Journal of Functional Analysis 282 (2022) 109400

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Balanced metrics for Kähler-Ricci solitons and quantized Futaki invariants

Louis Ioos

Max Planck Institute for Mathematics, Bonn 53111, Germany

A R T I C L E   I N F O

Article history:
Received 22 June 2021
Accepted 11 January 2022
Available online 24 January 2022
Communicated by Natasa Sesum

Keywords:
Kähler-Ricci solitons
Balanced metrics
Berezin-Toeplitz quantization
Futaki invariants

A B S T R A C T

We show that a Kähler-Ricci soliton on a Fano manifold can always be smoothly approximated by a sequence of relative anticanonically balanced metrics, also called quantized Kähler-Ricci solitons. The proof uses a semiclassical estimate on the spectral gap of an equivariant Berezin transform to extend a strategy due to Donaldson, and can be seen as the quantization of a method due to Tian and Zhu, using quantized Futaki invariants as obstructions for quantized Kähler-Ricci solitons. As corollaries, we recover the uniqueness of Kähler-Ricci solitons up to automorphisms, and show how our result also applies to Kähler-Einstein Fano manifolds with general automorphism group.

1. Introduction

The purpose of this paper is to use a relative extension of the notion of balanced metrics, first introduced by Donaldson in [13], in order to approximate Kähler-Ricci solitons on Fano manifolds.

E-mail address: ioos@mpim-bonn.mpg.de.
URL: https://louisioos.github.io.

https://doi.org/10.1016/j.jfa.2022.109400

Recall that a compact complex manifold X is a *Fano manifold* if its anticanonical line bundle $L := \det(T^{(1,0)}X)$ is *ample*. Thanks to a classical theorem of Kodaira, this means that L admits a *positive Hermitian metric* $h \in \mathrm{Met}^+(L)$, so that its Chern curvature $R_h \in \Omega^2(X, \mathbb{C})$ induces a Kähler form on X via the formula

$$\omega_h := \frac{\sqrt{-1}}{2\pi} R_h \,. \tag{1.1}$$

On the other hand, the group $\mathrm{Aut}(X)$ of holomorphic diffeomorphisms of X is a finite dimensional complex Lie group, inducing a complex embedding of its Lie algebra $\mathrm{Lie}\,\mathrm{Aut}(X)$ into the Lie algebra $\mathscr{C}^\infty(X, TX)$ of real vector fields over X, endowed with the complex structure $J \in \mathrm{End}(TX)$. A Kähler form $\omega_h \in \Omega^2(X, \mathbb{R})$ is called a *Kähler-Ricci soliton* with respect to $\xi \in \mathrm{Lie}\,\mathrm{Aut}(X)$ if it satisfies $L_{J\xi}\omega_h = 0$ and

$$\mathrm{Ric}(\omega_h) - \omega_h = L_\xi\, \omega_h \,, \tag{1.2}$$

where $\mathrm{Ric}(\omega_h) \in \Omega^2(X, \mathbb{R})$ denotes the Ricci form of ω_h, and L_η denotes the Lie derivative along $\eta \in \mathrm{Lie}\,\mathrm{Aut}(X)$. This definition coincides with the definition of Tian and Zhu in [43]. In the case $\xi = 0$, we recover the notion of a *Kähler-Einstein metric*.

Fix now $p \in \mathbb{N}^*$, and consider the space $H^0(X, L^p)$ of holomorphic sections of the p-th tensor power $L^p := L^{\otimes p}$. Let $h^p \in \mathrm{Met}^+(L^p)$ be a positive Hermitian metric on L^p, and consider the induced L^2-Hermitian inner product $L^2(h^p)$ defined on s_1, $s_2 \in H^0(X, L^p)$ by

$$\langle s_1, s_2 \rangle_{L^2(h^p)} := \int_X \langle s_1(x), s_2(x) \rangle_{h^p}\, d\nu_h(x) \,, \tag{1.3}$$

where $d\nu_h$ is the *anticanonical volume form* associated with the induced metric $h \in \mathrm{Met}^+(L)$ on L, defined over any contractible open subset $U \subset X$ by the formula

$$d\nu_h := \sqrt{-1}^{\,n^2} \frac{\theta \wedge \bar{\theta}}{|\theta|^2_{h^{-1}}} \,, \tag{1.4}$$

independent of $\theta \in \mathscr{C}^\infty(U, \det(T^{(1,0)*}X))$ non-vanishing, where $h^{-1} \in \mathrm{Met}(L^*)$ denotes the dual Hermitian metric on $L^* = \det(T^{(1,0)*}X)$. On the other hand, by definition of L as an ample line bundle, it induces an embedding of X into the projective space of hyperplanes in $H^0(X, L^p)$ for any $p \in \mathbb{N}^*$ large enough, called Kodaira embedding. Given a Hermitian inner product H on $H^0(X, L^p)$, we can then endow L^p with the positive Hermitian metric $\mathrm{FS}(H) \in \mathrm{Met}^+(L^p)$ induced by the Fubini-Study metric. A positive Hermitian metric $h^p \in \mathrm{Met}^+(L^p)$ is called *anticanonically balanced relative to* $\xi \in \mathrm{Lie}\,\mathrm{Aut}(X)$ if it satisfies $L_{J\xi}\omega_{h^p} = 0$ and

$$\omega_{h^p} = \phi^*_{\xi/2p}\, \omega_{\mathrm{FS}(L^2(h^p))} \,, \tag{1.5}$$

where $\phi_\eta \in \mathrm{Aut}(X)$ exponentiates $\eta \in \mathrm{Lie}\,\mathrm{Aut}(X)$. In the case $\xi = 0$, we recover the usual notion of an anticanonically balanced metrics, introduced by Donaldson in [14, §2.2.2].

