the family $\{T_{F,p}\}_{p\in\mathbb{N}^*}$ of operators acting on $\mathscr{C}^\infty(X, E_p)$ for any $p \in \mathbb{N}^*$ by

$$T_{F,p} = P_p F P_p, \tag{3.2.13}$$

where F denotes the operator acting by pointwise multiplication by F. Then $T_{F,p}$ has smooth Schwartz kernel $T_p(\cdot,\cdot) \in \mathscr{C}^\infty(X \times X, E_p \boxtimes E_p^*)$, given for any $x, y \in X$ by

$$T_{F,p}(x, y) = \int_X P_p(x, w)F(w)P_p(w, y)dv_X(w). \tag{3.2.14}$$

For any $x_0 \in X$, we will write $\langle\cdot,\cdot\rangle_{x_0}$ and $|\cdot|_{x_0}$ for the Hermitian product and norm on E_{x_0} induced by h^E. For any $\sigma > 0$, we use the notation $O(p^{-\sigma})$ as $p \to +\infty$ in the usual sense with respect to $|\cdot|_{x_0}$ and uniformly in $x_0 \in X$. The notation $O(p^{-\infty})$ means $O(p^{-\sigma})$ for any $\sigma > 0$. Unless otherwise stated, we also use the convention to sum on free indices appearing twice in a single term.

3.2.2 Local model

Let $(u, v) := (u_1, \ldots, u_n, v_1, \ldots, v_n) \in \mathbb{R}^{2n}$ be the canonical symplectic coordinates associated with the standard symplectic form Ω on $\mathbb{R}^{2n}$ given by

$$\Omega = \sum_{j=1}^n du_j \wedge dv_j. \tag{3.2.15}$$

We write $\mathbb{R}^n \times \{0\} = \{(u, 0) \in \mathbb{R}^{2n} \mid u \in \mathbb{R}^n\}$ and $\{0\} \times \mathbb{R}^n = \{(0, v) \in \mathbb{R}^{2n} \mid v \in \mathbb{R}^n\}$ for the two canonical oriented Lagrangian subspaces of $(\mathbb{R}^{2n}, \Omega)$ and write $\langle\cdot,\cdot\rangle, |\cdot|$ for the canonical scalar product and norm of $\mathbb{R}^{2n}$. To match with the notations of [50], we will write $Z := (u, v) \in \mathbb{R}^{2n}$, and use the same notation for the radial vector field of $\mathbb{R}^{2n}$. For any $\varepsilon > 0$, we denote by $B^{\mathbb{R}^{2n}}(0, \varepsilon)$ the ball of center 0 and radius ε in $\mathbb{R}^{2n}$, and for any linear subspace $\Sigma \subset \mathbb{R}^{2n}$, we write $B^\Sigma(0, \varepsilon) := B^{\mathbb{R}^{2n}}(0, \varepsilon) \cap \Sigma$.

For any $m \in \mathbb{N}$, let $|\cdot|_{\mathscr{C}^m}$ denotes the $\mathscr{C}^m$-norm on $E_p \boxtimes E_p^*$ over $X \times X$ induced by h^L, h^E, ∇^L, ∇^E, and let $d^X(\cdot,\cdot)$ be the Riemannian distance on (X, g^{TX}).

Proposition 3.2.3. *[50, § 1.1] For any $m, k \in \mathbb{N}$, $\varepsilon > 0$ and $\theta \in\ ]0, 1[$, there is $C_{m,k,\theta,\varepsilon} > 0$ such that for all $p \in \mathbb{N}^*$ and $x, x' \in X$ satisfying $d^X(x, x') > \varepsilon p^{-\theta/2}$,*

$$|P_p(x, x')|_{\mathscr{C}^m} \leqslant C_{m,k,\theta,\varepsilon}\, p^{-k}. \tag{3.2.16}$$

Let us now take $x_0 \in X$, $\varepsilon_0 > 0$, $V \subset X$ open neighbourhood of x_0 and

$$\phi_{x_0} : B^{\mathbb{R}^{2n}}(0, \varepsilon_0) \subset \mathbb{R}^{2n} \to V \tag{3.2.17}$$

a diffeomorphism sending 0 to x_0, such that its differential at 0 identifies Ω and $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^{2n}$ with ω and g^{TX}_ω on $T_{x_0}X$. Let us make such a choice of diffeomorphisms (3.2.17) for any x_0 in a small open set, smoothly in x_0. We cover X with such open sets, and choose

$\varepsilon_0 > 0$ which does not depend on $x_0 \in X$. As two Riemannian metrics induce equivalent distances in a continuous way with respect to parameters, there exist $0 < a < b$ such that for any $x_0 \in X$ and $Z, Z' \in B^{\mathbb{R}^{2n}}(0, \varepsilon_0)$,

$$a|Z - Z'| < d^X(\phi_{x_0}(Z), \phi_{x_0}(Z')) < b|Z - Z'|. \tag{3.2.18}$$

Then by (3.2.18), we get the following corollary of Proposition 3.2.3.

Corollary 3.2.4. *For any $\varepsilon > 0$, $m, k \in \mathbb{N}$ and $\theta \in]0, 1[$, there is $C_{m,k,\theta,\varepsilon} > 0$ such that for all $x_0 \in X$, $p \in \mathbb{N}^*$ and $Z, Z' \in B^{\mathbb{R}^{2n}}(0, \varepsilon_0)$ such that $|Z - Z'| > \varepsilon p^{-\theta/2}$,*

$$|P_p(\phi_{x_0}(Z), \phi_{x_0}(Z'))|_{\mathscr{C}^m} \leqslant C_{m,k,\theta,\varepsilon'} p^{-k}. \tag{3.2.19}$$

We use the following explicit local model on $\mathbb{R}^{2n}$ for the Bergman kernel, as defined in [51, (3.25)] for any $Z, Z' \in \mathbb{R}^{2n}$,

$$\mathscr{P}_{x_0}(Z, Z') = \det\left(\dot{R}^L_{x_0}/2\pi\right) \exp\left(-\frac{\pi}{2}|Z - Z'|^2 - \pi\sqrt{-1}\Omega(Z, Z')\right). \tag{3.2.20}$$

Note that the difference of (3.2.20) with [51, (3.25)] comes from the fact that $\langle\cdot,\cdot\rangle$ is identified with g^{TX}_ω via (3.2.17) instead of g^{TX} via the exponential map as in [51, § 3.2].

Let dZ be the canonical Lebesgue measure of $\mathbb{R}^{2n}$, and let $\kappa_{x_0} \in \mathscr{C}^\infty(B^{\mathbb{R}^{2n}}(0, \varepsilon_0), \mathbb{R})$ be the smooth function satisfying, for any $Z \in B^{\mathbb{R}^{2n}}(0, \varepsilon_0)$ in the chart (3.2.17),

$$dv_{X,\omega}(Z) = \kappa_{x_0}(Z) dZ. \tag{3.2.21}$$

Then $\kappa_{x_0}(0) = 1$. In the chart (3.2.17), we identify E, L over $B^{\mathbb{R}^{2n}}(0, \varepsilon_0)$ with E_{x_0}, L_{x_0} through parallel transport with respect to ∇^E, ∇^L along radial lines of $B^{\mathbb{R}^{2n}}(0, \varepsilon_0)$. For any x_0 in a small open set, we identify L_{x_0} with $\mathbb{C}$ using any unit local frame of L.

For any $f \in \mathscr{C}^\infty(X, E)$, we write $f_{x_0} \in \mathscr{C}^\infty(B^{\mathbb{R}^{2n}}(0, \varepsilon_0), E_{x_0})$ for the restriction of f to $B^{\mathbb{R}^{2n}}(0, \varepsilon_0)$ in this trivialization. Similarly, for any $T_p(\cdot,\cdot) \in \mathscr{C}^\infty(X \times X, E_p^* \otimes E_p^*)$, we denote by $T_{p,x_0}(Z, Z') \in \operatorname{End}(E_{x_0})$ its image evaluated at $Z, Z' \in B^{\mathbb{R}^{2n}}(0, \varepsilon_0)$ in this trivialization. If $Q(Z, Z')$ is a polynomial in $Z, Z' \in \mathbb{R}^{2n}$, we write $Q\mathscr{P}_{x_0}(Z, Z') := Q(Z, Z')\mathscr{P}_{x_0}(Z, Z')$.

Recall that we chose a family of charts $\{\phi_{x_0}\}_{x_0 \in W}$ as in (3.2.17) smoothly in $x_0 \in W$, where W is a small open set of X. Then $P_{p,x_0}(Z, Z')$ can be seen as a smooth section of $\pi^* \operatorname{End}(E)$ over $W \times B^{\mathbb{R}^{2n}}(0, \varepsilon_0) \times B^{\mathbb{R}^{2n}}(0, \varepsilon_0)$ evaluated in $x_0 \in W$, $Z, Z' \in B^{\mathbb{R}^{2n}}(0, \varepsilon_0)$, where $\pi : W \times B^{\mathbb{R}^{2n}}(0, \varepsilon_0) \times B^{\mathbb{R}^{2n}}(0, \varepsilon_0) \to W$ is the first projection. Let us write $|\cdot|_{\mathscr{C}^m(X)}$ for the $\mathscr{C}^m$-norm on $\pi^* \operatorname{End}(E)$ induced by h^E and derivation by $\nabla^{\pi^* \operatorname{End}(E)}$ in the direction of $x_0 \in W$. We are now ready to state the following result, which was first proved in [20, Th.4.18'] in the case of the spin^c Dirac operator, and which in the following form comes essentially from [44, Th.2.1].

Lemma 3.2.5. *For any $m, k \in \mathbb{N}$, $\varepsilon > 0$ and $\delta \in]0, 1[$, there is $C > 0$ and $\theta \in]0, 1[$ such that for all $x_0 \in X$, $p \in \mathbb{N}^*$ and $|Z|, |Z'| < \varepsilon p^{-\theta/2}$,*

$$\left|p^{-n} P_{p,x_0}(Z, Z') - \sum_{r=0}^{k} p^{-r/2} J_{r,x_0} \mathscr{P}_{x_0}(\sqrt{p}Z, \sqrt{p}Z') \kappa_{x_0}^{-1/2}(Z) \kappa_{x_0}^{-1/2}(Z')\right|_{\mathscr{C}^m(X)} \leqslant C p^{-\frac{k+1}{2} + \delta}, \tag{3.2.22}$$

where $\{J_{r,x_0}(Z,Z')\}_{r\in\mathbb{N}}$ *is a family of polynomials in* $Z,Z'\in\mathbb{R}^{2n}$ *of the same parity as* r *and with values in* $\mathrm{End}(E_{x_0})$*, depending smoothly on* $x_0\in X$*. Furthermore, we have*

$$J_{0,x_0}(Z,Z')\equiv \mathrm{Id}_{E_{x_0}}. \tag{3.2.23}$$

Parallel to Proposition 3.2.3 and Lemma 3.2.5, we have the following result on the asymptotic expansion in $p\in\mathbb{N}^*$ of the Berezin-Toeplitz operator (3.2.13). It was first proved in [51, Lem.4.6] in the spin^c case, and in this form is a consequence of Lemma 2.3.3.

Lemma 3.2.6. *Let* $F\in\mathscr{C}^\infty(X,\mathrm{End}(E))$*. Then for any* $0<\varepsilon\leqslant\varepsilon_0$*,* $m,k\in\mathbb{N}$ *and* $\theta\in\,]0,1[$*, there is* $C_{m,k,\theta,\varepsilon}>0$ *such that for all* $x_0\in X$*,* $p\in\mathbb{N}^*$*,* $Z,Z'\in\mathbb{R}^{2n}$*,* $|Z-Z'|>\varepsilon p^{-\theta/2}$*,*

$$|T_{F,p}(\phi_{x_0}(Z),\phi_{x_0}(Z'))|_{\mathscr{C}^m}\leqslant C_{m,k,\theta,\varepsilon}p^{-k}. \tag{3.2.24}$$

Furthermore, for any $m,k\in\mathbb{N}$*,* $\varepsilon>0$ *and* $\delta\in]0,1[$*, there is* $C>0$ *and* $\theta\in]0,1[$ *such that for all* $x_0\in X$*,* $p\in\mathbb{N}^*$*,* $|Z|,|Z'|<\varepsilon p^{-\theta/2}$*,*

$$\Big|p^{-n}T_{F,p,x_0}(Z,Z')-\sum_{r=0}^{k}p^{-r/2}\mathscr{Q}_{r,x_0}\mathscr{P}_{x_0}(\sqrt{p}Z,\sqrt{p}Z')\kappa_{x_0}^{-1/2}(Z)\kappa_{x_0}^{-1/2}(Z')\Big|_{\mathscr{C}^m(X)}\leqslant Cp^{-\frac{k+1}{2}+\delta}, \tag{3.2.25}$$

where $\{\mathscr{Q}_{r,x_0}(Z,Z')\}_{r\in\mathbb{N}}$ *is a family of polynomials in* $Z,Z'\in\mathbb{R}^{2n}$ *of the same parity as* r *and with values in* $\mathrm{End}(E_{x_0})$*, depending smoothly on* $x_0\in X$*. Furthermore, we have*

$$\mathscr{Q}_{0,x_0}(Z,Z')\equiv F_{x_0}. \tag{3.2.26}$$

3.2.3 Gaussian integrals

We now recall some well-known facts about Gaussian integrals, which will be used for local computations in the next Sections. For any $k\in\mathbb{N}^*$, let $\langle\cdot,\cdot\rangle$ denote the canonical scalar product of $\mathbb{R}^k$. For any positive symmetric matrix C acting on $\mathbb{R}^k$, we recall the following classical formula for the Gaussian integral,

$$\int_{\mathbb{R}^k}\exp(-\pi\langle Z,CZ\rangle)dZ=\det{}^{-\frac{1}{2}}C. \tag{3.2.27}$$

By analytic continuation, this formula is still valid when C is a symmetric matrix with complex coefficients, providing the integral is well defined along a path in the space of symmetric matrices joining C with a real positive symmetric matrix. Specifically, for A positive symmetric matrix and B real symmetric matrix, we will consider the path

$$\begin{aligned}\gamma:[0,1]&\to \mathrm{GL}_k(\mathbb{C})\\ t&\mapsto A+t\sqrt{-1}B.\end{aligned} \tag{3.2.28}$$

Then (3.2.27) holds for $C=A+\sqrt{-1}B$, with the determination of the square root given by continuation along the image of (3.2.28) by the application $\det^{-1}:GL_n(\mathbb{C})\to\mathbb{C}$. Henceforth, we will always use this determination of the square root of the determinant for $C=A+\sqrt{-1}B$ as above.

3.3 Isotropic states

Through all this section, we use the context and notations of Section 3.2. In particular, recall that (X, ω) is a compact symplectic manifold of dimension $2n$, and that the curvature of ∇^L on (L, h^L) over X satisfies (3.1.1).

3.3.1 Bohr-Sommerfeld submanifolds

An immersed submanifold $\iota : \Lambda \to X$ is said to be *isotropic* if $\iota^*\omega = 0$. If in addition $\dim \Lambda = n$, it is said to be *Lagrangian*. We write $\nabla^{\iota^*L}, |\cdot|_{\iota^*L}$ for the connection and norm induced by ∇^L, h^L on the pullback line bundle ι^*L over Λ. Note that by (3.1.1), the condition $\iota^*\omega = 0$ implies that ∇^{ι^*L} is *flat*. This observation motivates the following definition.

Definition 3.3.1. A properly immersed oriented isotropic submanifold $\iota : \Lambda \to X$ is said to satisfy the *Bohr-Sommerfeld condition* if there exists a non-vanishing smooth section $\zeta \in \mathscr{C}^\infty(\Lambda, \iota^*L)$ satisfying

$$\nabla^{\iota^*L}\zeta = 0. \tag{3.3.1}$$

Taking ζ satisfying further $|\zeta(x)|_{\iota^*L} = 1$ for any $x \in \Lambda$, the data of (Λ, ι, ζ) is called a *Bohr-Sommerfeld submanifold* of X, or a *Bohr-Sommerfeld Lagrangian* if in addition $\dim \Lambda = n$.

Note that this definition depends only on the symplectic structure on (X, ω) and the prequantization condition (3.1.1) on (L, h^L, ∇^L). Furthermore, as ∇^L is hermitian, up to renormalisation we can always assume that $\zeta \in \mathscr{C}^\infty(\Lambda, \iota^*L)$ satisfying (3.3.1) is such that $|\zeta(x)|_{\iota^*L} = 1$ for any $x \in \Lambda$. Finally, from the compactness of X, the properness hypothesis on ι is equivalent to the compactness of Λ.