The main result of this paper is the following Theorem, which we prove in Section 6. For any $m \in \mathbb{N}$, let $|\cdot|_{\mathscr{C}^m}$ be a fixed $\mathscr{C}^m$-norm on $\Omega^2(X,\mathbb{R})$, and write $\mathrm{Aut}_0(X)$ for the identity component of $\mathrm{Aut}(X)$.

Theorem 1.1. *Let $\omega_{h_\infty} \in \Omega^2(X,\mathbb{R})$ be a Kähler-Ricci soliton with respect to $\xi_\infty \in \mathrm{Lie}\,\mathrm{Aut}(X)$. Then for any $m \in \mathbb{N}$, there exists $C_m > 0$ and anticanonically balanced metrics $h^p \in \mathrm{Met}^+(L^p)$ relative to $\xi_p \in \mathrm{Lie}\,\mathrm{Aut}(X)$ for all $p \in \mathbb{N}^*$ big enough such that*

$$\xi_p \xrightarrow{p\to+\infty} \xi_\infty \quad \text{and} \quad \left|\frac{1}{p}\omega_{h^p} - \omega_{h_\infty}\right|_{\mathscr{C}^m} \leqslant \frac{C_m}{p}. \tag{1.6}$$

Furthermore, if $\widetilde{h}^p \in \mathrm{Met}^+(L^p)$ is another anticanonically balanced metric relative to $\widetilde{\xi}_p \in \mathrm{Lie}\,\mathrm{Aut}(X)$ for some $p \in \mathbb{N}^*$ big enough, then there exists $\phi \in \mathrm{Aut}_0(X)$ such that $\phi^*\widetilde{\xi}_p = \xi_p$ and $\phi^*\omega_{\widetilde{h}^p} = \omega_{h^p}$.

Theorem 1.1 answers a question of Donaldson in [14, §2.2.2]. This question has also been studied in previous works of Berman and Witt Nyström in [3, Th. 1.7] and Takahashi in [41, Th. 1.2], where the convergence in Theorem 1.1 is established in the weak sense of currents, and under the assumption that a modified Ding functional over the infinite dimensional space $\mathrm{Met}^+(L)$ is coercive modulo $\mathrm{Aut}_0(X)$. By contrast, our proof closely follows the finite dimensional method of Donaldson in [13], and relies on methods of *Berezin-Toeplitz quantization*. We hope that our approach can help to shed light on the different notions of stability in this context, following the work of Saito and Takahashi in [38]. Relative anticanonically balanced metrics were introduced in [3, §4.2.2] under the name of *quantized Kähler-Ricci solitons*.

As a straightforward consequence of Theorem 1.1, we get an alternative proof of the following result of Tian and Zhu in [43, Th. 1.1] and [44, Th. 3.2], which does not rely on solving a Monge-Ampère equation.

Corollary 1.2. *Let $\omega_h, \omega_{\widetilde{h}} \in \Omega^2(X,\mathbb{R})$ be Kähler-Ricci solitons with respect to $\xi, \widetilde{\xi} \in \mathrm{Lie}\,\mathrm{Aut}(X)$ respectively. Then there exists $\phi \in \mathrm{Aut}_0(X)$ such that $\phi^*\widetilde{\xi} = \xi$ and $\phi^*\omega_{\widetilde{h}} = \omega_h$.*

Let us point out that the coercivity assumption used in the proofs of [3, Th. 1.7] and [41, Th. 1.2] was shown to be a consequence of the existence of a Kähler-Ricci soliton by Darvas and Rubinstein in [11, Th. 8.1], but this last result actually uses Corollary 1.2, so that this does not lead to an alternative proof.

The proof of Theorem 1.1 follows the general strategy of Donaldson in [13], who established an analogue of Theorem 1.1 in the case of $\mathrm{Aut}(X)$ discrete, showing that a polarized Kähler metric of constant scalar curvature can always be approximated by a sequence of *balanced metrics*, defined as in (1.5) for $\xi = 0$ using the usual Liouville form instead of the anticanonical volume form (1.4) in the L^2-Hermitian product (1.3). Specifically, our Proposition 6.2 uses Donaldson's method of constructing approximately balanced metrics via the asymptotic expansion of the *Bergman kernel* along the diagonal, which we recall in Theorem 2.9, and our Proposition 6.7 is a straightforward adaptation of a fundamental Lemma of Donaldson on the convergence of the gradient flow of the norm squared of the associated *moment map* close to a zero.

The main difficulty lies instead in the most delicate part of Donaldson's strategy, given in [13, §3.2] in order to establish the key estimate [13, Cor. 22]. This part of the proof gives an estimate from below of the derivative of the associated moment map, and has already been improved and clarified by Phong and Sturm in [34, Th. 2]. In the situation of [13,34], the derivative of the moment map has a natural geometric interpretation, and this gives a natural approach to estimate the lower bound. Unfortunately, this geometric interpretation does not carry directly to the anticanonical case considered in Theorem 1.1. This difficulty was overcome only recently by Takahashi in [42, Th. 1.3], who established Theorem 1.1 in the case of $\mathrm{Aut}(X)$ discrete. The further extension of this geometric interpretation to the case of general $\mathrm{Aut}(X)$ is an interesting problem, but this has not been achieved yet.

The main novelty of our method is to replace this geometric interpretation by the use of the asymptotics of the *spectral gap of the Berezin transform*, which were first established in [22, Th. 3.1] and which we extend to the equivariant case in Theorem 5.9. These asymptotics are used in a crucial way in Section 6.2 to obtain the necessary estimate from below of the derivative of the appropriate moment map in this context. More precisely, we relate in Proposition 6.4 the derivative of the moment map with the equivariant *Berezin-Toeplitz quantum channel* of Definition 5.3, and we then use our asymptotics of the spectral gap to give an estimate of the Berezin-Toeplitz quantum channel in Theorem 6.5. As shown in Corollary 6.6, this produces the desired lower bound, and shows as a by-product that our estimate is optimal. This extends the strategy used in [21] to establish Theorem 1.1 in the case of $\mathrm{Aut}(X)$ discrete, and allows to bypass the geometric interpretation mentionned above, which has not yet been worked out in the case of Kähler-Ricci solitons. Note on the other hand that the asymptotics of the spectral gap are based on the asymptotic expansion of the Berezin transform recalled in Theorem 2.14, which is in turn a consequence of the asymptotic expansion of the Bergman kernel *outside* the diagonal. Our approach thus gives a unified way, entirely based on Berezin-Toeplitz quantization, to treat both the construction of approximately balanced metrics and the lower bound of the derivative of the moment map, instead of using an additional delicate geometric interpretation for the later.

The advantage of our method of proof of Theorem 1.1 is that it can be adapted in a systematic way to various choices of a volume form in the Hilbert product (1.3), leading to various different notions of balanced metrics. In [21, §2], we described a general set-up in which our method can be applied, which includes the original notion of balanced metrics of [13], but also the *ν-balanced metrics* on Calabi-Yau manifolds and the *canonically balanced metrics* on manifolds with ample canonical line bundle, introduced by Donaldson in [14], as well as the notion of twisted balanced metrics studied by Keller in [25] and Dervan in [12, §2.2]. It can also be extended to the case of balanced metrics over vector bundles, following Wang in [46], and to the case of coupled Kähler-Einstein metrics as in [42]. In all these cases, a new geometric interpretation as in [34] was needed to adapt the proof of [13] successfully. By contrast, our proof gives a general method to deal with this key step, using the asymptotics of the spectral gap of the corresponding Berezin transform, as given in Section 5 following the strategy described in [22, §3].

In Section 3, we study the quantization of the action of the automorphism group to establish a quantized counterpart of the method of Tian and Zhu in [44]. Namely, we show that the holomorphic vector fields $\xi_p \in \operatorname{Lie}\operatorname{Aut}(X)$ of Theorem 1.1 are determined a priori for all $p \in \mathbb{N}^*$, regardless of the existence of a relative balanced metric. As a first step, we show in Corollary 3.4 that for any $p \in \mathbb{N}^*$ large enough, there exists a unique vector field $\xi \in \operatorname{Lie}\operatorname{Aut}(X)$ such that the associated *quantized Futaki invariant* $\operatorname{Fut}_p^\xi : \operatorname{Lie}\operatorname{Aut}(X) \to \mathbb{C}$ vanishes. Following Berman and Witt Nyström in [3, §4.1.1], it is defined for any $\eta \in \operatorname{Lie}\operatorname{Aut}(X)$ by the formula

$$\operatorname{Fut}_p^\xi(\eta) := \operatorname{Tr}\left[L_\eta e^{L_\xi/p}\right], \tag{1.7}$$

where $L_\eta \in \operatorname{End}(H^0(X, L^p))$ denotes the natural action of η on the holomorphic sections of the p-th tensor power of the anticanonical line bundle $\det(T^{(1,0)}X)$. As a second step, we show in Proposition 4.7 that if there exists an anticanonically balanced metric $h^p \in \operatorname{Met}^+(L^p)$ relative to $\xi \in \operatorname{Lie}\operatorname{Aut}(X)$, then the associated quantized Futaki invariant $\operatorname{Fut}_p^\xi : \operatorname{Lie}\operatorname{Aut}(X) \to \mathbb{C}$ vanishes identically. This can be seen as the quantization of Proposition 2.5, which is due to Tian and Zhu in [44, Prop. 3.1], showing that the vector field $\xi_\infty \in \operatorname{Lie}\operatorname{Aut}(X)$ of Theorem 1.1 is determined a priori as the unique holomorphic vector field for which the associated *modified Futaki invariant* vanishes, regardless of the existence of a Kähler-Ricci soliton. This characterization of the vector fields ξ_p for all $p \in \mathbb{N}^*$ plays a crucial role in our proof of Theorem 1.1, and has no analogue in Donaldson's approach in [13], since it assumes $\operatorname{Aut}(X)$ discrete. In particular, we show in Corollary 3.4 that the vector fields ξ_p admit an asymptotic expansion as $p \to +\infty$ with highest order coefficient equal to ξ_∞, which shows the first identity of (1.6) and is used in a crucial way for the construction in Proposition 6.2 of approximately balanced metrics.

As remarked in [3, Rmk. 4.8], it is a consequence of the equivariant Riemman-Roch formula that the vanishing of the quantized Futaki invariants (1.7) relative to $\xi = 0$ for all $p \in \mathbb{N}^*$ big enough is equivalent to the vanishing of all *higher order Futaki invariants* [17] of X. We thus recover the fact, first established by Saito and Takahashi in [38, Lem. 3.2], that the higher order Futaki invariants are an obstruction for the existence of anticanonically balanced metrics in the usual sense, for all $p \in \mathbb{N}^*$ big enough. On the other hand, Ono, Sano and Yotsutani exhibit in [33, Th. 1.5] an example of a toric

Kähler-Einstein Fano manifold with non-vanishing higher order Futaki invariants. This example thus implies the following corollary of Theorem 1.1.

Corollary 1.3. *There is a Fano manifold X with Kähler-Einstein metric $\omega_{h_\infty} \in \Omega^2(X,\mathbb{R})$ such that the anticanonically balanced metrics $h^p \in \mathrm{Met}^+(L^p)$ relative to $\xi_p \in \mathrm{Lie}\,\mathrm{Aut}(X)$ of Theorem 1.1 satisfy $\xi_p \neq 0$ for all $p \in \mathbb{N}^*$ big enough.*

In particular, we recover the fact from [38, Ex. 5.6] that the toric example of [33] does not admit any anticanonically balanced metric in the usual sense, for all $p \in \mathbb{N}^*$ big enough. This was also established in [41, Cor. 1.1] in the sense of currents and under a coercivity assumption on the Ding functional. Corollary 1.3 illustrates the fact that Theorem 1.1 is already interesting in the case of Kähler-Einstein metrics. In fact, it is shown in [2,21] that a Kähler-Einstein metric on a Fano manifold X with $\mathrm{Aut}(X)$ discrete can always be approximated by anticanonically balanced metrics. Corollary 1.3 then shows that the assumption of $\mathrm{Aut}(X)$ discrete is necessary for such a result to hold and that Theorem 1.1 extends this result using relative anticanonically balanced metrics.