Remark 3.3.2. As noted above, if $\iota : \Lambda \to X$ is isotropic, then ∇^{ι^*L} is flat over Λ, hence determined by its holonomy $\mathrm{hol}_{\iota^*L} : \pi_1(\Lambda) \to S^1 \subset \mathbb{C}$. We can then reformulate (3.3.1) by saying that $\iota : \Lambda \to X$ satisfies the Bohr-Sommerfeld condition if and only if $\mathrm{hol}_{\iota^*L} = \{1\}$. Now if the order of hol_{ι^*L} is finite, then there exists a finite covering $j : \hat{\Lambda} \to \Lambda$ such that $\mathrm{hol}_{j^*\iota^*L} = \{1\}$, so that $\iota \circ j : \hat{\Lambda} \to X$ satisfies the Bohr-Sommerfeld condition. In particular, if there is $k \in \mathbb{N}$ such that $\iota : \Lambda \to X$ satisfies the Bohr-Sommerfeld condition for L^k instead of L, then the order of hol_{ι^*L} divides k, thus is finite. Such a $\iota : \Lambda \to X$ is called a *Bohr-Sommerfeld submanifold of order* k, and up to finite covering, Definition 3.3.1 also accounts for these. In the same line of thought, if $\iota : \Lambda \to X$ is not orientable, we can always work on the orientation double cover of Λ.

Let us now set some notations. We write ι^L, ι^E and ι_p for the natural maps covering $\iota : \Lambda \to X$ on the respective total spaces of L, E and E_p for any $p \in \mathbb{N}^*$. If ζ is any section of ι^*L, we write ζ^p for the p-th power of ζ defined as a section of ι^*L^p. If additionally f is a section of ι^*E, we write $\zeta^p f$ for the induced tensor product in ι^*E_p.

From now on, we fix an almost complex structure J on TX compatible with ω and an auxiliary Hermitian vector bundle (E, h^E) with Hermitian connection ∇^E.

Definition 3.3.3. The *isotropic state* associated to (Λ, ι, ζ) and $f \in \mathscr{C}^\infty(\Lambda, \iota^*E)$ is the family of sections $\{s_{f,p} \in \mathscr{H}_p\}_{p \in \mathbb{N}^*}$ defined for any $x \in X$ by the formula

$$s_{f,p}(x) = \int_\Lambda P_p(x, \iota(y))\iota_p.\zeta^p f(y) dv_\Lambda(y). \tag{3.3.2}$$

As ι is locally an embedding, when working locally we will often omit the mention of ι, considering locally Λ as a submanifold of X. With this convention, equation (3.3.2) writes

$$s_{f,p}(x) = \int_\Lambda P_p(x, y)\zeta^p f(y) dv_\Lambda(y). \tag{3.3.3}$$

We list the basic properties of isotropic states in the following proposition, which holds for any $p \in \mathbb{N}^*$.

Proposition 3.3.4. *For any $f_1, f_2 \in \mathscr{C}^\infty(\Lambda, \iota^*E)$, we have the following additivity property,*

$$s_{f_1+f_2,p} = s_{f_1,p} + s_{f_2,p}. \tag{3.3.4}$$

For any $s \in \mathscr{H}_p$, we have the following reproducing property,

$$\langle s, s_{f,p}\rangle_p = \int_\Lambda \langle s(\iota(x)), \iota_p.\zeta^p f(x)\rangle_{E_p} \, dv_\Lambda(x). \tag{3.3.5}$$

For any $f \in \mathscr{C}^\infty(\Lambda, \iota^*E)$ and any $F \in \mathscr{C}^\infty(X, \mathrm{End}(E))$, the action of $T_{F,p}$ on $s_{f,p}$ is given for any $x \in X$ by the formula

$$T_{F,p}s_{f,p} = \int_\Lambda T_{F,p}(x, \iota(y))\iota_p.\zeta^p f(y) dv_\Lambda(y). \tag{3.3.6}$$

Proof. First, the additivity property (3.3.4) is obvious from (3.3.2). Next, recall that P_p is self-adjoint with respect to $\langle\cdot,\cdot\rangle_p$ for any $p \in \mathbb{N}^*$, and restricts to the identity on $\mathscr{H}_p$. Then using (3.2.12), (3.3.3) and Fubini, we compute for any $s \in \mathscr{H}_p$,

$$\begin{aligned}\langle s, s_{f,p}\rangle_p &= \int_X \left\langle s(y), \int_\Lambda P_p(y, \iota(x))\iota_p.\zeta^p f(x) dv_\Lambda(x)\right\rangle_{E_p} dv_X(y)\\ &= \int_\Lambda \left\langle \int_X P_p(\iota(x), y)s(y) dv_X(y), \iota_p.\zeta^p f(x)\right\rangle_{E_p} dv_\Lambda(x)\\ &= \int_\Lambda \langle s(\iota(x)), \iota_p.\zeta^p f(x)\rangle_{E_p} \, dv_\Lambda(x).\end{aligned} \tag{3.3.7}$$

The reproducing property (3.3.5) follows from (3.3.7). Finally, from (3.2.13), we get for any $f \in \mathscr{C}^\infty(\Lambda, \iota^*E)$ and $F \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ that $T_{F,p}s_{f,p} = P_p F s_{f,p}$. Then by (3.2.14), (3.3.2) and using Fubini, we get for any $x \in X$,

$$\begin{aligned}(T_{F,p}s_{f,p})(x) &= \int_X \int_\Lambda P_p(x, w)F(w)P_p(w, \iota(y))\iota_p.\zeta^p f(y) dv_\Lambda(y) dv_X(w)\\ &= \int_\Lambda T_{F,p}(w, \iota(y))\iota_p.\zeta^p f(y) dv_\Lambda(y).\end{aligned} \tag{3.3.8}$$

From (3.3.8), we get (3.3.6). ∎

3.3.2 Asymptotic expansion of isotropic states

In this section, we establish the first semi-classical properties of isotropic states. In particular, we show that the L^2-norm of an isotropic state admits an asymptotic expansion in $p \in \mathbb{N}^*$, and we compute the highest order term.

For any $p \in \mathbb{N}^*$, we write $|\cdot|_{E_p}$ for the norm on E_p induced by h^L and h^E. We show in the following proposition how an isotropic state concentrates around the image of the associated isotropic submanifold as p tends to infinity.

Proposition 3.3.5. *Let $f \in \mathscr{C}^\infty(\Lambda, \iota^*E)$. For any closed subset $K \subset X$ such that $K \cap \iota(\Lambda) = \emptyset$ and for any $k \in \mathbb{N}$, there exists $C_k > 0$ such that for any $x \in K$ and all $p \in \mathbb{N}^*$,*

$$|s_{f,p}(x)|_{E_p} < C_k p^{-k}. \tag{3.3.9}$$

Proof. This is a direct consequence of Proposition 3.2.3 and formula (3.3.2). □

For any $p \in \mathbb{N}^*$, we denote by $\|\cdot\|_p$ the norm on $\mathscr{C}^\infty(X, E_p)$ induced by $\langle\cdot,\cdot\rangle_p$, and by $|\cdot|_{\iota^*E}$ the norm on ι^*E over Λ induced by h^E. The rest of the section is dedicated to the proof of the following theorem.

Theorem 3.3.6. *Let $f \in \mathscr{C}^\infty(\Lambda, \iota^*E)$, and set $d = \dim \Lambda$. Then there exist $b_r \in \mathbb{R}$, $r \in \mathbb{N}$, such that for any $k \in \mathbb{N}$ and as $p \to +\infty$,*

$$\|s_{f,p}\|_p^2 = p^{n-\frac{d}{2}} \sum_{r=0}^{k} p^{-r} b_r + O(p^{n-\frac{d}{2}-(k+1)}). \tag{3.3.10}$$

Furthermore, we have

$$b_0 = 2^{d/2} \int_\Lambda |f|^2_{\iota^*E} \det\left(\dot{R}^L/2\pi\right) \frac{dv_\Lambda}{dv_{\Lambda,\omega}} dv_\Lambda. \tag{3.3.11}$$

In particular, if $\dim \Lambda = n$, then $b_0 = 2^{n/2} \int_\Lambda |f|^2_{\iota^*E} dv_{\Lambda,\omega}$.

Additionally, for any $F \in \mathscr{C}^\infty(X, \mathrm{End}(E))$, the product $\langle T_{F,p} s_{f,p}, s_{f,p}\rangle_p$ satisfies the expansion of (3.3.10) with $b_r \in \mathbb{C}$, $r \in \mathbb{N}$, and

$$b_0 = 2^{d/2} \int_\Lambda \langle Ff, f\rangle_{\iota^*E} \det\left(\dot{R}^L/2\pi\right) \frac{dv_\Lambda}{dv_{\Lambda,\omega}} dv_\Lambda. \tag{3.3.12}$$

Proof. Note first that the reproducing property (3.3.5) gives

$$\|s_{f,p}\|_p^2 = \int_\Lambda \langle s_{f,p}(\iota(x)), \zeta^p f(x)\rangle_{E_p} dv_\Lambda(x). \tag{3.3.13}$$

Using (3.3.13), we are reduced to evaluate $s_{f,p}$ on the image of $\iota : \Lambda \to X$. Let then $x_0 \in X$ be in the image of ι. As $\iota : \Lambda \to X$ is an immersion, there is an integer $m \in \mathbb{N}$ such that for any small enough connected neighbourhood V of x_0 in X, there are m disjoint connected open sets $U_1, \dots, U_m \subset \Lambda$ such that $\iota^{-1}(V) = \cup_{j=1}^m U_j$. Using Proposition 3.2.3, we can then localize the problem in the following way,

$$\begin{aligned} s_{f,p}(x_0) &= \int_\Lambda P_p(x_0, \iota(x))\zeta^p f(x) dv_\Lambda(x) \\ &= \sum_{j=1}^m \int_{U_j} P_p(x_0, \iota(x))\zeta^p f(x) dv_\Lambda(x) + O(p^{-\infty}). \end{aligned} \tag{3.3.14}$$

In view of (3.3.10), (3.3.13) and (3.3.14), we can assume that f has compact support around $\cup_{j=1}^m U_j$. Using (3.3.4) and (3.3.13), we are reduced further to the case where f has compact support around one of the U_j for some j. As $U := U_j$ is embedded in X through ι, we can consider U as a submanifold of X, and (3.3.14) translates to

$$s_{f,p}(x_0) = \int_U P_p(x_0, x)\zeta^p f(x) dv_\Lambda(x) + O(p^{-\infty}). \tag{3.3.15}$$

Take $\varepsilon > 0$, $V \subset X$ and $\phi_{x_0} : B^{\mathbb{R}^{2n}}(0, \varepsilon) \to V$ as in (3.2.17), identifying $U \subset V$ with $B^\Sigma(0, \varepsilon)$, where Σ is a vector subspace of $\mathbb{R}^{2n}$. Then Σ is an isotropic subspace of $(\mathbb{R}^{2n}, \Omega)$. We identify E, L over $B^{\mathbb{R}^{2n}}(0, \varepsilon)$ with E_{x_0}, L_{x_0} as in Section 3.2.2. In particular, we use the unitary vector $\zeta(x_0)$ to identify L_{x_0} with $\mathbb{C}$, where $\zeta \in \mathscr{C}^\infty(\Lambda, \iota^* L)$ is the section associated to (Λ, ι, ζ) as in Definition 3.3.1. As ζ is parallel with respect to $\nabla^{\iota^* L}$ along Λ, it is identified with $1 \in \mathbb{C}$ over $B^\Sigma(0, \varepsilon)$ in this trivialization.

Let du be the Lebesgue measure of Σ, and define the function $h \in \mathscr{C}^\infty(B^\Sigma(0, \varepsilon), \mathbb{R})$ for all $u \in B^\Sigma(0, \varepsilon)$ by

$$dv_\Lambda(u) = h(u) du. \tag{3.3.16}$$

Then $h(0) = (dv_\Lambda / dv_{\Lambda,\omega})(x_0)$. Using Corollary 3.2.4 and Lemma 3.2.5, for any $\delta \in]0, 1[$, we get $\theta \in]0, 1[$ such that

$$\begin{aligned} &\langle s_{f,p}(x_0), \zeta^p f(x_0)\rangle_{E_p} \\ &\quad = \int_{B^\Sigma(0, \varepsilon p^{-\theta/2})} \langle P_p(x_0, \phi_{x_0}(u))\zeta^p f(\phi_{x_0}(u)), \zeta^p f(x_0)\rangle_{E_p} dv_\Lambda(u) + O(p^{-\infty}) \\ &\quad = p^n \int_{B^\Sigma(0, \varepsilon p^{-\theta/2})} \sum_{r=0}^k p^{-r/2} \langle J_{r,x_0} \mathscr{P}_{x_0}(0, \sqrt{p}u) f_{x_0}(u), f(x_0)\rangle_{x_0} \kappa_{x_0}^{-1/2}(u) dv_\Lambda(u) \\ &\qquad + p^n \int_{B^\Sigma(0, \varepsilon p^{-\theta/2})} O(p^{-\frac{k+1}{2}+\delta}) dv_\Lambda(u) + O(p^{-\infty}) \\ &\quad = p^n \int_{B^\Sigma(0, \varepsilon p^{-\theta/2})} \sum_{r=0}^k p^{-r/2} \langle J_{r,x_0} \mathscr{P}_{x_0}(0, \sqrt{p}u) f_{x_0}(u), f(x_0)\rangle_{x_0} \kappa_{x_0}^{-1/2}(u) h(u) du \\ &\qquad + p^n p^{-\frac{d\theta}{2}} O(p^{-\frac{k+1}{2}+\delta}). \end{aligned} \tag{3.3.17}$$

Let us write $g_{x_0} = h\kappa_{x_0}^{1/2} f_{x_0} \in \mathscr{C}^\infty(B^\Sigma(0, \varepsilon), E_{x_0})$. Then from (3.2.21) and (3.3.16), we get the following Taylor expansion in $u \in \mathbb{R}^n$ up to order $k \in \mathbb{N}$,

$$\begin{aligned} g_{x_0}(u) &= f(x_0)(dv_\Lambda/dv_{\Lambda,\omega})(x_0) + \sum_{1 \leqslant |\alpha| \leqslant k} \frac{\partial^{|\alpha|} g_{x_0}}{\partial u^\alpha} \frac{u^\alpha}{\alpha!} + O(|u|^{k+1}) \\ &= f(x_0)(dv_\Lambda/dv_{\Lambda,\omega})(x_0) + \sum_{1 \leqslant |\alpha| \leqslant k} p^{-\alpha/2} \frac{\partial^{|\alpha|} g_{x_0}}{\partial u^\alpha} \frac{(\sqrt{p}u)^\alpha}{\alpha!} + p^{-\frac{k+1}{2}} O(|\sqrt{p}u|^{k+1}). \end{aligned} \tag{3.3.18}$$

On another hand, recall from Lemma 3.2.5 that $J_{r,x_0}(0, \sqrt{p}u) \in \mathrm{End}(E_{x_0})$ is polynomial in $\sqrt{p}u$ of the same parity as $r \in \mathbb{N}$. Let M_k be the supremum of the degree of J_{r,x_0} for all $1 \leqslant r \leqslant k$, and write $\delta' = \delta + (M_k + k + 1 + d)(1-\theta)/2$. We deduce from (3.3.17) and (3.3.18) the existence of a sequence $\{G_r\}_{r \in \mathbb{N}}$ of polynomials in one variable of $\mathbb{R}^n$ of the same parity as r, with values in $\mathbb{C}$, and with $G_0(Z, Z') = |f(x_0)|_{x_0}(dv_\Lambda/dv_{\Lambda,\omega})(x_0)$, such that

$$\begin{aligned} \langle s_{f,p}(x_0), &\zeta^p f(x_0)\rangle_{E_p} \\ &= p^n \sum_{r=0}^{k} p^{-r/2} \int_{B^\Sigma(0,\varepsilon p^{-\theta/2})} G_r(\sqrt{p}u) \mathscr{P}_{x_0}(0, \sqrt{p}u) du + O(p^{n - \frac{d+k+1}{2} + \delta'}) \\ &= p^{n-\frac{d}{2}} \sum_{r=0}^{k} p^{-r/2} \int_{B^\Sigma(0,\varepsilon p^{(1-\theta)/2})} G_r(u) \mathscr{P}_{x_0}(0, u) du + O(p^{n - \frac{d+k+1}{2} + \delta'}). \end{aligned} \tag{3.3.19}$$

Recall from (3.2.20) that

$$\mathscr{P}_{x_0}(0, u) = \det\left(\dot{R}^L_{x_0}/2\pi\right) \exp\left(-\frac{\pi}{2}|u|^2\right). \tag{3.3.20}$$

Thus as $1 - \theta > 0$, we deduce from (3.3.20) that for any $l \in \mathbb{N}$, there is $C_l > 0$ such that for any $u \in \Sigma$ outside $B^\Sigma(0, \varepsilon p^{(1-\theta)/2})$,

$$\mathscr{P}_{x_0}(0, u) \leqslant C_l p^{-l}. \tag{3.3.21}$$

We then deduce from (3.3.19) and (3.3.21) that

$$\langle s_{f,p}(x_0), \zeta^p f(x_0)\rangle_{E_p} = p^{n-\frac{d}{2}} \sum_{r=0}^{k} p^{-r/2} \int_\Sigma G_r(u) \mathscr{P}_{x_0}(0, u) du + O(p^{n - \frac{d+k+1}{2} + \delta'}). \tag{3.3.22}$$

As G_r is of the same parity as r, we immediately deduce from (3.3.20) that for any $m \in \mathbb{N}$,

$$\int_\Sigma G_{2m+1}(u) \mathscr{P}_{x_0}(0, u) du = 0. \tag{3.3.23}$$

Finally, as $G_0(Z, Z') = |f(x_0)|_{x_0}$, we get from (3.3.20) the following formula for the highest order term of (3.3.22),

$$\begin{aligned} \int_\Sigma \mathscr{P}_{x_0}(0, u) &|f(x_0)|_{x_0} (dv_\Lambda/dv_{\Lambda,\omega})(x_0) du \\ &= \det\left(\dot{R}^L_{x_0}/2\pi\right) |f(x_0)|_{x_0} (dv_\Lambda/dv_{\Lambda,\omega})(x_0) \int_\Sigma \exp(-\frac{\pi}{2}|u|^2) du \\ &= 2^{d/2} \det\left(\dot{R}^L_{x_0}/2\pi\right) |f(x_0)|_{x_0} (dv_\Lambda/dv_{\Lambda,\omega})(x_0). \end{aligned} \tag{3.3.24}$$

Then recalling that all the estimates above are uniform in $x_0 \in X$, and by (3.2.5), (3.3.13) and (3.3.23), it suffices to integrate (3.3.22) and (3.3.24) over $x_0 \in \Lambda$ with respect to dv_Λ to get (3.3.10) and (3.3.11). Now if Λ is Lagrangian, we know that

$$dv_{\Lambda,\omega} = \det{}^{1/2}\left(\dot{R}^L/2\pi\right) dv_\Lambda. \tag{3.3.25}$$

Using Lemma 3.2.6 and (3.3.6), the computation of $\langle T_{F,p}s_{f,p}, s_{f,p}\rangle_p$ is completely analogous to the one above. This achieves the proof of Theorem 3.3.6. □

3.4 Isotropic intersections

Let us consider two Bohr-Sommerfeld submanifolds $(\Lambda_j, \iota_j, \zeta_j)$ together with $f_j \in \mathscr{C}^\infty(\Lambda_j, \iota_j^*E)$, for $j = 1, 2$, and set $d_j = \dim \Lambda_j$. In this section, we establish the existence of an asymptotic expansion in $p \in \mathbb{N}^*$ of the Hermitian product $\langle s_{f_1,p}, s_{f_2,p}\rangle_p$ of the two associated isotropic states, and we compute the highest order term, which depends only on the geometry of the intersection. Note that the case $\{s_{f_1,p}\}_{p\in\mathbb{N}^*} = \{s_{f_2,p}\}_{p\in\mathbb{N}^*}$ is precisely the result of Theorem 3.3.6.