In [37], Rubinstein, Tian and Zhang introduce a notion of anticanonically balanced metrics depending on a parameter $\delta > 0$, which coincides with the usual notion of an anticanonically balanced metric when $\delta = 1$. In [37, Prop. 5.10], they show that these balanced metrics with $\delta < 1$ can be used to approximate Kähler-Einstein metrics on Fano manifolds with general $\mathrm{Aut}(X)$ as $\delta \to 1$, with convergence in the sense of currents. In the same way as in Theorem 1.1, our method readily extends to establish smooth convergence. On the other hand, our notion of relative anticanonically balanced metric coincides with the quantized Kähler-Ricci solitons of [3], where the more general notion of a *Kähler-Ricci g-soliton* is considered. Although, we restrict to usual Kähler-Ricci solitons for simplicity, our proof extends to this more general case without any difficulty.

A relative version of balanced metrics has first been introduced by Mabuchi in [30] to study *extremal Kähler metrics*, and can be seen as the relative version of constant scalar curvature metrics in the case when $\mathrm{Aut}(X)$ is not discrete. Instead, the notion of relative anticanonically balanced metrics used in this paper is an analogue of the *relative balanced metrics* introduced by Sano and Tipler in [39], defined as in (1.5) with the anticanonical volume form (1.4) replaced by the usual Liouville form in the L^2-Hermitian product (1.3). Theorem 1.1 is then an anticanonical version of [39, Th. 1.1], where the extremal metric is replaced by a Kähler-Ricci soliton, and Propositions 3.6 and 6.2 were inspired by [39, Lem. 4.5, Th. 5.5]. Closely related notions of relative balanced metrics as quantizations of extremal Kähler metrics have also been introduced by Hashimoto in [20], Mabuchi in [31] and Seyyedali in [40]. All these works establish an analogue of Theorem 1.1 for extremal metrics, extending the geometric interpretation of [13,34] for the lower bound of the moment map in their respective relative settings. We refer to [20, §6] for a detailed comparison between these different notions. We only point out here that the approach of [20] is a quantization of the fact that extremal Kähler metrics are critical points of the *Calabi functional*. In particular, this approach does not extend to our case, since

Kähler-Ricci solitons are not critical points of the anticanonical analogue of the Calabi functional, which is the *Ding functional*.

The theory of Berezin-Toeplitz quantization was first developed by Bordemann, Meinrenken and Schlichenmaier in [6], using the work of Boutet de Monvel and Sjöstrand on the Szegö kernel in [8] and the theory of Toeplitz structures of Boutet de Monvel and Guillemin in [7]. This paper is based instead on the theory of Ma and Marinescu in [28], using the off-diagonal asymptotic expansion of the Bergman kernel established by Dai, Liu and Ma in [10, Th. 4.18']. A comprehensive introduction for this theory can be found in [27]. The point of view of quantum measurement theory on Berezin-Toeplitz quantization adopted in this paper has been advocated by Polterovich in [35,36].

Acknowledgments. The author wishes to thank Pr. Xiaonan Ma for his constant support and Pr. Leonid Polterovich for helpful discussions. The author also wishes to thank the anonymous referees for useful comments and suggestions. This work was supported by the European Research Council Starting grant 757585.

2. Setting

Let X be a compact complex manifold with complex structure $J \in \mathrm{End}(TX)$, and write

$$TX_{\mathbb{C}} = T^{(1,0)}X \oplus T^{(0,1)}X \tag{2.1}$$

for the splitting of the complexification of the tangent bundle TX of X into the eigenspaces of J corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$ respectively. For any vector field $\xi \in \mathscr{C}^\infty(X, TX)$, we will write $\xi^{1,0}$, $\xi^{1,0} \in \mathscr{C}^\infty(X, T^{(1,0)}X)$ for its components with respect to this splitting.

In this paper, we will always assume that X is a Fano manifold, so that the space $\mathrm{Met}^+(L)$ of positive Hermitian metrics on $L := \det(T^{(1,0)}X)$ is not empty. For any $h \in \mathrm{Met}^+(L)$ and $p \in \mathbb{N}^*$, we write $h^p \in \mathrm{Met}^+(L^p)$ for the induced positive Hermitian metric on the p-th tensor power L^p. Conversely, any $h^p \in \mathrm{Met}^+(L^p)$ uniquely determines a positive Hermitian metric $h \in \mathrm{Met}^+(L)$. We will also write $\mathrm{Met}(L)$ for the space of Hermitian metrics on L. We write $\mathscr{C}^\infty(X, L^p)$ for the space of smooth sections of L^p and $H^0(X, L^p) \subset \mathscr{C}^\infty(X, L^p)$ for the subspace of holomorphic sections of L^p over X.

Recall that for any $h \in \mathrm{Met}^+(L)$, the 2-form $\omega_h \in \Omega^2(X, \mathbb{R})$ defined by formula (1.1) is a *Kähler form*, meaning that the following formula defines a Riemannian metric on X,

$$g_h^{TX} := \omega_h(\cdot, J\cdot)\,. \tag{2.2}$$

Note by definition (1.4) of the associated anticanonical volume $d\nu_h$ that for any $f \in \mathscr{C}^\infty(X, \mathbb{R})$, we have

$$d\nu_{e^f h} = e^f \, d\nu_h\,. \tag{2.3}$$

2.1. Action of the automorphism group

Recall that the group $\mathrm{Aut}(X)$ of holomorphic diffeomorphisms of X is a finite dimensional complex Lie group, and so that there is a natural embedding $\mathrm{Lie}\,\mathrm{Aut}(X) \subset \mathscr{C}^\infty(X, TX)$. For any $\xi \in \mathrm{Lie}\,\mathrm{Aut}(X)$, we write $\phi_{t\xi} \in \mathrm{Aut}(X)$, $t \in \mathbb{R}$, for the flow generated by $\xi \in \mathrm{Lie}\,\mathrm{Aut}(X)$. The holomorphic action of $\mathrm{Aut}(X)$ on X lifts naturally to $L := \det(T^{(1,0)}X)$, and for any $\xi \in \mathrm{Lie}\,\mathrm{Aut}(X)$, we write L_ξ for the induced differential operator acting on a smooth section $s \in \mathscr{C}^\infty(X, L)$ by

$$L_\xi s := \frac{\partial}{\partial t}\Big|_{t=0} \phi_{t\xi}^* \, s \,. \tag{2.4}$$

Recall also that definition (1.4) of the anticanonical volume form $d\nu_h$ associated with $h \in \mathrm{Met}(L)$ does not depend on $\theta \in \mathscr{C}^\infty(U, \det(T^{(1,0)*}X))$, so that for all $t \in \mathbb{R}$,

$$\phi_{t\xi}^* \, d\nu_h = \sqrt{-1}^{\,n^2} \frac{\phi_{t\xi}^* \, \theta \wedge \phi_{t\xi}^* \, \bar{\theta}}{|\phi_{t\xi}^* \, \theta|^2_{\phi_{t\xi}^* \, h^{-1}}} = d\nu_{\phi_{t\xi}^* h} \,. \tag{2.5}$$

For any $h \in \mathrm{Met}^+(L)$, write ∇^h for the *Chern connection* of (L, h). We then have the following complex version of the *Kostant formula*.

Definition 2.1. For any $h \in \mathrm{Met}^+(L)$, the associated *holomorphy potential* of $\xi \in \mathrm{Lie}\,\mathrm{Aut}(X)$ is the function $\theta_h(\xi) \in \mathscr{C}^\infty(X, \mathbb{C})$ defined for any $s \in \mathscr{C}^\infty(X, \mathbb{C})$ by the formula

$$\theta_h(\xi) \, s := L_\xi \, s - \nabla^h_\xi \, s \,. \tag{2.6}$$

In the same way as the usual Kostant formula for moment maps, formula (2.6) gives a well-defined scalar function $\theta_h(\xi) \in \mathscr{C}^\infty(X, \mathbb{C})$, which via formula (1.1) for the Kähler form $\omega_h \in \Omega^2(X, \mathbb{R})$ satisfies

$$\iota_{\xi^{1,0}} \, \omega_h = \frac{\sqrt{-1}}{2\pi} \bar{\partial} \, \theta_h(\xi) \,. \tag{2.7}$$

Thanks to the Kodaira vanishing theorem (see for instance [1, Prop. 3.72, (1)] with $\mathscr{L} := K_X$), recall that a Fano manifold X satisfies $H^1(X, \mathbb{C}) = 0$. Then as explained in [18], a fundamental result of Fujiki [15] implies in that case that $\mathrm{Aut}_0(X)$ is a linear complex algebraic group. In particular, it includes the complexification $K_\mathbb{C} \subset \mathrm{Aut}_0(X)$ of any connected compact subgroup $K \subset \mathrm{Aut}_0(X)$, and we have a natural decomposition

$$\mathrm{Lie}\, K_\mathbb{C} = \mathrm{Lie}\, K \oplus \sqrt{-1}\, \mathrm{Lie}\, K \subset \mathrm{Lie}\,\mathrm{Aut}(X) \,. \tag{2.8}$$

The following proposition gives some basic properties of the holomorphy potential.

Proposition 2.2. *For any* $h \in \mathrm{Met}^+(K_X^*)$ *and* $\xi \in \mathrm{Lie}\,\mathrm{Aut}(X)$*, we have*

$$\frac{\partial}{\partial t}\Big|_{t=0} \phi_{t\xi}^* h = -2\,\mathrm{Re}\,\theta_h(\xi)\,h\,. \tag{2.9}$$

Furthermore, we have

$$\int_X \theta_h(\xi)\,d\nu_h = 0\,, \tag{2.10}$$

and for any $f \in \mathscr{C}^\infty(X,\mathbb{R})$*, we have*

$$\theta_{e^f h}(\xi) = \theta_h(\xi) - df.\xi^{1,0}\,. \tag{2.11}$$

Finally, if $K \subset \mathrm{Aut}_0(X)$ *is a compact subgroup preserving* $h \in \mathrm{Met}^+(L)$*, then the map* $\theta_h : \mathrm{Lie}\,K_{\mathbb{C}} \to \mathscr{C}^\infty(X,\mathbb{C})$ *is a* $\mathbb{C}$*-linear embedding, and for any* $\xi \in \sqrt{-1}\,\mathrm{Lie}\,K$*, we have* $\theta_h(\xi) \in \mathscr{C}^\infty(X,\mathbb{R})$*.*