We will use the following definition of the intersection of immersions, which requires a natural regularity assumption, assumed throughout the section.

Definition 3.4.1. We say that two proper immersions $\iota_j : \Lambda_j \to X$, $j = 1, 2$ are intersecting *cleanly* if for any $x \in \iota_1(\Lambda_1) \cap \iota_2(\Lambda_2)$, any $y_j \in \Lambda_j$ such that $\iota_1(y_1) = \iota_2(y_2) = x$ and any small enough neighbourhoods $U_j \subset \Lambda_j$ of y_j, their intersection $\iota_1(U_1) \cap \iota_2(U_2)$ is a submanifold of X satisfying $T_x\iota_1(U_1) \cap T_x\iota_2(U_2) = T_x(\iota_1(U_1) \cap \iota_2(U_2))$.

In that case, we define their *intersection* as the fibred product of $\iota_1 : \Lambda_1 \to X$ and $\iota_2 : \Lambda_2 \to X$ over X, which is given by the data of a manifold $\Lambda_1 \cap \Lambda_2$ together with two immersion $j_i : \Lambda_1 \cap \Lambda_2 \to \Lambda_i$, $i = 1, 2$, such that $\iota_1 \circ j_1 = \iota_2 \circ j_2$ and universal for this property.

3.4.1 Asymptotic expansion of discrete intersections

In this section, we deal with the case of discrete intersection. We consider first the easy case when the intersection is empty.

Proposition 3.4.2. *Suppose that $\Lambda_1 \cap \Lambda_2 = \emptyset$, and let $F \in \mathscr{C}^\infty(X, \mathrm{End}(E))$. Then for any $k \in \mathbb{N}$, there exists $C_k > 0$ such that for all $p \in \mathbb{N}^*$,*

$$|\langle T_{F,p}s_{f_1,p}, s_{f_2,p}\rangle_p| < C_k p^{-k}. \tag{3.4.1}$$

Proof. Using the reproducing property (3.3.5), we get for any $p \in \mathbb{N}^*$,

$$\langle T_{F,p}s_{f_1,p}, s_{f_2,p}\rangle_p = \int_\Lambda \langle T_{F,p}s_{f_1,p}(\iota_2(x)), \zeta_2^p f_2(x)\rangle_{E_p} dv_{\Lambda_2}(x). \tag{3.4.2}$$

In particular, choosing $K = \iota_2(\Lambda_2)$ in Proposition 3.3.5, we deduce (3.4.1) from (3.4.2). □

In view of Proposition 3.4.2, we will assume from now on that $\Lambda_1 \cap \Lambda_2$ is not empty. In the statement of the following theorem, the immersions $\iota_i : \Lambda_i \to X$ and $j_i : \Lambda_1 \cap \Lambda_2 \to \Lambda_i,\ i = 1, 2$, are implicit, and we omit to mention them for simplicity.

Theorem 3.4.3. *Suppose that $(\Lambda_1, \iota_1, \zeta_1)$ and $(\Lambda_2, \iota_2, \zeta_2)$ intersect cleanly, and that their intersection $\Lambda_1 \cap \Lambda_2$ in the sense above is discrete. Set $m = \#\,\Lambda_1 \cap \Lambda_2$ and write $\Lambda_1 \cap \Lambda_2 = \{x_1, \dots, x_m\}$. Then for any $F \in \mathscr{C}^\infty(X, \mathrm{End}(E))$, there exist $b_{q,r} \in \mathbb{C},\ r \in \mathbb{N},\ 1 \leqslant q \leqslant m$, such that for any $k \in \mathbb{N}$ and as $p \to +\infty$,*

$$\langle T_{F,p} s_{f_1,p}, s_{f_2,p} \rangle_p = p^{n - \frac{d_1+d_2}{2}} \sum_{q=1}^{m} \lambda_q^p \sum_{r=0}^{k} p^{-r} b_{q,r} + O(p^{n - \frac{d_1+d_2}{2} - (k+1)}), \tag{3.4.3}$$

where $\lambda_q = \langle \zeta_1(x_q), \zeta_2(x_q) \rangle_L$. Furthermore, if $\dim \Lambda_1 = n$, we have

$$\begin{aligned} b_{q,0} = 2^{n/2} \langle F_{x_q} f_1(x_q), f_2(x_q) \rangle_{x_q} \det{}^{1/2} \left(\dot{R}^L_{x_q} / 2\pi \right) \frac{dv_{\Lambda_2}}{dv_{\Lambda_2,\omega}}(x_q) \\ \det{}^{-\frac{1}{2}} \left\{ \sqrt{-1} \sum_{k=1}^{n} h^{TX}_\omega(e_k, \nu_i) \omega(e_k, \nu_j) \right\}_{i,j=1}^{d_2}, \end{aligned} \tag{3.4.4}$$

where $\langle e_i \rangle_{i=1}^n, \langle \nu_j \rangle_{j=1}^{d_2}$ are oriented orthonormal bases for g^{TX}_ω of the tangent spaces of Λ_1, Λ_2 in X at x_q, and the square root of the determinant is determined by (3.2.28).

Proof. We will prove Theorem 3.4.3 for $F = \mathrm{Id}_E$ (so that $T_{F,p} = P_p$), the proof of the general case being totally analogous by Lemma 3.2.6 and (3.3.6). First, using the reproducing property (3.3.5), we get for any $p \in \mathbb{N}^*$,

$$\langle s_{f_1,p}, s_{f_2,p} \rangle_p = \int_\Lambda \langle s_{f_1,p}(\iota_2(x)), \zeta_2^p f_2(x) \rangle_{E_p} dv_{\Lambda_2}(x). \tag{3.4.5}$$

We can then reproduce the argument in the proof of Proposition 3.4.2 using Proposition 3.3.5 to reduce the proof to the case of f_2 with compact support in any neighbourhood of $\iota_2^{-1}(\iota_1(\Lambda_1) \cap \iota_2(\Lambda_2)) = j_2(\Lambda_1 \cap \Lambda_2)$, which is a finite set by assumption. Symmetrically, using the reproducing property of $s_{f_1,p}$ instead of $s_{f_2,p}$, we can assume further that f_1 has compact support in any neighbourhood of $\iota_1^{-1}(\iota_1(\Lambda_1) \cap \iota_2(\Lambda_2)) = j_1(\Lambda_1 \cap \Lambda_2)$. By (3.3.4) and (3.4.3), we are further reduced to the case of f_i with compact support in a neighborhood of only one point $y_i \in j_i(\Lambda_1 \cap \Lambda_2)$ for any $i = 1, 2$. Using Proposition 3.3.5, we are finally reduced to the case $\iota_1(y_1) = \iota_2(y_2)$, or equivalently, of $j_i(x_q) = y_i$ for any $i = 1, 2,\ x_q \in \Lambda_1 \cap \Lambda_2,\ 1 \leqslant q \leqslant m$. Set $\iota_1(y_1) = \iota_2(y_2) =: x_0 \in X$.

Let U_j be so small that it is embedded in X by ι_j for any $j = 1, 2$, so that we can consider them as submanifolds of X intersecting cleanly at $x_0 \in X$ only. In particular, using (3.3.3), equation (3.4.5) becomes

$$\begin{aligned} \langle s_{1,p}, s_{2,p} \rangle_p &= \int_{U_2} \langle s_{f_1,p}(x), \zeta_2^p f_2(x) \rangle_{E_p} dv_{\Lambda_2}(x) + O(p^{-\infty}) \\ &= \int_{U_2} \int_{U_1} \langle P_p(x,y) \zeta_1^p f_1(y), \zeta_2^p f_2(x) \rangle_{E_p} dv_{\Lambda_1}(y) dv_{\Lambda_2}(x) + O(p^{-\infty}). \end{aligned} \tag{3.4.6}$$

Take $\varepsilon > 0$, $V \subset X$ such that $V \cup \Lambda_j = U_j$ and $\phi_{x_0} : B^{\mathbb{R}^{2n}}(0,\varepsilon) \to V$ as in (3.2.17), identifying U_j with $B^{\Sigma_j}(0,\varepsilon)$ for any $j = 1,2$, where Σ_1 and Σ_2 are isotropic subspaces of $(\mathbb{R}^{2n},\Omega)$. As U_1 and U_2 intersect cleanly at x_0 only, we have $\Sigma_1 \cap \Sigma_2 = \{0\}$. We identify E, L over $B^{\mathbb{R}^{2n}}(0,\varepsilon)$ with E_{x_0}, L_{x_0} as in Section 3.2.2, and use the unitary vector $\zeta_1(x_0)$ to identify L_{x_0} with $\mathbb{C}$. Then ζ_1 is identified with $1 \in \mathbb{C}$ over $B^{\Sigma_1}(0,\varepsilon)$ in this trivialization. As ζ_2 is parallel with respect to $\nabla^{\iota_2^* L}$ over U_2, it is identified with $\bar{\lambda} \in \mathbb{C}$ over $B^{\Sigma_2}(0,\varepsilon)$, where $\lambda = \langle \zeta_1(x_0), \zeta_2(x_0)\rangle_L$.

Then for all $p \in \mathbb{N}^*$, (3.4.6) writes

$$\langle s_{1,p}, s_{2,p}\rangle_p = \lambda^p \int_{B^{\Sigma_2}(0,\varepsilon)} \int_{B^{\Sigma_1}(0,\varepsilon)} \langle P_p(\phi_{x_0}(Z), \phi_{x_0}(Z')) f_{1,x_0}(Z'), f_{2,x_0}(Z)\rangle_{x_0} \, dv_{\Lambda_1}(Z') dv_{\Lambda_2}(Z) + O(p^{-\infty}). \tag{3.4.7}$$

Let du and dw be the Lebesgue measures of Σ_1 and Σ_2 respectively. For any $j = 1,2$, define the functions $h_j \in \mathscr{C}^\infty(B^{\Sigma_j}(0,\varepsilon),\mathbb{R})$ in the chart (3.2.17) for any $u \in B^{\Sigma_1}(0,\varepsilon)$, $w \in B^{\Sigma_2}(0,\varepsilon)$ by

$$dv_{\Lambda_1}(u) = h_1(u)du \quad \text{and} \quad dv_{\Lambda_2}(w) = h_2(w)dw. \tag{3.4.8}$$

Then for $j = 1,2$, the functions h_j satisfy $h_j(0) = (dv_{\Lambda_j}/dv_{\Lambda_j,\omega})(x_0)$. Recalling (3.2.18) and the fact that $|\lambda^p| = 1$ for all $p \in \mathbb{N}^*$, we can use Corollary 3.2.4 and Lemma 3.2.5, to get $\theta \in]0,1[$ for any $k \in \mathbb{N}$, $\delta \in]0,1[$, such that (3.4.7) becomes

$$\begin{aligned}
\langle s_{1,p}, s_{2,p}\rangle_p = &\lambda^p \int_{B^{\Sigma_2}(0,\varepsilon p^{-\theta/2})} \int_{B^{\Sigma_1}(0,\varepsilon p^{-\theta/2})} \\
&\langle P_p(\phi_{x_0}(Z), \phi_{x_0}(Z')) f_{1,x_0}(Z'), f_{2,x_0}(Z)\rangle_{x_0} dv_{\Lambda_1}(Z') dv_{\Lambda_2}(Z) + O(p^{-\infty}) \\
= &\lambda^p p^n \sum_{r=0}^k p^{-r/2} \int_{B^{\Sigma_2}(0,\varepsilon p^{-\theta/2})} \int_{B^{\Sigma_1}(0,\varepsilon p^{-\theta/2})} \\
&\langle J_{r,x_0} \mathscr{P}_{x_0}(\sqrt{p}Z, \sqrt{p}Z') f_{1,x_0}(Z'), f_{2,x_0}(Z)\rangle_{x_0} \\
&\kappa_{x_0}^{-1/2}(Z') \kappa_{x_0}^{-1/2}(Z) dv_{\Lambda_1}(Z') dv_{\Lambda_2}(Z) \\
&+ p^n \int_{B^{\Sigma_2}(0,\varepsilon p^{-\theta/2})} \int_{B^{\Sigma_2}(0,\varepsilon p^{-\theta/2})} O(p^{-\frac{k+1}{2}+\delta}) dv_{\Lambda_1}(Z') dv_{\Lambda_2}(Z) + O(p^{-\infty}) \\
= &\lambda^p p^n \sum_{r=0}^k p^{-r/2} \int_{B^{\Sigma_2}(0,\varepsilon p^{-\theta/2})} \int_{B^{\Sigma_1}(0,\varepsilon p^{-\theta/2})} \\
&\langle J_{r,x_0} \mathscr{P}_{x_0}(\sqrt{p}w, \sqrt{p}u) f_1(u), f_2(w)\rangle_{x_0} \\
&\kappa_{x_0}^{-1/2}(u) \kappa_{x_0}^{-1/2}(w) h_1(u) h_2(w) du dw + p^n p^{-\frac{(d_1+d_2)\theta}{2}} O(p^{-\frac{k+1}{2}+\delta}).
\end{aligned} \tag{3.4.9}$$

Consider now the Taylor expansion up to order $k \in \mathbb{N}$ of $g_j = h_j \kappa_{x_0}^{-1/2} f_{j,x_0} \in \mathscr{C}^\infty(B^{\Sigma_j}(0,\varepsilon),\mathbb{C})$ for $j = 1,2$ as in (3.3.18). By Lemma 3.2.5 and following the proof of Theorem 3.4.3, we get $\delta' > 0$ and a sequence $\{G_r\}_{r\in\mathbb{N}}$ of polynomials in two variables of $\mathbb{R}^{2n}$ with values in $\mathbb{C}$, of the same parity as r with

$$G_0(Z,Z') = \langle f_1(x_0), f_2(x_0)\rangle_{x_0} \frac{dv_{\Lambda_1}}{dv_{\Lambda_1,\omega}} \frac{dv_{\Lambda_2}}{dv_{\Lambda_2,\omega}}(x_0), \tag{3.4.10}$$

such that (3.4.9) becomes

$$\langle s_{1,p}, s_{2,p}\rangle_p = \lambda^p p^{n-\frac{d_1+d_2}{2}} \sum_{r=1}^{k} p^{-r/2} \int_{B^{\Sigma_2}(0,\varepsilon p^{(1-\theta)/2})} \int_{B^{\Sigma_1}(0,\varepsilon p^{(1-\theta)/2})} G_r \mathscr{P}_{x_0}(w,u) du dw + O(p^{n-\frac{d_1+d_2+k+1}{2}+\delta'}). \quad (3.4.11)$$

As $\Sigma_1 \cap \Sigma_2 = \{0\}$ and as $1-\theta > 0$, we get from (3.2.20) the existence of $C_l > 0$ for any $l \in \mathbb{N}$ such that for any $u \in \Sigma_1,\ w \in \Sigma_2$ outside $B^{\Sigma_1}(0, \varepsilon p^{(1-\theta)/2}),\ B^{\Sigma_2}(0, \varepsilon p^{(1-\theta)/2})$ respectively,

$$\mathscr{P}_{x_0}(w,u) \leqslant C_l p^{-l}. \quad (3.4.12)$$

From (3.4.12), equation (3.4.11) then becomes

$$\langle s_{1,p}, s_{2,p}\rangle_p = \lambda^p \sum_{r=1}^{k} p^{-r/2} \int_{\Sigma_2} \int_{\Sigma_1} G_r \mathscr{P}_{x_0}(w,u) du dw + O(p^{-\frac{k+1}{2}+\delta'}). \quad (3.4.13)$$

Let us now evaluate the integrals in (3.4.13). Up to a linear symplectic transformation, the canonical symplectic basis $\{e_j, f_j\}_{j=1}^n$ of $(\mathbb{R}^{2n}, \Omega)$ can be chosen such that $\Sigma_1 = \langle e_1, \ldots, e_{d_1}\rangle$ as an oriented isotropic subspace. Let $\nu_1, \ldots, \nu_{d_2} \in \Sigma_2$ form an oriented orthonormal basis of Σ_2 for the metric induced by $\langle \cdot, \cdot \rangle$. Consider the matrices A and B given by

$$\begin{aligned} A &= (a_i^j)_{1\leqslant i\leqslant n, 1\leqslant j\leqslant d_2} \quad \text{with} \quad a_i^j = \Omega(e_i, \nu_j), \\ B &= (b_i^j)_{1\leqslant i\leqslant n, 1\leqslant j\leqslant d_2} \quad \text{with} \quad b_i^j = \langle e_i, \nu_j\rangle. \end{aligned} \quad (3.4.14)$$

As $\Omega(e_i, \nu_j) = \langle f_i, \nu_j\rangle$ for all $1 \leqslant i \leqslant d_1, 1 \leqslant j \leqslant d_2$, we know that for any $1 \leqslant j \leqslant d_2$,

$$\nu_j = \sum_{i=1}^{n} b_i^j e_i + \sum_{i=1}^{n} a_i^j f_i. \quad (3.4.15)$$

Let us write $dt := dt_1 \ldots dt_{d_2}$ for the Lebesgue measure of $\mathbb{R}^{d_2}$, and let φ be any measurable function with compact support on $\mathbb{R}^{2n}$. Setting $w = t_i \nu_i$ for any $w \in \Sigma_2$, integration of φ along Σ_2 for its Lebesgue measure dw becomes

$$\int_{\Sigma_2} \varphi(w) dw = \int_{\mathbb{R}^{d_2}} \varphi\Big(\sum_{j=1}^{d_2} t_j \nu_j\Big) dt. \quad (3.4.16)$$

Let us use the convention of Section 3.2.1, summing i from 1 to d_1 and k, j from 1 to d_2 whenever they appear as free indices. From the explicit expression (3.2.20), taking

Fourier transform and performing a change of variables, we compute

$$
\begin{aligned}
&\int_{\Sigma_2}\int_{\Sigma_1} G_r(w,u)\mathscr{P}_{x_0}(w,u)dudw = \int_{\mathbb{R}^{d_2}}\int_{\mathbb{R}^{d_1}} G_r(t_j\nu_j,u_ie_i)\mathscr{P}_{x_0}(t_j\nu_j,u_ie_i)dudt \\
&= \det\left(\dot{R}^L_{x_0}/2\pi\right)\int_{\mathbb{R}^{d_2}}\int_{\mathbb{R}^{d_1}} G_r(t_j\nu_j,u_ie_i)\exp\left(-\frac{\pi}{2}\sum_{i=d_1+1}^{n}(t_jb_i^j)^2+(t_ja_i^j)^2\right) \\
&\exp\left(-\frac{\pi}{2}\sum_{i=1}^{d_1}\left((u_i-t_jb_i^j)^2+(t_ja_i^j)^2+2\sqrt{-1}u_it_ja_i^j\right)\right)dudt \\
&= \det\left(\dot{R}^L_{x_0}/2\pi\right)\int_{\mathbb{R}^{d_2}}\int_{\mathbb{R}^{d_1}} G'_r(t)\exp\left(-\frac{\pi}{2}\sum_{i=d_2+1}^{n}(t_jb_i^j)^2+(t_ja_i^j)^2\right) \\
&\exp\left(-\frac{\pi}{2}\sum_{i=1}^{n}\left(u_i^2+(t_ja_i^j)^2+2\sqrt{-1}u_it_ja_i^j+2\sqrt{-1}t_kb_i^ka_i^jt_j\right)\right)dudt \\
&= 2^{d_2/2}\det\left(\dot{R}^L_{x_0}/2\pi\right)\int_{\mathbb{R}^{d_2}} G''_r(t)\exp\left(-\frac{\pi}{2}\sum_{i=d_1+1}^{n}(t_jb_i^j)^2+(t_ja_i^j)^2\right) \\
&\exp\left(-\pi\sum_{i=1}^{d_1}\left((t_ja_i^j)^2+\sqrt{-1}t_kb_i^ka_i^jt_j\right)\right)dt,
\end{aligned}
\tag{3.4.17}
$$

where $G'_r(t), G''_r(t)$ are polynomials in $t\in\mathbb{R}^{d_1}$ of the same parity as r. Using that $\Sigma_1\cap\Sigma_2=\{0\}$, we get the convergence of the integral in (3.4.17), and as the integrand is of the same parity as r, the integral vanishes if r is odd. Together with (3.4.13), this proves (3.4.3).

Let us now compute the first coefficient of (3.4.13) in the case $\dim\Lambda_1=n$. From (3.4.17), we get

$$
\begin{aligned}
&\int_{\Sigma_2}\int_{\Sigma_1}\mathscr{P}_{x_0}(u,w)dudw \\
&\qquad = 2^{n/2}\det\left(\dot{R}^L_{x_0}/2\pi\right)\int_{\mathbb{R}^{d_2}}\exp\left(-\pi\sum_{i=1}^{n}\left((t_ja_i^j)^2+\sqrt{-1}t_kb_i^ka_i^jt_j\right)\right)dt.
\end{aligned}
\tag{3.4.18}
$$

As $\langle\nu_1,\dots,\nu_{d_2}\rangle$ is the basis of an isotropic submanifold, we get that $\omega(\nu_j,\nu_k)=0$ for all $1\leqslant j,k\leqslant d_2$, which is equivalent through (3.4.15) to the fact that B^TA is symmetric. Then summing i from 1 to n, the matrix $(a_i^ka_i^j+\sqrt{-1}b_i^ka_i^j)_{k,j=1}^{d_2}=A^TA+\sqrt{-1}B^TA$ is symmetric, and its real part A^TA is strictly positive as A has maximal rank. Thus from (3.4.18) and using (3.2.27), we get

$$
\int_{\Sigma_2}\int_{\Sigma_1}\mathscr{P}_{x_0}(u,w)dudw = 2^{n/2}\det{}^{-\frac{1}{2}}(\sqrt{-1}(B-\sqrt{-1}A)^TA)\det\left(\dot{R}^L_{x_0}/2\pi\right). \tag{3.4.19}
$$

Then (3.4.4) follows from (3.2.4), (3.2.5), (3.3.25), (3.4.10), (3.4.14) and (3.4.18). □

3.4.2 Asymptotic expansion of clean intersections

In this section, we deal with the case of general clean intersection in the sense of Definition 3.4.1. As in Section 3.4.1, the immersions $\iota_i : \Lambda_i \to X$ and $j_i : \Lambda_1 \cap \Lambda_2 \to \Lambda_i,\ i = 1, 2$, are implicit in the statement of the following theorem, and we omit to mention them for simplicity.

Theorem 3.4.4. *Suppose that* $(\Lambda_1, \iota_1, \zeta_1)$ *and* $(\Lambda_2, \iota_2, \zeta_2)$ *intersect cleanly. Let* $\Lambda_1 \cap \Lambda_2 = \cup_{q=1}^m Y_m$ *be the decomposition into connected components of their intersection in the sense above, and set* $l_q = \dim Y_q$*. Then for any* $F \in \mathscr{C}^\infty(X, \mathrm{End}(E))$*, there exist* $b_{q,r} \in \mathbb{C},\ r \in \mathbb{N},\ 1 \leqslant q \leqslant m$*, such that for any* $k \in \mathbb{N}$ *and as* $p \to +\infty$*,*

$$\langle T_{F,p} s_{f_1,p}, s_{f_2,p} \rangle_p = \sum_{q=1}^m p^{n - \frac{d_1+d_2}{2} + \frac{l_q}{2}} \lambda_q^p \sum_{r=0}^k p^{-r} b_{q,r} + O(p^{n - \frac{d_1+d_2}{2} + \frac{l_q}{2} - (k+1)}), \tag{3.4.20}$$

where $\lambda_q \in \mathbb{C}$ *is the value of the constant function on* Y_q *defined for any* $x \in Y_q$ *by* $\lambda_q(x) = \langle \zeta_1(x), \zeta_2(x) \rangle_L$*. If* $\dim \Lambda_1 = n$*, we have*

$$\begin{aligned} b_{q,0} = 2^{n/2} \int_{Y_q} \langle F f_1(x), f_2(x) \rangle_E \det{}^{1/2} \left(\dot{R}^L / 2\pi \right) \frac{dv_{\Lambda_2}}{dv_{\Lambda_2,\omega}}(x) \\ \det{}^{-\frac{1}{2}} \left\{ \sqrt{-1} \sum_{k=1}^{n-l_q} h_\omega^{TX}(e_k, \nu_i) \omega(e_k, \nu_j) \right\}_{i,j=1}^{d_2 - l_q} (x) |dv|_{Y_q,\omega}(x), \end{aligned} \tag{3.4.21}$$

where $\langle e_i \rangle_{i=1}^{n-d_q}, \langle \nu_j \rangle_{j=1}^{n-d_q}$ *are local orthonormal frames of the normal bundle of* Y_q *inside* $\Lambda_1,\ \Lambda_2$ *with respect to* $g_\omega^{T\Lambda_1}, g_\omega^{T\Lambda_2}$*, and* $|dv|_{Y_q,\omega}$ *is the Riemannian density of* $(Y_q, g_\omega^{TY_q})$*. The square root of the determinant is determined by* (3.2.28)*.*

Proof. Let us set $F = \mathrm{Id}_E$, the proof of the general case being totally analogous by Lemma 3.2.6 and (3.3.6). Using Proposition 3.3.5, (3.3.4) and (3.4.20), we can assume that $\Lambda_1 \cap \Lambda_2$ has a unique connected component Y, and that $f_j,\ j = 1, 2$, have compact support in any given open sets. The following computations are then local on Y, and we may assume Y oriented and embedded in Λ_2 by $j_2 : Y \to \Lambda_2$. We omit the mention of j_2 in the sequel. We set $l = \dim Y$.

Let N be the *normal bundle* of Y inside Λ_2, identified with the orthogonal complement of TY in $(T\Lambda_2, g_\omega^{T\Lambda_2})$, and let g_ω^N be the induced metric on N. Let $\varepsilon > 0$ be such that the exponential map $\exp_\omega^{\Lambda_2}$ of $(\Lambda_2, g_\omega^{T\Lambda_2})$ restricted to $B^N(0, \varepsilon) := \{w \in N \mid |w|_{g^N} < \varepsilon\}$ is a diffeomorphism. Then for any $x \in Y$ and with Y embedded in N as its zero section, the differential $d\exp_{\omega,x}^{\Lambda_2} : T_x Y \oplus N_x \to T_x \Lambda_2$ is the identity map, and $\exp_\omega^{\Lambda_2}(B^N(0, \varepsilon))$ is a tubular neighbourhood of Y in Λ_2.

Let dw be an Euclidean volume form on the fibres of (N, g_ω^N) such that the volume form $dw dv_{Y,\omega}$ on the total space of N is compatible with the orientation of X. Let $h_2 \in \mathscr{C}^\infty(B^N(0, \varepsilon), \mathbb{R})$ be such that for any $x \in \Lambda_2,\ w \in N$ with $|w|_{g_{\omega,x}^N} < \varepsilon$,

$$dv_{\Lambda_2}(x, w) = h_2(x, w)\ dw dv_{Y,\omega}(x). \tag{3.4.22}$$

Then $h_2(x,0) = (dv_{\Lambda_2}/dv_{\Lambda_2,\omega})(x)$. Let us now define $I(f_1,f_2) \in \mathscr{C}^\infty(B^N(0,\varepsilon),\mathbb{C})$ at $x \in Y, w \in N_x$ with $|w|_{g^N_{\omega,x}} < \varepsilon$, by the formula

$$I(f_1,f_2)(x,w) = \int_{\Lambda_1} \langle P_p((x,w),\iota_1(y))\iota_{1,p}.\zeta_1^p f_1(y), \zeta_2^p f_2(x_0,w)\rangle_{E_p} \\ h_2(x_0,w)dv_{\Lambda_1}(y). \quad (3.4.23)$$

Using (3.3.2), (3.3.4), (3.3.5) and Proposition 3.3.5, we get from (3.4.22) and (3.4.23),

$$\begin{aligned}\langle s_{1,p}, s_{2,p}\rangle_p &= \int_{\Lambda_2}\int_{\Lambda_1} \langle P_p(\iota_2(x),\iota_1(y))\iota_{1,p}.\zeta_1^p f_1(y), \iota_{2,p}.\zeta_2^p f_2(x)\rangle_{E_p} dv_{\Lambda_1}(y)dv_{\Lambda_2}(x)\\ &= \int_{\exp_\omega^{\Lambda_2}(B^N(0,\varepsilon))}\int_{\Lambda_1} \langle P_p(x,\iota_1(y))\iota_{1,p}.\zeta_1^p f_1(y), \zeta_2^p f_2(x)\rangle_{E_p}\\ &\qquad dv_{\Lambda_1}(y)dv_{\Lambda_2}(x) + O(p^{-\infty})\\ &= \int_{x\in Y}\int_{B^{N_x}(0,\varepsilon)} I(f_1,f_2)(x,w)dwdv_{Y,\omega}(x) + O(p^{-\infty}). \quad (3.4.24)\end{aligned}$$

Fix now $x_0 \in Y$. Take $\varepsilon > 0$, $U \subset \Lambda_1$ and a diffeomorphism $\phi^{\Lambda_1}_{x_0} : B^{\mathbb{R}^{d_1}}(0,\varepsilon) \to U$ sending 0 to x_0 and such that its differential at 0 identifies $\langle\cdot,\cdot\rangle$ with $g^{T\Lambda_1}_\omega$. As $\exp^{\Lambda_2}_\omega(B^{N_{x_0}}(0,\varepsilon))$ and Λ_1 intersect cleanly at x_0 only, for $\varepsilon > 0$ small enough we can extend the union $\exp^{\Lambda_2}_\omega \cup \phi^{\Lambda_1}_{x_0} : B^{N_{x_0}}(0,\varepsilon) \cup B^{\mathbb{R}^n}(0,\varepsilon) \to X$ to a diffeomorphism $\phi_{x_0} : B^{\mathbb{R}^{2n}}(0,\varepsilon) \to V$ as in (3.2.17), identifying U with $B^\Sigma(0,\varepsilon)$, where Σ is an isotropic subspace of $(\mathbb{R}^{2n},\Omega)$ and where the fibre $(N_{x_0}, g^N_{\omega,x_0})$ is seen as an Euclidean subspace of $(\mathbb{R}^{2n},\langle\cdot,\cdot\rangle)$.