Proof. By definition, the Chern connection ∇^h of any $h \in \mathrm{Met}(K_X^*)^+$ induces the holomorphic structure of L, while the lift of $\mathrm{Aut}(X)$ to $L := \det(T^{(1,0)}X)$ is holomorphic. This implies in particular that for all $h \in \mathrm{Met}(K_X^*)^+$ and all $\xi \in \mathrm{Lie}\,\mathrm{Aut}(X)$, we have $L_{\xi^{0,1}} = \nabla^h_{\xi^{0,1}}$, so that Definition 2.1 implies

$$\theta_h(J\xi) = \sqrt{-1}\theta_h(\xi)\,. \tag{2.12}$$

By the unitarity of the Chern connection, for any $s \in \mathscr{C}^\infty(X, K_X^*)$, Definition 2.1 gives

$$\begin{aligned} \frac{\partial}{\partial t}\Big|_{t=0} |s|^2_{\phi_{t\xi}^* h} &= L_\xi\langle s,s\rangle_h - \langle L_\xi s,s\rangle_h - \langle s, L_\xi s\rangle_h \\ &= \langle(\nabla^h_\xi - L_\xi)s,s\rangle_h + \langle s,(\nabla^h_\xi - L_\xi)s\rangle_h \\ &= -2\,\mathrm{Re}\,\theta_h(\xi)\,|s|^2_h\,. \end{aligned} \tag{2.13}$$

This shows the identity (2.9). Combining formulas (2.3), (2.5) and (2.13), we then get

$$-2\int_X \mathrm{Re}\,\theta_h(\xi)\,d\nu_h = \frac{\partial}{\partial t}\Big|_{t=0} \int_X \phi_{t\xi}^*\,d\nu_h = 0\,. \tag{2.14}$$

Using the fact from formula (2.12) that $\mathrm{Im}\,\theta_h(\xi) = -\mathrm{Re}\,\theta_h(J\xi)$, this implies the identity (2.10).

On the other hand, the identity (2.11) is an immediate consequence of Definition 2.1 and the fact that for any $f \in \mathscr{C}^\infty(X,\mathbb{R})$, we have

$$\nabla^{e^f h} = \nabla^h + df.\xi^{1,0}\,. \tag{2.15}$$

Finally, if $K \subset \mathrm{Aut}_0(X)$ preserves $h \in \mathrm{Met}^+(L)$, then formula (2.13) implies that for any $\xi \in \mathrm{Lie}\, K$, we have $\frac{\sqrt{-1}}{2\pi}\theta_h(\xi) \in \mathscr{C}^\infty(X,\mathbb{R})$, while formula (2.12) implies that the map $\theta_h : \mathrm{Lie}\, K_\mathbb{C} \to \mathscr{C}^\infty(X,\mathbb{C})$ is $\mathbb{C}$-linear for the standard complex structure on both spaces, and formula (2.7) shows that $\theta_h(\xi) \equiv 0$ if and only if $\xi = 0$ by non-degeneracy of ω_h. This concludes the proof. □

Remark 2.3. Note that if $K \subset \mathrm{Aut}_0(X)$ preserves $h \in \mathrm{Met}^+(L)$, Definition 2.1 for the map $\frac{\sqrt{-1}}{2\pi}\theta_h : \mathrm{Lie}\, K \to \mathscr{C}^\infty(X,\mathbb{R})$ reduces to the definition of a *moment map* for the action of K on the symplectic manifold (X,ω_h) via the usual Kostant formula. The usual condition for the moment map is recovered from the real part of formula (2.7) by $\mathbb{C}$-linearity of $\theta_h : \mathrm{Lie}\, K_\mathbb{C} \to \mathscr{C}^\infty(X,\mathbb{C})$.

We will establish the quantized counterpart of the following result of Tian and Zhu in Section 3.2.

Proposition 2.4. [44, Proof of Lemma 2.2, Prop. 2.1] *Let $K \subset \mathrm{Aut}_0(X)$ be a given compact subgroup. Then there exists a strictly convex and proper functional $F : \sqrt{-1}\,\mathrm{Lie}\, K \to \mathbb{R}$, such that for any $\xi \in \sqrt{-1}\,\mathrm{Lie}\, K$ and $h \in \mathrm{Met}^+(L)$, we have*

$$F(\xi) := \int_X e^{\theta_h(\xi)}\, \frac{\omega_h^n}{n!}\,. \tag{2.16}$$

Furthermore, for any $\xi \in \sqrt{-1}\,\mathrm{Lie}\, K$, the following formula for the associated modified Futaki invariant $\mathrm{Fut}_\xi : \sqrt{-1}\,\mathrm{Lie}\, K \to \mathbb{C}$ *at $\eta \in \sqrt{-1}\,\mathrm{Lie}\, K$ does not depend on $h \in$* $\mathrm{Met}^+(L)$

$$\mathrm{Fut}_\xi(\eta) := \int_X \theta_h(\eta)\, e^{\theta_h(\xi)}\, \frac{\omega_h^n}{n!}\,. \tag{2.17}$$

Finally, there exists a unique $\xi_\infty \in \sqrt{-1}\,\mathrm{Lie}\, K$ such that $\mathrm{Fut}_{\xi_\infty} : \sqrt{-1}\,\mathrm{Lie}\, K \to \mathbb{C}$ vanishes identically, which is given by the unique minimizer of $F : \sqrt{-1}\,\mathrm{Lie}\, K \to \mathbb{R}$ and satisfies $[\xi_\infty, \eta] = 0$ for all $\eta \in \mathrm{Lie}\, K_\mathbb{C}$.