Let us identify E, L over $B^{\mathbb{R}^{2n}}(0,\varepsilon)$ with E_{x_0}, L_{x_0} as in Section 3.2.2 and use $\zeta_1(x_0)$ to identify L_{x_0} with $\mathbb{C}$. Then ζ_1,ζ_2 are identified with $1,\overline{\lambda} \in \mathbb{C}$ over $B^{\mathbb{R}^{2n}}(0,\varepsilon)$, where $\lambda = \langle \zeta_1(x_0),\zeta_2(x_0)\rangle_L$. Let du be the Lebesgue measure of Σ and $h_1 \in \mathscr{C}^\infty(B^\Sigma(0,\varepsilon),\mathbb{R})$ be such that for $u \in B^\Sigma(0,\varepsilon)$,

$$dv_{\Lambda_1}(u) = h_1(u)du. \quad (3.4.25)$$

Then $h_2(0) = (dv_{\Lambda_1}/dv_{\Lambda_1,\omega})(x_0)$. By Corollary 3.2.4 and Lemma 3.2.5, for any $k \in \mathbb{N}$ and $\delta \in]0,1[$, we get $\theta \in]0,1[$ such that

$$\begin{aligned}&\int_{B^{N_{x_0}}(0,\varepsilon)} I(f_1,f_2)(x_0,w)dw\\ &\quad= \int_{B^{N_{x_0}}(0,\varepsilon)}\int_{B^\Sigma(0,\varepsilon)} \langle P_p(w,u)\zeta_1^p f_1(u), \zeta_2^p f_2(w)\rangle_{E_p} h_2(x_0,w)dv_{\Lambda_1}(u)dw\\ &\quad= \int_{B^{N_{x_0}}(0,\varepsilon p^{-\theta/2})}\int_{B^\Sigma(0,\varepsilon p^{-\theta/2})} \langle P_p(w,u)\zeta_1^p f_1(u), \zeta_2^p f_2(w)\rangle_{E_p}\\ &\qquad h_2(x_0,w)h_1(u)dudw + O(p^{-\infty})\\ &\quad= \lambda^p p^n \sum_{r=0}^k p^{-\frac{r}{2}} \int_{B^{N_{x_0}}(0,\varepsilon p^{-\theta/2})}\int_{B^\Sigma(0,\varepsilon p^{-\theta/2})}\\ &\qquad \langle J_{r,x_0}\mathscr{P}_{x_0}(\sqrt{p}w,\sqrt{p}u) f_{1,x_0}(u), f_{2,x_0}(w)\rangle_{x_0}\\ &\qquad \kappa_{x_0}^{-1/2}(w)\kappa_{x_0}^{-1/2}(u)h_2(x_0,w)h_1(u)dudw + p^{n-\frac{d_1+d_2}{2}+\frac{l}{2}}O(p^{-\frac{k+1}{2}+\delta}). \quad (3.4.26)\end{aligned}$$

Consider now the Taylor expansions up to order $k \in \mathbb{N}$ of $h_j \kappa_{x_0}^{-1/2} f_{j,x_0}$ for $j = 1, 2$ as in (3.3.18). As in the proof of Theorem 3.4.3, we get $\delta' > 0$ and a sequence $\{F_{x_0,r}\}_{r\in\mathbb{N}}$ of polynomials in two variables of $\mathbb{R}^{2n}$ with values in $\mathbb{C}$, of the same parity as r and with $F_{x_0,0}(Z, Z') = \langle f_1(x_0), f_2(x_0)\rangle_{x_0}(dv_{\Lambda_1}/dv_{\Lambda_1,\omega})(dv_{\Lambda_2}/dv_{\Lambda_2,\omega})(x_0)$, such that (3.4.11) becomes

$$\int_{B^{N_{x_0}}(0,\varepsilon)} I(f_1, f_2)(x_0, w)dw = p^{n-\frac{d_1+d_2}{2}+\frac{l}{2}}\lambda^p \\ \sum_{r=0}^{k} p^{-r/2} \int_{N_{x_0}} \int_{\Sigma} F_{x_0,r}(w, u)\mathscr{P}_{x_0}(w, u)dudw + p^{n-\frac{d_1+d_2}{2}+\frac{l}{2}} O(p^{-\frac{k+1}{2}+\delta'}). \quad (3.4.27)$$

Thus writing

$$b_r(x_0) = \int_{N_{x_0}} \int_{\Sigma} F_{x_0,r}(w, u)\mathscr{P}_{x_0}(w, u)dudw, \quad (3.4.28)$$

and recalling that the estimates are uniform in $x_0 \in Y$, we get from (3.4.23), (3.4.24) and (3.4.27),

$$\langle s_{1,p}, s_{2,p}\rangle_p = p^{n-\frac{d_1+d_2}{2}+\frac{l}{2}}\lambda^p \sum_{r=0}^{k} p^{-r/2} \int_Y b_r(x)dv_Y(x) + p^{n-\frac{d_1+d_2}{2}+\frac{l}{2}} O(p^{-\frac{k+1}{2}}). \quad (3.4.29)$$

Now, we can use (3.4.17) to compute (3.4.28) in general, and the argument of parity holds in the same way here, so that the coefficients b_r defined in (3.4.28) for $r \in \mathbb{N}$ vanish identically for r odd. By (3.4.29), this gives (3.4.20).

Assume now $\dim \Lambda_1 = n$, and let us compute

$$b_0(x_0) = \frac{dv_{\Lambda_1}}{dv_{\Lambda_1,\omega}}\frac{dv_{\Lambda_2}}{dv_{\Lambda_2,\omega}}(x_0)\langle f_1(x_0), f_2(x_0)\rangle_{x_0} \int_{N_{x_0}} \int_{\Sigma} \mathscr{P}_{x_0}(w, u)dudw. \quad (3.4.30)$$

In the same way than in the proof of Theorem 3.4.3, we can take the canonical symplectic basis $\{e_j, f_j\}_{j=1}^n$ of $(\mathbb{R}^{2n}, \Omega)$ such that $\Sigma = \mathbb{R}^n \times \{0\}$ and such that $\langle e_{n-l+1}, \dots, e_n\rangle$ is an oriented orthonormal basis of $(T_{x_0}Y, g_\omega^{TY})$ in the identification of $\mathbb{R}^{2n}$ with $T_{x_0}X$ via $d\phi_{x_0}$. Let $\nu_1, \dots, \nu_{d_2-l} \in N_{x_0}$ be such that $\langle \nu_1, \dots, \nu_{d_2-l}, e_{n-l+1}, \dots, e_n\rangle$ is an oriented orthonormal basis of the isotropic subspace $\Sigma_2 := N_{x_0} \oplus T_{x_0}Y$. Then for $1 \leqslant i \leqslant d_2 - l$ and $n - l + 1 \leqslant j \leqslant n$, we have that $\langle \nu_i, f_j\rangle = -\omega(\nu_i, e_j) = 0$. Thus setting

$$\begin{aligned} A &= (a_i^j)_{1\leqslant i\leqslant n-l,\, 1\leqslant j\leqslant d_2-l} \quad \text{with} \quad a_i^j = \omega(e_i, \nu_j), \\ B &= (b_i^j)_{1\leqslant i\leqslant n-l,\, 1\leqslant j\leqslant d_2-l} \quad \text{with} \quad b_i^j = \langle e_i, \nu_j\rangle, \end{aligned} \quad (3.4.31)$$

we get for all $1 \leqslant j \leqslant d_2 - l$,

$$\nu_j = \sum_{i=1}^{n-l} b_i^j e_i + \sum_{i=1}^{n-l} a_i^j f_i. \quad (3.4.32)$$

Write $dt := dt_1 \dots dt_{d_2-l}$ for the Lebesgue measure of $\mathbb{R}^{d_2-l}$. Using the summation convention of Section 3.2.1 with i from 1 to $n - l$ and j, k from 1 to $d_2 - l$ whenever they appear as free indices, we get

$$\begin{aligned}
\int_{N_{x_0}}\int_{\Sigma}\mathscr{P}_{x_0}(w,u)dudw &= \int_{\mathbb{R}^{d_2-l}}\int_{\mathbb{R}^n}\mathscr{P}_{x_0}(t_j\nu_j,u_ie_i)dudt\\
&= \det\left(\dot{R}^L_{x_0}/2\pi\right)\int_{\mathbb{R}^{d_2-l}}\int_{\mathbb{R}^n}\exp\left(-\frac{\pi}{2}\sum_{i=n-l+1}^{n}u_i^2\right)\\
&\exp\left(-\frac{\pi}{2}\sum_{i=1}^{n-l}\left((u_i-t_jb_i^j)^2+(t_ja_i^j)^2+2\sqrt{-1}u_it_ja_i^j\right)\right)dudt\\
&= 2^{l/2}\det\left(\dot{R}^L_{x_0}/2\pi\right)\int_{\mathbb{R}^{d_2-l}}\int_{\mathbb{R}^{n-l}}\exp\Big(-\frac{\pi}{2}\sum_{i=1}^{n-l}u_i^2+(t_ja_i^j)^2\\
&\quad +2\sqrt{-1}u_it_ja_i^j+2\sqrt{-1}t_kb_i^ka_i^jt_j\Big)du_1\dots du_{n-l}dt\\
&= 2^{n/2}\det\left(\dot{R}^L_{x_0}/2\pi\right)\int_{\mathbb{R}^{d_2-l}}\exp\left(-\frac{\pi}{2}\sum_{j=1}^{n-l}\left(2(t_ja_i^j)^2+2\sqrt{-1}t_kb_i^ka_i^jt_j\right)\right)dt. \quad (3.4.33)
\end{aligned}$$

As $\omega(\nu_j,\nu_k)=0$, for all $1\leqslant j,k\leqslant d_2-l$, we know from (3.4.32) that the matrix B^TA is symmetric. Then as in (3.4.19), we get

$$\int_{N_{x_0}}\int_{\Sigma}\mathscr{P}_{x_0}(w,u)dudw = 2^{n/2}\det{}^{-1/2}(\sqrt{-1}A(B-\sqrt{-1}A))\det\left(\dot{R}^L_{x_0}/2\pi\right). \quad (3.4.34)$$

Using the explicit definition of A and B above and from (3.2.4), (3.2.5), (3.3.25) and (3.4.34), we get (3.4.21).

□

Remark 3.4.5. Suppose that the first Chern class $c_1(TX)$ of (TX,J) is even in $H^2(X,\mathbb{Z})$. Then there exists a complex line bundle $K_X^{1/2}$ over X such that its second tensor power is equal to the canonical line bundle K_X of X. The choice of $K_X^{1/2}$ does not depend on J compatible with ω, and is called a *metaplectic structure* on (X,ω). Now if $\iota:\Lambda\to X$ is an immersed Lagrangian submanifold, then ι^*K_X is canonically isomorphic to $\det(T^*\Lambda_{\mathbb{C}})$ over Λ, and we call $\iota^*K_X^{1/2}$ the *half-form bundle* of Λ. We endow $K_X^{1/2}$ with the Hermitian structure induced by $h_\omega^{K_X}$ as in Section 3.2.1.

Consider now the setting of Theorem 3.4.4, with $\dim\Lambda_1=\dim\Lambda_2=n$. Via the isomorphism above, we define the *angle* of $\iota_j:\Lambda_j\to X,\ j=1,2$, as a function on any connected component Y of their intersection by the formula

$$\begin{aligned}
\det\{\Lambda_1,\Lambda_2\} &= h_\omega^{K_X}(dv_{\Lambda_1},dv_{\Lambda_2})^{-1}\\
&= \det\left\{h_\omega^{TX}(e_i,\nu_j)\right\}_{i,j=1}^{n-l}.
\end{aligned} \quad (3.4.35)$$

On another hand, following [11, Lem.3.1], we can construct a sesquilinear pairing $\#:\iota_1^*K_X^{1/2}|_Y\times\iota_2^*K_X^{1/2}|_Y\to\det(T^*Y_{\mathbb{C}})$ over Y, depending only on the metaplectic structure of (X, ω), which at any $x \in Y$ takes two square roots $dv_{\Lambda_j,\omega,x}^{1/2}$ of $dv_{\Lambda_j,\omega,x}$ for $j = 1, 2$ to

$$dv_{\Lambda_1,\omega,x}^{1/2} \# dv_{\Lambda_2,\omega,x}^{1/2} = \det{}^{-1/2}\{\omega(e_i, \nu_j)\}_{i,j=1}^{n-l} dv_{Y,\omega,x}, \tag{3.4.36}$$

for an Euclidean volume form $dv_{Y,\omega,x}$ of $(T_xY, g_{\omega,x}^{TY})$ and some coherent choice of square root induced by $dv_{\Lambda_1,\omega,x}^{1/2}$, $dv_{\Lambda_2,\omega,x}^{1/2}$ and $dv_{Y,\omega,x}$. Then taking $E = K_X^{1/2}$, Theorem 3.4.4 gives the following formula for b_0 on Y as in (3.4.21),

$$b_0 = 2^{\frac{n-l}{2}} e^{-\sqrt{-1}\frac{(n-l)\pi}{4}} \int_Y \det\{\Lambda_1, \Lambda_2\}^{-1} f_1 \# f_2. \tag{3.4.37}$$

In the particular case of (X, J, ω) Kähler with $c_1(TX)$ even, $g^{TX} = g_\omega^{TX}$ and $\dim \Lambda_1 = \dim \Lambda_2 = n$, this formula can be compared with the one appearing in [11, Prop.3.16]. In particular, they get $\det\{\Lambda_1, \Lambda_2\}^{-1/2}$ instead of $\det\{\Lambda_1, \Lambda_2\}^{-1}$ as in (3.4.37). This discrepancy is due to the fact that even though they use half-forms, their Lagrangian states take values in L^p and not in $L^p \otimes K_X^{1/2}$ as it is the case here.

Note that the proof of Theorem 3.4.4 delivers as well a formula for the first coefficient of (3.4.20) in the case Λ_1 and Λ_2 both not Lagrangian, although its geometric meaning is unclear, which is why we did not give it explicitly.

Note finally that without metaplectic structure on (X, ω), only the product of the square root of (3.4.35) with (3.4.36) make sense in general (see [58] for related results).

3.5 Extensions to non-compact manifolds and orbifolds

In this section, we show how one can adapt the results of the previous Sections in the case of non-compact manifolds and orbifolds. We will work for simplicity in the case of (X, J, ω) Kähler and $g^{TX} = g_\omega^{TX}$. Then as underlined in the introduction, the renormalized Bochner Laplacian (3.2.8) reduces to the Kodaira Laplacian on sections.

Note further that the existence of an expansion of the form (3.2.25) is a straightforward consequence of the existence of an expansion as in [51, (4.9)].

3.5.1 Non-compact case

Let (X, J, ω, g^{TX}) be a complete Kähler manifold with $\omega(\cdot,\cdot) = g^{TX}(J\cdot,\cdot)$, let (L, h^L) be a holomorphic line Hermitian bundle over X with Chern connection ∇^L satisfying (3.1.1), and let (E, h^E) be an auxiliary holomorphic Hermitian bundle with Chern connection ∇^E. For any $p \in \mathbb{N}^*$, let $H^0_{(2)}(X, E_p)$ denote the space of holomorphic sections of $E_p = L^p \otimes E$ which are square integrable with respect to the L^2-Hermitian product defined as in (3.2.9). Let P_p denote the orthogonal projection from the space of L^2-sections of E_p onto $H^0_{(2)}(X, E_p)$ with respect to this product. Then as noticed in [49, Rem.1.4.3], P_p has smooth Schwartz kernel $P_p(\cdot,\cdot) \in \mathscr{C}^\infty(X \times X, E_p \boxtimes E_p^*)$ with respect to the Riemannian volume form dv_X of (X, g^{TX}), and $P_p(\cdot,\cdot)$ is square integrable and holomorphic with respect to its first variable.

Let us write R^{det} for the curvature of the Chern connection of K_X^*. Then we have the following result.

Theorem 3.5.1. *[51, Th.5.2, Th.5.3] Suppose that there exists $C > 0$ such that for all $x \in X$ and $v \in T_xX$, the following inequality holds in the sense of endomorphisms of E,*

$$\sqrt{-1}(R^{det}\mathrm{Id}_E + R^E)(v, Jv) > -C\omega(v, Jv)\mathrm{Id}_E. \tag{3.5.1}$$

Then for any compact set $K \subset X$, Proposition 3.2.3 holds uniformly for any $x, x' \in K$ and Lemma 3.2.5 holds uniformly for any $x_0 \in X$.

If $F \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ has compact support, then Lemma 3.2.6 holds uniformly for any $x_0 \in X$.

From now on, we suppose that (3.5.1) is verified for X. Then Definition 3.3.1 still makes sense in this context, provided Λ is compact. Precisely, for (Λ, ι, ζ) Bohr-Sommerfeld manifold as in Definition 3.3.1 with Λ compact and for $f \in \mathscr{C}^\infty(\Lambda, \iota^*E)$, we define the associated isotropic state $\{s_{f,p}\}_{p\in\mathbb{N}}$ in the same way than in (3.3.2) for any $p \in \mathbb{N}^*$ and $x \in X$ by the formula

$$s_{f,p}(x) = \int_\Lambda P_p(x, \iota(y))\iota_p.\zeta^p f(y) dv_\Lambda(y). \tag{3.5.2}$$

Then as Λ is compact, we get that $s_{f,p} \in H^0_{(2)}(X, E_p)$. Furthermore, the following analogue of Proposition 3.3.4 holds.

Lemma 3.5.2. *Suppose that (X, J, ω, g^{TX}) is a complete Kähler manifold satisfying* (3.5.1)*, and let (Λ, ι, ζ) be a compact Bohr-Sommerfeld submanifold of X. Then for any $s \in H^0_{(2)}(X, E_p)$, the following reproducing property holds,*

$$\langle s, s_{f,p}\rangle_p = \int_\Lambda \langle s(\iota(x)), \iota_p.\zeta^p f(x)\rangle_{E_p}\, dv_\Lambda(y). \tag{3.5.3}$$

Furthermore, for any $F \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ with compact support, property (3.3.6) *holds.*

Proof. As Λ is compact, we can repeat the computations of (3.3.7), so that (3.5.3) holds. As $F \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ has compact support, we can repeat in the same way the computations of (3.3.8), and (3.3.6) holds as well in this context. □

With these preliminaries, we can state the following generalization of the results of Section 3.3.2, Section 3.4.1 and Section 3.4.2.

Theorem 3.5.3. *Suppose that (X, J, ω, g^{TX}) is a complete Kähler manifold satisfying* (3.5.1)*. If (Λ, ι, ζ) is a compact Bohr-Sommerfeld submanifold of (X, ω), then Theorem 3.3.6 holds.*

Furthermore, if $(\Lambda_j, \iota_j, \zeta_j)$, $j = 1, 2$, are two compact Bohr-Sommerfeld submanifolds of (X, ω) intersecting cleanly, then Theorem 3.4.4 hold.

Proof. Let $(\Lambda_j, \iota_j, \zeta_j)$, $j = 1, 2$, be two compact Bohr-Sommerfeld submanifolds of X, and consider $f_j \in \mathscr{C}^\infty(X, \iota_j^* E)$, $j = 1, 2$. By Theorem 3.5.1, we know that Proposition 3.3.5 is still true uniformly in any compact set $K \subset X$. Furthermore, using (3.5.2), (3.5.3) and omitting the immersions, we get for any $p \in \mathbb{N}^*$,

$$\begin{aligned}\langle s_{f_1,p}, s_{f_2,p} \rangle_p &= \int_{\Lambda_2} \langle s_{f_1,p}(x), \zeta_2^p f_2(x) \rangle_{E_p} dv_{\Lambda_2}(x) \\ &= \int_{\Lambda_2} \int_{\Lambda_1} \langle P_p(x,y) \zeta_1^p f_1(y), \zeta_2^p f_2(x) \rangle_{E_p} dv_{\Lambda_1}(y) dv_{\Lambda_2}(x).\end{aligned} \tag{3.5.4}$$

We can then choose the compact set K in Theorem 3.5.1 to contain $\iota(\Lambda_1) \cup \iota(\Lambda_2)$, and the proof of Theorem 3.5.3 goes along the lines of the proofs of Theorem 3.3.6, Theorem 3.4.3 and Theorem 3.4.4. By the second part of Lemma 3.5.2, the case of $\langle T_{F,p} s_{f_1,p}, s_{f_2,p} \rangle_p$ such that $F \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ has compact support is strictly analogous. $\square$

3.5.2 Orbifold case

In this section, we consider (X, J, ω, g^{TX}) complete Kähler orbifold satisfying (3.5.1), (L, h^L) a holomorphic Hermitian proper orbifold line bundle over X with Chern connection ∇^L satisfying (3.1.1), and (E, h^E) a holomorphic Hermitian proper orbifold vector bundle over X endowed with its Chern connection ∇^L. In order to give a precise meaning to these notions, we first state some notations and definitions from [49, § 5.4].

Definition 3.5.4. Let $\mathscr{M}$ be the category whose objects are the pairs (M, G), with M smooth connected manifold and G a finite group acting effectively on M, and whose morphisms $\Phi : (M, G) \to (M', G')$ are families of open embeddings $\varphi : M \to M'$ satisfying:

- For each $\varphi \in \Phi$, there is an injective group homomorphism $\lambda_\varphi : G \to G'$ such that φ is λ_φ-equivariant.
- For $g \in G'$ and $\varphi \in \Phi$, define $g\varphi : M \to M'$ by $(g\varphi)(x) = g\varphi(x)$ for any $x \in M$. If $(g\varphi)(M) \cap \varphi(M) = \emptyset$, then $g \in \lambda_\varphi(G)$.
- For $\varphi \in \Phi$, we have $\Phi = \{g\varphi \mid g \in G'\}$.

Definition 3.5.5. Let X be a paracompact Hausdorff space and let $\mathscr{U}_X$ be a covering of X consisting of connected open subsets, satisfying the condition

$$\begin{aligned}&\text{For any } U, U' \in \mathscr{U}_X \text{ and } x \in U \cap U', \\ &\text{there is } U'' \in \mathscr{U}_X \text{ such that } x \in U'' \subset U \cap U'.\end{aligned} \tag{3.5.5}$$

An *orbifold structure* $\mathscr{V}_X$ on X consists of the following datas:

- For any $U \in \mathscr{U}_X$, an object $(G_U, \widetilde{U})$ of $\mathscr{M}$ and a ramified covering $\tau_U : \widetilde{U} \to U$ which is G_U-invariant and induces a homeomorphism $U \simeq \widetilde{U}/G_U$.

- For any $U,\ V \in \mathscr{U}_X$ such that $U \subset V$, a morphism $\Phi_{VU} : (G_U, \widetilde{U}) \to (G_V, \widetilde{V})$ of $\mathscr{M}$, which covers the inclusion $U \subset V$ and satisfies $\Phi_{WU} = \Phi_{WV} \circ \Phi_{VU}$ for any $U,\ V,\ W \in \mathscr{U}_X$, with $U \subset V \subset W$.

If $\mathscr{U}'_X$ is a refinement of $\mathscr{U}_X$ satisfying the condition (3.5.5), then there is an orbifold structure $\mathscr{V}'_X$ associated to $\mathscr{U}'_X$ such that $\mathscr{V}_X \cap \mathscr{V}'_X$ is again an orbifold structure. We then say that $\mathscr{V}_X$ and $\mathscr{V}'_X$ are *equivalent*. An equivalence class is called an *orbifold structure* on X. In particular, we can suppose that $\mathscr{U}_X$ is arbitrarily fine. In the sequel, we will always consider the unique maximal representative in the equivalence class.

In the above definitions, we can replace the objects of $\mathscr{M}$ by manifolds with specified structures together with a group preserving these structures, and morphisms preserving these structures. In the case in hand, by structure we mean an orientation, a Riemannian metric, a symplectic structure, an almost-complex structure or a complex structure. Furthermore, we can realise Cartesian products of orbifolds in the obvious way.

Let $(X, \mathscr{V}_X)$ be an orbifold. For each $x \in X$, up to refinement of $\mathscr{V}_X$, there exists $U_x \in \mathscr{U}_X$ containing x and $\widetilde{x} \in \widetilde{U}$, $\tau_U(\widetilde{x}) = x$, such that $\widetilde{x}$ is a fixed point of G_U. Then by the second axiom of Definition 3.5.4, such a group is unique up to isomorphism, and we denote it by G_x^X. If $|G_x^X| = 1$, then X has a smooth structure in a neighbourhood of x, and we call such an x a *smooth point* of X. If $|G_x^X| > 1$, we call such an x a *singular point* of X. We denote $X_{sing} = \{x \in X \mid |G_x^X| > 1\}$ the *singular set* of X, and $X_{reg} = \{x \in X \mid |G_x^X| = 1\}$ the *regular set* of X. In the sequel, we always denote by $\widetilde{x} \in \widetilde{U}$ a lift of $x \in U \in \mathscr{U}_X$.

The next set of definitions are generalisations of [45, Def. 1.6, Def.1.7].

Definition 3.5.6. An *orbifold immersion* $I : (Y, \mathscr{V}_Y) \to (X, \mathscr{V}_X)$ is a continuous map $\iota : Y \to X$, such that for any $V \in \mathscr{U}_X$ and any $U \in \mathscr{U}_Y$ connected component of $\iota^{-1}(V)$, there is a family I_{UV} of immersions $\iota_{UV} : \widetilde{U} \to \widetilde{V}$ covering ι together with surjective group homomorphisms $\lambda_{UV} : G_V \to G_U$ such that ι_{UV} is λ_{UV}-equivariant. Furthermore, the families I_{UV} satisfy $I_{UV} = \{g\iota_{UV} \mid g \in G_U\}$ and are compatible with the orbifold structures in the obvious sense. In that case, we define the *stabilizer* of V in U by $K_{UV} = \operatorname{Ker} \lambda_{UV}$. Then $m_{X,Y} := |K_{UV}|$ is locally constant on Y, and is called the *relative multiplicity* on Y.

A *singular immersion* $\hat{I}$ from a smooth manifold Y to an orbifold $(X, \mathscr{V}_X)$ is a continuous map $\iota : Y \to X$, together with immersions $\widetilde{\iota}_V : U \to \widetilde{V}$ covering ι for any $V \in \mathscr{U}_X$, such that $g.\iota(U)$ intersects $\iota(U)$ cleanly in the sense of Definition 3.4.1 for all $g \in G_V$. In that case, we define the stabilizer of U in V by the subgroup $K_{UV} \subset G_V$ fixing each point of $\widetilde{\iota}_V(U)$. Then the *relative multiplicity* $m_{X,Y} = |K_{UV}|$ is again locally constant on Y.

An *orbifold submersion* $P : (M, \mathscr{V}_M) \to (X, \mathscr{V}_X)$ is a continuous map $\pi : M \to X$ such that $\pi(U) \in \mathscr{U}_X$ for any $U \in \mathscr{U}_M$, together with submersions $\pi_U : \widetilde{U} \to \widetilde{\pi(U)}$ covering π and surjective group homomorphisms $\lambda_U : G_U \to G_{\pi(U)}$ for any $U \in \mathscr{U}_X$ making π_U be λ_U-equivariant. Furthermore, we assume compatibility with the orbifold structures in the obvious sense.

Note that any $x \in X$ can be seen as an immersed orbifold with $m_{X,x} = |G_x|$. In both definitions of an immersion above, if $\iota^{-1}(X_{sing})$ has strictly positive measure for the density induced by any Riemannian metric, then G_V fixes $\iota(U)$ and $m_{X,Y}$ is strictly positive. The intersection of two orbifold immersions is still defined as in Definition 3.4.1 to be their fibred product over X, which gets a natural orbifold structure making all maps into orbifold immersions.

Finally, note that we can easily combine the definitions above to get the notion of a singular orbifold immersion, and the results of this section hold in this case as well. For simplicity and clarity, we will keep both notions separated from each other.

Definition 3.5.7. An *orbifold vector bundle* is an orbifold submersion $P : (E, \mathscr{V}_E) \to (X, \mathscr{V}_X)$ such that $E_U := \pi^{-1}(U)$ belongs to $\mathscr{U}_E$ for any $U \in \mathscr{U}_X$ and $\pi_{E_U} : \widetilde{E}_U \to \widetilde{U}$ are G_{E_U}-equivariant vector bundles. Furthermore, we ask the inclusions $\Phi_{E_V E_U}$ covering Φ_{VU} to be vector bundle maps, for any $U,\ V \in \mathscr{U}_X$ such that $U \subset V$.

If G_{E_U} acts effectively on $\widetilde{U}$ for all $U \in \mathscr{U}_X$, that is the group morphisms $\lambda_{E_U} : G_{E_U} \to G_U$ associated to P as in Definition 3.5.6 are isomorphisms, we say that E is *proper*.

We can then define the proper tangent orbifold bundle TX and the proper cotangent orbifold bundle T^*X over any orbifold $(X, \mathscr{V}_X)$ in the obvious way. We can as well form tensor products of vector bundles by taking the tensor products locally over each orbifold chart, and we check easily that this operation preserves properness. If E is a proper orbifold bundle over X and if $\Psi : (X, \mathscr{V}_X) \to (Y, \mathscr{V}_Y)$ is any of the orbifold maps of Definition 3.5.6, we can pullback E to Y by Ψ in the obvious way, and we write Ψ^*E for the pullback orbifold vector bundle, which is still proper.

If X is complete, we define a distance on X for any $x, y \in X$ by

$$\begin{aligned} d(x,y) = \inf_{\gamma} \Big\{ \sum_j \int_{t_{j-1}}^{t_{j+1}} |\frac{\partial}{\partial t}\widetilde{\gamma}_j(t)|dt \ \Big|\ & \gamma : [0,1] \to X,\ \gamma(0) = x,\ \gamma(1) = y, \\ & \text{such that there exists } t_0 = 0 < t_1 < \cdots < t_k = 1,\ \gamma([t_{j-1}, t_j]) \subset U_j, \\ & U_j \in \mathscr{U}_X, \text{ and a smooth map } \widetilde{\gamma}_j : [t_{j-1}, t_j] \to \widetilde{U}_j \text{ that covers } \gamma|_{[t_{j-1},t_j]} \Big\}. \end{aligned} \tag{3.5.6}$$

Let $E \to X$ be an orbifold vector bundle. An orbifold section $s : X \to E$ is called *smooth* if for each $U \in \mathscr{U}_X$, the restriction of s to U is covered by a G_U^E-equivariant smooth section $\widetilde{s}_U : \widetilde{U} \to \widetilde{E}_U$. In the same way, if X is a complex orbifold and E is a holomorphic orbifold vector bundle, we say s is holomorphic if it is locally covered by holomorphic sections. The space of smooth (resp. holomorphic) sections of E is denoted by $\mathscr{C}^\infty(X, E)$ (resp. $H^0(X, E)$).

If X is oriented and α is a smooth section of the exterior product orbifold bundle $\Lambda(T^*X)$ with support in $U \in \mathscr{U}$, we define

$$\int_X \alpha := \frac{1}{|G_U|} \int_{\widetilde{U}} \widetilde{\alpha}_U, \tag{3.5.7}$$

where $\widetilde{\alpha}_U$ is an invariant section covering α over $\widetilde{U}$. We extend this definition for general α using a partition of unity. In particular, if X is oriented and Riemannian, there is an induced Riemannian volume form dv_X on X, so that we can integrate functions.

Let now (X, J, ω) be a Kähler orbifold. As we can verify locally, for any Hermitian holomorphic proper orbifold bundle over X, its Chern connection is well-defined and unique. Let then (L, h^L) be a holomorphic Hermitian proper orbifold line bundle, such that its Chern connection satisfies (3.1.1). We write g^{TX} for the Riemannian metric on X satisfying (1.1.2), and dv_X for the associated Riemannian volume form. Let (E, h^E) be an auxiliary holomorphic Hermitian proper orbifold vector bundle on X.

We define the L^2-Hermitian product associated with all the previous datas on $\mathscr{C}^\infty(X, E_p)$ by the formula (1.3.20), and the *Bergman kernel* $P_p(\cdot,\cdot) \in \mathscr{C}^\infty(X \times X, E_p \boxtimes E_p^*)$ is the Schwartz kernel with respect to dv_X of the orthogonal projection P_p from $\mathscr{C}^\infty(X, E_p)$ to $H^0_{(2)}(X, E_p)$ as in (3.2.12). For any $V \in \mathscr{U}_X$ and all $p \in \mathbb{N}^*$, let $\widetilde{P}_p(\cdot,\cdot) \in \mathscr{C}^\infty(\widetilde{V} \times \widetilde{V}, \widetilde{E}_{p,V} \boxtimes \widetilde{E}^*_{p,V})$ be the $G_V \times G_V$-invariant lift of $P_p(\cdot,\cdot) \in \mathscr{C}^\infty(V \times V, E_p \boxtimes E_p^*)$. More generally, for any object on $V \in \mathscr{U}_X$, we add a superscript $\widetilde{\ }$ to denote the corresponding object on $\widetilde{V}$.

For any $m \in \mathbb{N}$, let $|\cdot|_{\mathscr{C}^m}$ denote the $\mathscr{C}^m$-norm on $E_p \boxtimes E_p^*$ over $X \times X$ induced by h^L, h^E and ∇^L, ∇^E. The following result is the version of Lemma 3.2.5 for orbifolds. It uses the fact, noticed in [45], that the finite propagation speed of the wave equation on orbifolds holds.

Proposition 3.5.8. *[51, § 6.2],[49, Rem.5.4.12.b] Proposition 3.2.3 holds in the case of* (X, J, ω, g^{TX}) *complete Kähler orbifold satisfying* (3.5.1). *Moreover, for any* $V \in \mathscr{U}_X$, *there exists a section* $F(\widetilde{D}_p)(\cdot,\cdot) \in \mathscr{C}^\infty(\tilde{V} \times \tilde{V}, \widetilde{E}_{p,V} \boxtimes \widetilde{E}^*_{p,V})$ *satisfying the following properties:*

For any $\widetilde{x},\ \widetilde{y} \in \widetilde{V}$ *and* $g \in G_V$,

$$(g,1)F(\widetilde{D}_p)(g^{-1}\widetilde{x}, \widetilde{y}) = (1, g^{-1})F(\widetilde{D}_p)(\widetilde{x}, g\widetilde{y}). \tag{3.5.8}$$

For any $m,\ l \in \mathbb{N}$, *there is* $C_{m,l} > 0$ *such that for any* $\widetilde{x},\ \widetilde{y} \in \widetilde{V}$ *and all* $p \in \mathbb{N}^*$,

$$|\widetilde{P}_p(\widetilde{x}, \widetilde{y}) - \sum_{g \in G_U} (1, g^{-1})F(\widetilde{D}_p)(\widetilde{x}, g\widetilde{y})|_{\mathscr{C}^m} \leqslant C_{m,l}\, p^{-l}. \tag{3.5.9}$$

$F(\widetilde{D}_p)(\cdot,\cdot)$ *satisfies the expansion of Lemma 3.2.5 at any* $x_0 \in \widetilde{V}$.

With all these prerequisites in hand, Definition 3.3.1 still makes sense in this context replacing the immersion ι by an orbifold immersion or singular immersion I as in Definition 3.5.6. In the second case, we talk about a *singular* Bohr-Sommerfeld submanifold. In any case, if Λ is compact, the associated isotropic state as in (3.3.2) is well defined and Proposition 3.3.4 still holds. We will use the additivity property (3.3.4) to assume that the section f of Definition 3.3.3 has compact support in some given open set $U \in \mathscr{U}_\Lambda$.