2.2. Kähler-Ricci solitons

A Kähler form $\omega \in \Omega^2(X,\mathbb{R})$ on a compact complex manifold X induces a natural Hermitian metric $h_\omega \in \mathrm{Met}(L)$ on $L = \det(T^{(1,0)}X)$, defined using the anticanonical volume form (1.4) by the formula

$$\frac{\omega^n}{n!} = d\nu_{h_\omega}\,. \tag{2.18}$$

The associated *Ricci form* is defined by the formula $\mathrm{Ric}(\omega) := \frac{\sqrt{-1}}{2\pi} R_{h_\omega}$ where $R_{h_\omega} \in \Omega^2(X,\mathbb{C})$ is the Chern curvature of the Hermitian metric $h_\omega \in \mathrm{Met}(L)$. Using Cartan's formula and by definition (2.7) of the holomorphy potential, we see that $h \in \mathrm{Met}^+(L)$ induces a Kähler-Ricci soliton $\omega_h \in \Omega^2(X,\mathbb{R})$ with respect to $\xi \in \mathrm{Lie}\,\mathrm{Aut}(X)$ in the sense of formula (1.2) if and only if

$$\mathrm{Ric}(\omega_h) = \omega_h + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\,\theta_h(\xi)\,. \tag{2.19}$$

By definition of the Chern curvature, this means that there exists a constant $c > 0$ such that $h_{\omega_h} = ce^{\theta_h(\xi)}h$, so that formula (2.18) shows that $\omega_h \in \Omega^2(X,\mathbb{R})$ is a Kähler-Ricci soliton with respect to $\xi \in \mathrm{Lie}\,\mathrm{Aut}(X)$ if and only if there is a constant $c > 0$ such that

$$d\nu_h = ce^{\theta_h(\xi)}\frac{\omega_h^n}{n!}\,. \tag{2.20}$$

Let now $\omega_h \in \Omega^2(X,\mathbb{R})$ be a Kähler-Ricci soliton with respect to $\xi \in \mathrm{Lie}\,\mathrm{Aut}(X)$, and recalling formula (2.2) for the associated Riemannian metric, let $K \subset \mathrm{Aut}_0(X)$ be a connected subgroup of holomorphic isometries of (X, g_h^{TX}). As $L_{J\xi}\omega_h = 0$ by definition, we get that $\xi \in \sqrt{-1}\,\mathrm{Lie}\,K$ in the decomposition (2.8). The following result of Tian and Zhu shows that the modified Futaki invariant of Proposition 2.4 is an obstruction for the existence of Kähler-Ricci solitons. It will play a key role in the construction of approximately balanced metrics in Section 6.1.

Proposition 2.5. [44, Prop. 1.3] *Let $\omega_h \in \Omega^2(X,\mathbb{R})$ be a Kähler-Ricci soliton with respect to $\xi \in \mathrm{Lie}\,\mathrm{Aut}(X)$, and let $K \subset \mathrm{Aut}_0(X)$ be a connected subgroup of holomorphic isometries of (X, g_h^{TX}). Then the associated modified Futaki invariant $\mathrm{Fut}_\xi : \sqrt{-1}\,\mathrm{Lie}\,K \to \mathbb{C}$ of Proposition 2.4 vanishes identically.*

For any compact subgroup $K \subset \mathrm{Aut}_0(X)$, we write $\mathrm{Met}^+(L)^K$ for the space of K-invariant positive Hermitian metrics, and $\mathscr{C}^\infty(X,\mathbb{C})^K$ for the space of K-invariant functions over X. For any $h \in \mathrm{Met}^+(L)$, write Δ_h for the Riemannian Laplacian of (X, g_h^{TX}) acting on $\mathscr{C}^\infty(X,\mathbb{R})$.

Lemma 2.6. *Let $K \subset \mathrm{Aut}_0(X)$ be a connected compact subgroup and let $T \subset K$ be the identity component of its center. For any $h \in \mathrm{Met}^+(L)^K$ and $\xi \in \sqrt{-1}\,\mathrm{Lie}\,T$, the operator $\Delta_h^{(\xi)}$ acting on $f \in \mathscr{C}^\infty(X,\mathbb{R})^K$ by the formula*

$$\Delta_h^{(\xi)} f := \frac{1}{4\pi}\Delta_h f - df.\xi^{1,0}\,, \tag{2.21}$$

is positive and essentially self-adjoint with respect to the scalar product $\langle\cdot,\cdot\rangle_{L^2(h,\xi)}$ defined on $f, g \in \mathscr{C}^\infty(X,\mathbb{R})^K$ by the formula

$$\langle f, g\rangle_{L^2(h,\xi)} = \int_X f\,g\,e^{\theta_h(\xi)}\,\frac{\omega_h^n}{n!}\,. \tag{2.22}$$

Furthermore, we have $\operatorname{Ker}\Delta_h^{(\xi)} = \mathbb{C}$*.*

Proof. Fix $h \in \mathrm{Met}^+(L)^K$ and $\xi \in \sqrt{-1}\,\mathrm{Lie}\,T$, and recall that by definition, we have $J\xi \in \mathrm{Lie}\,T$. Then all $f \in \mathscr{C}^\infty(X,\mathbb{R})^K$ satisfy $df.\xi = 2df.\xi^{1,0}$, and for all $\eta \in \mathrm{Lie}\,K$, we have $[\xi,\eta] = 0$ and $[\Delta_h,\eta] = 0$, so that $\Delta_h^{(\xi)}$ preserves $\mathscr{C}^\infty(X,\mathbb{R})^K$ inside $\mathscr{C}^\infty(X,\mathbb{C})$. Using Proposition 2.2, the imaginary part of the holomorphy potential equation (2.7) gives $2\pi\iota_{J\xi}\omega_h = -d\theta_h(\xi)$. Then writing $\langle\cdot,\cdot\rangle_{g_h^{TX}}$ for the pointwise scalar product on T^*X induced by the Riemannian metric (2.2) and using an integration by part from formulas (2.21) and (2.22), for any $f, g \in \mathscr{C}^\infty(X,\mathbb{C})^K$ we get

$$\langle\Delta_h^{(\xi)} f, g\rangle_{L^2(h,\xi)} = \int_X \langle df, dg\rangle_{g_h^{TX}}\, e^{\theta_h(\xi)}\,\frac{\omega_h^n}{n!}\,. \tag{2.23}$$

This shows that the operator $\Delta_h^{(\xi)}$ given by formula (2.21) is essentially self-adjoint and positive with respect to the scalar product $L^2(h,\xi)$ on $\mathscr{C}^\infty(X,\mathbb{R})^K$, and that its kernel is reduced to the constant functions. □