Theorem 3.5.9. *Let (X, J, ω, g^{TX}) be a complete Kähler orbifold satisfying* (1.1.2)*, let (L, h^L) be a holomorphic Hermitian proper orbifold line bundle such that the curvature of its Chern connection satisfies* (3.1.1)*, and let (E, h^E) be a holomorphic Hermitian proper orbifold vector bundle. Suppose that (X, J, ω, g^{TX}) satisfies* (3.5.1)*.*

If (Λ, I, ζ) is a compact Bohr-Sommerfeld submanifold of X and $F \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ has compact support, then Theorem 3.3.6 holds, with the following formula for the first coefficient of (3.3.12)*,*

$$b_0 = 2^{d/2} m_{X,\Lambda} \int_\Lambda \langle Ff, f\rangle_{\iota^* E} dv_\Lambda. \tag{3.5.10}$$

If $(\Lambda_j, I_j, \zeta_j)$, $j = 1, 2$, are two compact Bohr-Sommerfeld submanifolds of X intersecting cleanly and if $F \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ has compact support, then the expansion of Theorem 3.4.4 holds. If $\dim \Lambda_1 = n$, then the first coefficients $b_{q,0}$ of (3.4.20) *satisfy the formula* (3.4.4) *multiplied by*

$$m_{\Lambda_2, Y_q}/m_{X,\Lambda_1}. \tag{3.5.11}$$

Finally, the above holds for compact singular Bohr-Sommerfeld submanifolds of X, provided their intersection locus is away from the singular set.

Proof. Let (Λ, I, ζ) be a compact Bohr-Sommerfeld submanifold, and let $f \in \mathscr{C}^\infty(\Lambda, I^*E)$ have compact support in a sufficiently small open set $U \in \mathscr{U}_\Lambda$, connected component of $\iota^{-1}(V)$ for some $V \in \mathscr{U}_X$. Then using (3.5.7) and (3.5.9), for any $\widetilde{x} \in \widetilde{V}$ we have

$$\begin{aligned}
\widetilde{s}_{f,p}(\widetilde{x}) &= \frac{1}{|G_U|} \int_{\widetilde{U}} \widetilde{P}_p(\widetilde{x}, \iota_{UV}(\widetilde{y})) \iota_{p,UV}.\widetilde{f}\widetilde{\zeta}^p(\widetilde{y}) dv_{\widetilde{U}}(\widetilde{y}) \\
&= \frac{1}{|G_U|} \int_{\widetilde{U}} \sum_{g \in G_V} (1, g^{-1}) F(\widetilde{D}_p)(\widetilde{x}, g\iota_{UV}(\widetilde{y})) \iota_{p,UV}.\widetilde{f}\widetilde{\zeta}^p(\widetilde{y}) dv_{\widetilde{U}}(\widetilde{y}) + O(p^{-\infty}) \\
&= \frac{1}{|G_U|} \int_{\widetilde{U}} \sum_{g \in G_V} F(\widetilde{D}_p)(\widetilde{x}, \iota_{UV}(\widetilde{y})) \iota_{p,UV}.(g.\widetilde{f}\widetilde{\zeta}^p(g^{-1}\widetilde{y})) dv_{\widetilde{U}}(\widetilde{y}) + O(p^{-\infty}) \\
&= \frac{|G_V|}{|G_U|} \int_{\widetilde{U}} F(\widetilde{D}_p)(\widetilde{x}, \iota_{UV}(\widetilde{y})) \iota_{p,UV}.\widetilde{f}\widetilde{\zeta}^p(\widetilde{y}) dv_{\widetilde{U}}(\widetilde{y}) + O(p^{-\infty}).
\end{aligned} \tag{3.5.12}$$

Here $\iota_{UV} : \widetilde{U} \to \widetilde{V}$ is any member of the family of maps in I_{UV}. Now by Definition 3.5.6, we have $|G_V|/|G_U| = m_{X,\Lambda}$. By Proposition 3.5.8, $F(\widetilde{D}_p)(\cdot, \cdot)$ satisfies the expansion of Lemma 3.2.5 at any $x_0 \in \widetilde{V}$, so that we can follow the proof of Theorem 3.3.6 to deduce from (3.5.12) an asymptotic expansion in $p \in \mathbb{N}^*$ of the form (3.3.10) for the norm of $s_{f,p}$, with highest coefficient given by (3.5.10) in the case $F = \mathrm{Id}_E$.

For any $j = 1, 2$, let $(\Lambda_j, I_j, \zeta_j)$ be compact Bohr-Sommerfeld submanifolds and let $f_j \in \mathscr{C}^\infty(\Lambda, I^*E)$ have compact support in a sufficiently small open set $U_j \in \mathscr{U}_\Lambda$, connected component of $\iota^{-1}(V)$ for some $V \in \mathscr{U}_X$. Then as the reproducing property (3.3.5) still holds, analogous to (3.4.6), (3.5.12), using (3.5.7), (3.5.9), and omiting the immersion maps, we have

$$\begin{aligned}\langle s_{1,p},s_{2,p}\rangle_p &= \frac{1}{|G_{U_2}|}\int_{\widetilde{U}_2}\langle\widetilde{\zeta}_{f_1,p}(\widetilde{x}),\widetilde{\zeta}_2^p\widetilde{f}_2(\widetilde{x})\rangle_{E_p}dv_{\widetilde{U}_2}(\widetilde{x})\\ &= \frac{1}{|G_{U_1}|}\frac{1}{|G_{U_2}|}\int_{\widetilde{U}_2}\int_{\widetilde{U}_1}\langle\widetilde{P}_p(\widetilde{x},\widetilde{y})\widetilde{\zeta}_1^p\widetilde{f}_1(\widetilde{y}),\widetilde{\zeta}_2^p\widetilde{f}_2(\widetilde{x})\rangle_{E_p}dv_{\widetilde{U}_1}(\widetilde{y})dv_{\widetilde{U}_2}(\widetilde{x})\\ &= \frac{|G_V|}{|G_{U_1}||G_{U_2}|}\int_{\widetilde{U}_2}\int_{\widetilde{U}_1}\langle F(\widetilde{D}_p)(\widetilde{x},\widetilde{y})\widetilde{\zeta}_1^p\widetilde{f}_1(\widetilde{y}),\widetilde{\zeta}_2^p\widetilde{f}_2(\widetilde{x})\rangle_{E_p}dv_{\widetilde{U}_1}(\widetilde{y})dv_{\widetilde{U}_2}(\widetilde{x}).\end{aligned} \tag{3.5.13}$$

By Definition 3.5.6, we have $m_{X,\Lambda_2} = |G_V|/|G_{U_2}|$, and then $m_{\Lambda_1,y} = |G_y^{\Lambda_1}| = |G_{U_1}|$ for U_1 small enough. In the case of discrete intersection, we take $y \in \iota_2^{-1}(\iota_1(\Lambda_1)\cap\iota_2(\Lambda_2))$ and $V \in \mathscr{U}_X$ to be a small enough neighbourhood of $\iota_1(y) \in X$ to get (3.5.11) in the case $F = \mathrm{Id}_E$ and discrete intersection.

Recall Definition 3.4.1. Let now $\widetilde{W}$ be the lift of some open set $W \in \mathscr{U}_Y$, where is Y the connected component of $\Lambda_1 \cap \Lambda_2$ such that its image by j_1 intersects the support of f_1, and set $l = \dim Y$. In the case of clean intersection, we can follow the proof of Theorem 3.4.4 until (3.4.29) to get an asymptotic expansion of the form (3.4.20), and get from (3.5.13) a sequence $b_r \in \mathscr{C}^\infty(Y,\mathbb{C}),\ r \in \mathbb{N}$ such that

$$\begin{aligned}\langle s_{1,p},s_{2,p}\rangle_p &= \frac{|G_V|}{|G_{U_1}||G_{U_2}|}p^{\frac{l}{2}}\lambda^p\sum_{r=0}^k p^{-r/2}\int_{\widetilde{W}}\widetilde{b}_r(\widetilde{x})dv_{\widetilde{W}}(\widetilde{x}) + O(p^{\frac{l}{2}-\frac{k+1}{2}})\\ &= \frac{|G_V|}{|G_{U_2}|}\frac{|G_W|}{|G_{U_1}|}p^{\frac{l}{2}}\lambda^p\sum_{r=0}^k p^{-r/2}\int_W b_r(x)dv_Y(x) + O(p^{\frac{l}{2}-\frac{k+1}{2}})\\ &= \frac{m_{X,\Lambda_2}}{m_{\Lambda_1,Y}}p^{\frac{l}{2}}\lambda^p\sum_{r=0}^k p^{-r/2}\int_W b_r(x)dv_Y(x) + O(p^{\frac{l}{2}-\frac{k+1}{2}}).\end{aligned} \tag{3.5.14}$$

We can then go on to the proof of Theorem 3.4.4 to get (3.5.11) in the case $F = \mathrm{Id}_E$. Now for the general case, if $F \in \mathscr{C}^\infty(X,\mathrm{End}(E))$ has compact support, we can define its Berezin-Toeplitz quantization by (3.2.13), and it is showed in [51, Lem.6.10] that it satisfies Lemma 3.2.6 as well. Furthermore, the formula (3.3.6) holds in the same way.

Finally, let us consider the case of singular Bohr-Sommerfeld submanifolds. Following (3.5.12)-(3.5.14), it suffices to prove the case $m_{X,Y} = 1$, and as we assumed the intersection locus away from the singular set, we need only to prove the analogue of (3.3.10), and suppose that f has compact support in some $U \in \mathscr{U}_X$.

First recall that the reproducing property gives

$$\begin{aligned}&\|s_{f,p}\|_p^2 = \int_\Lambda\langle s_{f,p}(\iota(x)),\iota_p.\zeta^p f(x)\rangle_{E_p}dv_\Lambda(x)\\ &= \int_U\int_U\langle\widetilde{P}_p(\widetilde{\iota}_V(x),\widetilde{\iota}_V(y))\widetilde{\iota}_p.\widetilde{\zeta}^p\widetilde{f}(y),\widetilde{\iota}_p.\widetilde{\zeta}^p\widetilde{f}(x)\rangle_{E_p}dv_\Lambda(y)dv_\Lambda(x)\\ &= \sum_{g\in G_V}\int_U\int_U\langle F(\widetilde{D}_p)(\widetilde{\iota}_V(x),g\widetilde{\iota}_V(y))g.\widetilde{\iota}_p.\widetilde{\zeta}^p\widetilde{f}(y),\widetilde{\iota}_p.\widetilde{\zeta}^p\widetilde{f}(x)\rangle_{E_p}dv_\Lambda(y)dv_\Lambda(x).\end{aligned} \tag{3.5.15}$$

Now, as G_V acts on $\widetilde{V}$ preserving all the structures and by Definition 3.5.6, the immersion $g\widetilde{\iota}_V$ is an isotropic immersion intersecting $\widetilde{\iota}_V$ cleanly, for any $g \in G_V$. As $F(\widetilde{D}_p)(\cdot,\cdot)$ satisfies the expansion of Lemma 3.2.5, we can then apply Theorem 3.4.4 to compute each term of the last line of (3.5.15). We then have an asymptotic expansion of the form (3.3.12).

To compute the first order term, note that if $g\widetilde{\iota}_V$ and $\widetilde{\iota}_V$ do not coincide, the highest order of the corresponding expansion (3.3.12) is strictly smaller than $n/2$. Thus we need only to consider the subgroup of G_V fixing the image of ι, which contains at least the identity element of G_V. Summing the contributions of all the elements of this subgroup and by (3.4.21), we get a function $b_U \in \mathscr{C}^\infty(U,\mathbb{C})$, depending on f only locally, such that the highest order term of (3.5.15) is given by integration of b_U along U. Now, as $\iota^{-1}(X_{sing})$ is of measure 0, we can pick a sequence $U_n \subset U$, $n \in \mathbb{N}$, of open sets in $\mathscr{U}_\Lambda$ containing $\iota^{-1}(X_{sing})$ and whose measure tends to 0. We can then repeat (3.5.15) replacing U by U_n and use (3.5.10) on the regular part of V to get the following formula for the highest order term, for all $n \in \mathbb{N}$,

$$b_0 = 2^{d/2} \int_{\Lambda \setminus U_n} \langle Ff(x), f(x) \rangle_{\iota^* E} dv_\Lambda(x) + \int_{U_n} b_U(x) dv_\Lambda(x). \tag{3.5.16}$$

As the second term can be made arbitrarily small, we can take the limit of (3.5.16) at n tends to infinity, so that formula (3.4.21) holds for singular Bohr-Sommerfeld submanifolds. □

3.6 Application to relative Poincaré series

Recall that the special linear group

$$SL_2(\mathbb{R}) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a,b,c,d \in \mathbb{R},\ ad - bc = 1 \right\} \tag{3.6.1}$$

acts on the Poincaré upper-half plane $\mathbb{H} := \left\{ z = x + \sqrt{-1}y \in \mathbb{C} \;\middle|\; y > 0 \right\}$ by the formula

$$g.z := \frac{az+b}{cz+d}. \tag{3.6.2}$$

The induced action on the canonical line bundle $K_{\mathbb{H}} = T^{*(1,0)}\mathbb{H}$ over $\mathbb{H}$ is given on the canonical section dz by

$$g.dz = (cz+d)^2 dz =: j(g,z)^2 dz. \tag{3.6.3}$$

Let $g^{T\mathbb{H}}$ be the *hyperbolic metric* on $\mathbb{H}$, defined by the formula

$$g^{T\mathbb{H}} = \frac{dx^2 + dy^2}{y^2}, \tag{3.6.4}$$

so that the associated Kähler metric $\omega_{\mathbb{H}}$ satisfies

$$\omega_{\mathbb{H}} = \frac{\sqrt{-1}}{2}\frac{dz \wedge d\bar{z}}{y^2}. \tag{3.6.5}$$

Let us write $|\cdot|_{K_{\mathbb{H}}}$ for the Hermitian norm induced by $g^{T\mathbb{H}}$ on $K_{\mathbb{H}}$, which is given by

$$|dz|_{K_{\mathbb{H}}} = y. \tag{3.6.6}$$

Note that the group $SL_2(\mathbb{R})$ acts on $\mathbb{H}$ by holomorphic isometries. Thus if Γ is a discrete subgroup of $SL_2(\mathbb{R})$, the quotient $X := \mathbb{H}/\Gamma$ has an induced structure of a Kähler orbifold, and its canonical line bundle K_X is the quotient of $K_{\mathbb{H}}$ by the induced action (3.6.3). We denote g^{TX} and ω_X for the quotient metric and quotient Kähler form on X respectively.

Let $\mathscr{F}$ be a measurable fundamental domain of Γ in $\mathbb{H}$. Through the natural identification $\mathscr{C}^\infty(X, K_X) \simeq \mathscr{C}^\infty(\mathbb{H}, K_{\mathbb{H}})^\Gamma$ and trivializing $K_{\mathbb{H}}$ using its canonical section dz, we have from (3.6.3) and for any $p \in \mathbb{N}^*$ the following natural identification,

$$\begin{split} H^0_{(2)}(X, K_X^p) \simeq \Big\{ f \in \mathscr{C}^\infty(\mathbb{H}) \ \Big| \ f \text{ holomorphic}, \\ f(g.z) = f(z) j(g,z)^{2p}, \ \int_{\mathscr{F}} |f(z)|^2 y^{2p-2} dx dy < \infty \Big\}. \end{split} \tag{3.6.7}$$

This identification will be used implicitly throughout the rest of this section.

Remark 3.6.1. Assume $\mathrm{Vol}(X) < +\infty$, that is Γ is a *Fuchsian group of the first kind*. As explained in [2, § 6], the space $H^0_{(2)}(X, K_X^p)$ is then identified through the identification (3.6.7) with the space $S_{2p}(\Gamma)$ of *cusp forms of weight* $2p$, the space of holomorphic functions on $\mathbb{H}$ satisfying the equivariance property of (3.6.7) and vanishing at infinity. Such spaces are of particular interest in arithmetic.

To set the previous discussion in the context of Section 3.2, we use the classical fact that the curvature of the Chern connection $\nabla^{K_{\mathbb{H}}}$ on $K_{\mathbb{H}}$ satisfies the condition (3.1.1) for the renormalized Kähler form $\omega_{\mathbb{H}}/2\pi$. As $SL_2(\mathbb{R})$ acts by holomorphic isometries, (3.1.1) holds for K_X as well. Furthermore, as $R^{det} = -R^{K_X}$ is proportional to $\sqrt{-1}\omega_X$, it is easily seen that K_X satisfies (3.5.1). Therefore, setting $L = K_X$ and $E = \mathbb{C}$, we are precisely in the context of the previous Sections for the renormalized Kähler form $\omega = \omega_X/2\pi$, with $g^{TX}_\omega = g^{TX}/2\pi$.

Recall that a smooth path $\gamma : [0, l] \to X$, $l > 0$, is said to be a *closed loop* if it induces a (singular) immersion $\widetilde{\gamma} : S^1 \to X$ by identification of 0 with L. The following lemma describes the class of (singular) Bohr-Sommerfeld submanifolds we will be interested in.

Lemma 3.6.2. *For $l > 0$, let $\gamma : [0, l] \to X$ be a closed loop in X parametrized by arclength with respect to g^{TX}, and suppose that the holonomy of K_X along γ with respect to $\nabla^{K_{\mathbb{H}}}$ is trivial. Then the immersion $\widetilde{\gamma} : S^1 \to X$, obtained from γ by identification of* 0 *and l, satisfies the Bohr-Sommerfeld condition of Definition 3.3.1.*

Proof. As ω_X is a 2-form, any smooth map $f : S^1 \to X$ satisfies $f^*\omega = 0$. Thus as $\dim X = 2$, any immersion $\iota : S^1 \to X$ is Lagrangian. By Remark 3.3.2, it satisfies the Bohr-Sommerfeld condition if and only if the holonomy of the pullback connection is trivial, which is exactly the hypothesis of Lemma 3.6.2 by Remark 3.3.2. □

In any case, such a path $\gamma : [0,l] \to X,\ l > 0$, is called a *Bohr-Sommerfeld curve.* The orientation on $\widetilde{\gamma} : S^1 \to X$ is determined by the canonical vector field $\frac{\partial}{\partial t}$ on $[0,L]$. Following Remark 3.3.2, if $\gamma : [0,l] \to X,\ l > 0$, is a smooth closed loop such that its holonomy is a k-th root of unity for some $k \in \mathbb{N}$, we can take a cover of degree k of this loop to get a Bohr-Sommerfeld curve $\gamma_k : [0,kl] \to X$.

Note that as X is a complex orbifold with $\dim_{\mathbb{C}} X = 1$ and as Γ acts on $\mathbb{H}$ holomorphically, the singular set X_{sing} is necessarily a discrete set. By Definition 3.5.6 and as S^1 is a manifold, the stabilizer of $\widetilde{\gamma}$ is then necessarily trivial in any case.

Corollary 3.6.3. *A closed geodesic loop $\gamma : [0,l] \to X,\ l > 0$, parametrized by arclength, is a Bohr-Sommerfeld curve.*

Proof. Recall that $K_X = T^{*(1,0)}X$ is equipped with the Hermitian metric and connection h^{K_X}, ∇^{K_X} induced by g^{TX}, ∇^{TX} via (3.2.1). For any $t \in [0,l]$, let $\dot{\gamma}_t \in T_{\gamma(t)}X$ denote the vector tangent to the curve $\gamma : [0,l] \to X$, inducing $\dot{\gamma}_t^{(0,1)} \in T^{(0,1)}X$ via (3.2.1). We write $\dot{\gamma}_t^{(0,1),*} \in K_{X,\gamma(t)}$ for its metric dual. As $\gamma : [0,l] \to X$ is geodesic, we know that $\nabla^{TX}_{\dot{\gamma}}\dot{\gamma} = 0$, so that $\nabla^{K_X}_{\dot{\gamma}}\dot{\gamma}^{(0,1),*} = 0$, which means precisely that $\widetilde{\gamma} : S^1 \to X$ satisfies the Bohr-Sommerfeld condition with associated section $\gamma^{(0,1),*} \in \mathscr{C}^\infty(S^1, \widetilde{\gamma}^*K_X)$.

Now if X is an orbifold and if $z \in X$ is a singular point of X, then its associated group G_z^X preserves the Riemannian structure, and sends a geodesic through z to another geodesic through z, which intersect transversally bu unicity of the geodesics. Thus $\gamma : [0,l] \to X$ satisfies the definition of a singular immersion as in Definition 3.5.6. □

Let $\gamma : [0,l] \to X,\ l > 0$, be a Bohr-Sommerfeld curve together with a unitary flat section $\zeta \in \mathscr{C}^\infty([0,L], \gamma^*K_X)$, inducing a (possibly singular) Bohr-Sommerfeld submanifold $(S^1, \widetilde{\gamma}, \zeta)$ as above. For any $p \in \mathbb{N}^*$, we define $s_{\gamma,p} \in H^0_{(2)}(X, K_X^p)$ by

$$s_{\gamma,p}(x) = \int_0^L P_p^X(x, \gamma(t))\gamma_p.\zeta^p(t)dt, \tag{3.6.8}$$

for any $x \in X$, where $P_p^X(\cdot,\cdot)$ is the Bergman kernel with respect to dv_X of the orthogonal projection on $H^0_{(2)}(X, K_X^p)$. Then $s_{\gamma,p}$ is precisely the Lagrangian state associated to $(S^1, \widetilde{\gamma}, \zeta)$ and $f = 1$, in the sense of Definition 3.3.3.

We can then apply Theorem 3.5.3 and Theorem 3.5.9 to get the following specialisation of (3.3.11) and (3.4.4), where we adopt the convention that $\sqrt{-a} = \sqrt{-1}\sqrt{a}$ if $a > 0$.

Theorem 3.6.4. *Let $\gamma : [0,l] \to X,\ l > 0$, be a Bohr-Sommerfeld curve, and let $\{s_{\gamma,p}\}_{p\in\mathbb{N}^*}$ be as in* (3.6.8)*. Then*

$$\|s_{\gamma,p}\|^2_{L^2} = \left(\frac{p}{\pi}\right)^{1/2} l + O(p^{-1/2}). \tag{3.6.9}$$

Furthermore, if γ_1 and γ_2 are two Bohr-Sommerfeld curves intersecting cleanly away from the singular set, we get

$$\langle s_{\gamma_1,p}, s_{\gamma_2,p}\rangle = \sqrt{2}\sum_{z\in\gamma_1\cap\gamma_2}\sum_{\substack{t_1,t_2>0,\\ \gamma_1(t_1)=\gamma_2(t_2)=z}}\lambda_{t_1,t_2}^p\frac{e^{\sqrt{-1}(\theta_z/2-\pi/4)}}{\sqrt{\sin(\theta_z)}} + O(p^{-1}), \tag{3.6.10}$$

where $\theta_z \in [0, 2\pi[$ is the oriented angle from γ_1 to γ_2 at z and where for all $t_1, t_2 > 0$ such that $\gamma_1(t_1) = \gamma_2(t_2)$, we define $\lambda_{t_1,t_2} = \langle \gamma_1^L.\zeta_1(t_1), \gamma_2^L.\zeta_2(t_2)\rangle_{K_X}$.

Proof. In the case X smooth and compact, (3.6.9) and (3.6.10) are standard computations from (3.3.24) and (3.4.4). We will indicate how to modify directly the argument to get the case $g^{TX} = 2\pi g^{TX}_\omega$ from the case $g^{TX} = g^{TX}_\omega$ in all generality.

For any $p \in \mathbb{N}^*$, let us write $P_{p,\omega}$ for the orthogonal projection to $H^0_{(2)}(X, K_X^p)$ with respect to the L^2-Hermitian product induced by g^{TX}_ω. Then $P_{p,\omega} = P_p^X$, but $dv_{X,\omega} = dv_X/2\pi$, so that the associated Bergman kernel with respect to $dv_{X,\omega}$ satisfies $P_{p,\omega}(\cdot,\cdot) = 2\pi P_p^X(\cdot,\cdot)$. On another hand, the Riemannian volume form dt_ω on $[0, L[$ induced by g^{TX}_ω satisfies $dt_\omega = dt/\sqrt{2\pi}$. Thus, writing $\{s_{\omega,\gamma,p}\}_{p\in\mathbb{N}^*}$ for the Lagrangian state obtained replacing g^{TX} by g^{TX}_ω, we get from (3.3.2) that $s_{\omega,\gamma,p} = \sqrt{2\pi}s_{\gamma,p}$ for any $p \in \mathbb{N}^*$.

Consider now two Bohr-Sommerfeld curves γ_1 and γ_2. Following the above notations, we get for any $p \in \mathbb{N}^*$,

$$\langle s_{\gamma_1,p}, s_{\gamma_2,p}\rangle_p = \frac{1}{2\pi}\int_X \langle s_{\omega,\gamma_1,p}, s_{\omega,\gamma_2,p}\rangle_{K_X^p} dv_X = \langle s_{\omega,\gamma_1,p}, s_{\omega,\gamma_2,p}\rangle_{\omega,p}, \tag{3.6.11}$$

where $\langle\cdot,\cdot\rangle_{\omega,p}$ denote the L^2-Hermitian product with respect to g^{TX}_ω. Noticing finally that $\mathrm{Vol}_\omega(\gamma) = l/\sqrt{2\pi}$ for any $\gamma : [0, l] \to X$, $l > 0$ parametrized by arclength with respect to g^{TX}, we recover (3.6.9) and (3.6.10) as in the case of X smooth and compact. □

In the case where X is a compact Riemann surface, so that in particular Γ acts freely on $\mathbb{H}$, Theorem 3.6.4 is the result of [11, Th.4.4]. As shown in Proposition 3.6.6, formulas (3.6.9) and (3.6.10) are especially interesting in the case of curves $\gamma : \mathbb{R} \to \mathbb{H}$ such that there exists $l > 0$, $g_0 \in \Gamma$ satisfying $g_0.\gamma(t) = \gamma(t + l)$ for any $t \in \mathbb{R}$. We say that γ is *associated with* g_0.

In particular, if γ is a closed geodesic, then γ is associated with an *hyperbolic* element $g_0 \in \Gamma$, that is satisfying $\mathrm{Tr}(g_0) > 2$, unique up to conjugation. Closed geodesics belong to a larger class of hyperbolic curves called *hypercycles*.

If $g_0 \in \Gamma$ is *parabolic*, that is satisfying $\mathrm{Tr}(g_0) = 2$, then its action has no fixed points in $\mathbb{H}$, and it occurs in Γ only in the case of X non-compact. The most interesting associated curves in that case are the so-called *horocycles*, which are isometric to a horizontal line in $\mathbb{H}$.

If $g_0 \in \Gamma$ is *elliptic*, that is satisfying $\mathrm{Tr}(g_0) < 2$, then g_0 fixes a unique point $z \in \mathbb{H}$, which descends to a singular point of X. The most interesting associated curves in that case are circles with center the fixed point of g_0 in $\mathbb{H}$. Note that Γ acts freely on $\mathbb{H}$ if and only if it contains no elliptic elements.

Our next goal is to identify the Lagrangian states associated with such curves. The following result is classical and follows for instance from [28, Prop.5.3, § II.1].

Proposition 3.6.5. *For any $p \in \mathbb{N}^*$, the Bergman kernel of $H^0_{(2)}(\mathbb{H}, K^p_{\mathbb{H}})$ satisfies the formula*

$$P_p^{\mathbb{H}}(z,w) = (-1)^p \frac{2^{2p-2}(2p-1)}{\pi(z-\overline{w})^{2p}} dz^p d\overline{w}^p, \tag{3.6.12}$$

for any $z, w \in \mathbb{H}$, where $d\overline{w} \in \overline{K}_{\mathbb{H},w} \simeq K^*_{\mathbb{H},w}$ denotes the metric dual of $dw \in K_{\mathbb{H},w}$. Furthermore, through the identification (3.6.7)*, we have*

$$P_p^X(z,w) = \sum_{g\in\Gamma} P_p^{\mathbb{H}}(z, g.w) j(g,w)^{2p}, \tag{3.6.13}$$

where the convergence of the right-hand side is absolute and uniform in z, w in any compact set of $\mathbb{H}$.

The series (3.6.13) is an example of *Poincaré series*, and is a standard method to construct functions in $S_{2p}(\Gamma)$ as in Remark 3.6.1. A fundamental problem of the theory of cusp forms is to decide whether a given series vanishes identically or not.

If $\Gamma_0 \subset \Gamma$ is a subroup of Γ, let us write Γ/Γ_0 for the set of equivalence classes $[g] := \{gg_0 \in \Gamma \mid g_0 \in \Gamma_0\}$ for all $g \in \Gamma$. Recall that if g_0 is hyperbolic or parabolic, it generates a free group $\Gamma_0 \subset \Gamma$, whereas if g_0 is elliptic, it generates a cyclic subgroup $\Gamma_0 \subset \Gamma$.

Using Proposition 3.6.5 and a classical unfolding technique, we get explicit formulas for the Lagrangian states associated with remarkable curves. This is described in the next result.

Proposition 3.6.6. *Let $g_0 \in \Gamma$, and let $\gamma : \mathbb{R} \to \mathbb{H}$ be a smooth curve on $\mathbb{H}$ parametrized by arclength, together with a unitary flat section $\zeta \in \gamma^* K_{\mathbb{H}}$, such that there is an $l > 0$ satisfying $g_0.\gamma(t) = \gamma(t+l)$ and $g_0.\zeta(t) = \zeta(t+l)$ for all $t \in \mathbb{R}$. Write $\Gamma_0 \subset \Gamma$ for the subgroup generated by g_0.*

If g_0 is hyperbolic or parabolic, then the Lagrangian state $\{s_{\gamma,p}\}_{p\in\mathbb{N}^*}$ associated to γ is given through (3.6.7) *and for any $p \in \mathbb{N}^*$ by*

$$s_{\gamma,p}(z) = (-1)^p \frac{2^{2p-2}(2p-1)}{\pi} \sum_{[g]\in\Gamma/\Gamma_0} \int_{-\infty}^{+\infty} \left(z - \overline{g.\gamma(t)}\right)^{-2p} \zeta(t) j(g,\gamma(t))^{2p} dt. \tag{3.6.14}$$

If g_0 is elliptic, then letting $n \in \mathbb{N}$ be the order of Γ_0, the Lagrangian state $\{s_{\gamma,p}\}_{p\in\mathbb{N}^*}$ is given through (3.6.7) *and for any $p \in \mathbb{N}^*$ by*

$$s_{\gamma,p}(z) = (-1)^p \frac{2^{2p-2}(2p-1)}{\pi} \sum_{[g]\in\Gamma/\Gamma_0} \int_0^n \left(z - \overline{g.\gamma(t)}\right)^{-2p} \zeta(t) j(g,\gamma(t))^{2p} dt. \tag{3.6.15}$$

The convergence of the series in (3.6.14) *and* (3.6.15) *are absolute and uniform in z in any compact set of $\mathbb{H}$.*

Proof. First note that by definition, for $g, g' \in SL_2(\mathbb{R})$ and $w \in \mathbb{H}$, we have $j(gg', w) = j(g, g'w)$. Then from (3.6.8) and from the uniform convergence of (3.6.13), if $g_0 \in \Gamma$ is hyperbolic or parabolic, we get

$$\begin{aligned} s_{\gamma,p}(z) &= \sum_{g\in\Gamma} \int_\gamma P_p^{\mathbb{H}}(z, g.\gamma(t))\zeta(t) j(g,\gamma(t))^{2p} dt \\ &= \sum_{[g]\in\Gamma/\Gamma_0} \sum_{n\in\mathbb{Z}} \int_0^L P_p^{\mathbb{H}}(z, gg_0^n.\gamma(t))\zeta(t) j(gg_0^n,\gamma(t))^{2p} dt \\ &= \sum_{[g]\in\Gamma/\Gamma_0} \sum_{n\in\mathbb{Z}} \int_{nL}^{(n+1)L} P_p^{\mathbb{H}}(z, g.\gamma(t))\zeta(t) j(g,\gamma(t))^{2p} dt \\ &= \sum_{[g]\in\Gamma/\Gamma_0} \int_{-\infty}^{+\infty} P_p^{\mathbb{H}}(z, g.\gamma(t))\zeta(t) j(g,\gamma(t))^{2p} dt, \end{aligned} \tag{3.6.16}$$

and we conclude by (3.6.12). Note that the sums in (3.6.16) do not depend on the choice of the representatives $g \in \Gamma$ of any $[g] \in \Gamma/\Gamma_0$. The elliptic case (3.6.15) is strictly analogous. □

The series (3.6.14) and (3.6.15) are called *relative Poincaré series*. We can now state our main theorem, which is a consequence of Theorem 3.6.4.

Theorem 3.6.7. *If $\gamma : \mathbb{R} \to \mathbb{H}$ satisfying the hypotheses of Proposition 3.6.6 descends to a Bohr-Sommerfeld curve, then there is a $p_0 \in \mathbb{N}$ such that the associated series* (3.6.14) *or* (3.6.15) *do not vanish identically for $p > p_0$. This holds in particular if $\gamma : \mathbb{R} \to \mathbb{H}$ is a closed geodesic.*

Proof. By (3.6.9), we know that there is $p_0 \in \mathbb{N}$ such that $s_{\gamma,p}$ is non-vanishing for $p > p_0$, so that we may conclude by Corollary 3.6.3 and Proposition 3.6.6. □

In general, there are simple numerical criterions for horocycles, circles and hypercycles to satisfy the Bohr-Sommerfeld condition, and the integral in the sums (3.6.14) and (3.6.15) can be computed explicitly using Proposition 3.6.5 and elementary complex analysis. In particular, as computed in [11, Th.4.11], if $g_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a hyperbolic element of Γ, the series (3.6.14) for γ closed geodesic associated with g_0 takes the form

$$s_{\gamma,p}(z) = \sum_{[g]\in\Gamma/\Gamma_0} j(g,z)^{-2p} \Big(c(g.z)^2 + (d-a)(g.z) - b\Big)^{-p}, \tag{3.6.17}$$

where the convergence is uniform in z in any compact set of $\mathbb{H}$, and we recover (up to normalisation) the relative Poincaré series associated to closed hyperbolic geodesics by Katok [40, § 1]. Furthermore, we get from Theorem 3.6.4 a formula for the highest order term in $p \in \mathbb{N}$ of the intersection product of two closed geodesics, recovering a result of [40, Th.3]. As showed in [40, Th.1], if Γ is a Fuchsian group of the first kind, the series associated to the primitive hyperbolic elements of Γ as above generate the whole space $S_{2p}(\Gamma)$.

Finally, note that there are many discrete subgroups $\Gamma \subset SL_2(\mathbb{R})$ of interest containing elliptic points and leading to non-compact quotients $X = \mathbb{H}/\Gamma$, even in the case of Γ Fuchsian group of the first kind. The most famous examples are the classical modular curves.

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