Let $\omega_{h_\infty} \in \Omega^2(X,\mathbb{R})$ be a Kähler-Ricci soliton with respect to $\xi_\infty \in \mathrm{Lie}\,\mathrm{Aut}(X)$, let $K \subset \mathrm{Aut}_0(X)$ be a connected compact subgroup of isometries of $(X, g_{h_\infty}^{TX})$ and let $T \subset K$ be the identity component of its center. Via Definition 2.1, Proposition 2.5 implies in particular that for any $\eta \in \mathrm{Lie}\,K$ and any $\xi \in \sqrt{-1}\,\mathrm{Lie}\,T$, we have

$$\frac{\partial}{\partial t}\Big|_{t=0}\phi_{t\eta}^*\,\theta_{h_\infty}(\xi) = \theta_{h_\infty}([\eta,\xi]) = 0\,, \tag{2.24}$$

so that Proposition 2.2 implies that $\eta \in \sqrt{-1}\,\mathrm{Lie}\,K$ belongs to $\sqrt{-1}\,\mathrm{Lie}\,T$ if and only if $\theta_{h_\infty}(\eta) \in \mathscr{C}^\infty(X,\mathbb{R})^K$. As explained by Futaki in [16, §4] and using formula (2.20) for Kähler-Ricci solitons, this implies the following result via a straightforward generalization of a theorem of Lichnerowicz and Matsushima in [26,32] restricted to the subspace $\mathscr{C}^\infty(X,\mathbb{R})^K \subset \mathscr{C}^\infty(X,\mathbb{C})$. We also refer to [43, Lem. 2.2] for a self-contained proof.

Proposition 2.7. [16, Prop. 4.1] *Let $h_\infty \in \mathrm{Met}^+(L)$ be a Kähler-Ricci soliton with respect to $\xi_\infty \in \mathrm{Lie}\,\mathrm{Aut}(X)$, let $K \subset \mathrm{Aut}_0(X)$ be the identity component of the group of holomorphic isometries of $(X, g_{h_\infty}^{TX})$ and let $T \subset K$ be the identity component of its center. Then the first positive eigenvalue $\lambda_1(h_\infty,\xi_\infty) > 0$ of $\Delta_{h_\infty}^{(\xi_\infty)}$ acting on $\mathscr{C}^\infty(X,\mathbb{R})^K$ as in Lemma 2.6 satisfies $\lambda_1(h_\infty,\xi_\infty) = 1$, and the associated eigenspace satisfies*

$$\operatorname{Ker}\left(\Delta_{h_\infty}^{(\xi_\infty)} - \mathrm{Id}\right) = \langle\theta_{h_\infty}(\xi)\mid \xi \in \sqrt{-1}\,\mathrm{Lie}\,T\rangle\,. \tag{2.25}$$

2.3. Berezin-Toeplitz quantization

In this Section, we fix a positive Hermitian metric $h \in \mathrm{Met}^+(L)$ on the line bundle $L := \det(T^{(1,0)}X)$, and use the associated volume form $d\nu_h$ given by formula (1.4). For any $p \in \mathbb{N}^*$, we consider the Hermitian product $L^2(h^p)$ on $H^0(X, L^p)$ defined for any $s_1,\ s_2 \in \mathscr{C}^\infty(X, L^p)$ by

$$\langle s_1, s_2 \rangle_{L^2(h^p)} := \int_X \langle s_1(x), s_2(x) \rangle_{h^p} \, d\nu_h(x) \,. \tag{2.26}$$

We write

$$\mathscr{H}_p := \left(H^0(X, L^p), \langle \cdot, \cdot \rangle_{L^2(h^p)} \right) \,, \tag{2.27}$$

for the associated Hilbert space of holomorphic sections, and set $n_p := \dim \mathscr{H}_p$. We write $\mathscr{L}(\mathscr{H}_p)$ for the space of Hermitian endomorphisms of $\mathscr{H}_p$.

By definition of L ample and for all $p \in \mathbb{N}^*$ big enough, the *Kodaira map*

$$\begin{aligned} \mathrm{Kod}_p : X &\longrightarrow \mathbb{P}(H^0(X, L^p)^*) \,, \\ x &\longmapsto \{ \, s \in H^0(X, L^p) \mid s(x) = 0 \, \} \end{aligned} \tag{2.28}$$

is well-defined and an embedding. In this section, we will always implicitly assume $p \in \mathbb{N}^*$ big enough so that this is verified.

Definition 2.8. The *coherent state projector* is the map

$$\Pi_{h^p} : X \longrightarrow \mathscr{L}(\mathscr{H}_p) \tag{2.29}$$

sending $x \in X$ to the orthogonal projector satisfying

$$\operatorname{Ker} \Pi_{h^p}(x) = \{ \, s \in \mathscr{H}_p \mid s(x) = 0 \, \} \,. \tag{2.30}$$

The *Rawnsley function* is the unique positive function $\rho_{h^p} \in \mathscr{C}^\infty(X, \mathbb{R})$ defined for any $s_1,\ s_2 \in \mathscr{H}_p$ and $x \in X$ by

$$\rho_{h^p}(x) \, \langle \Pi_{h^p}(x) s_1, s_2 \rangle_{L^2(h^p)} = \langle s_1(x), s_2(x) \rangle_{h^p} \,. \tag{2.31}$$

From the well-definition of the Kodaira map (2.28), formula (2.30) implies that $\Pi_{h^p}(x)$ is a rank-1 projector for all $x \in X$, so that for any orthonormal basis $\{s_j\}_{j=1}^{n_p}$ of $\mathscr{H}_p$, formula (2.31) implies

$$\rho_{h^p} = \rho_{h^p} \ \mathrm{Tr}[\Pi_{h^p}] = \sum_{j=1}^{n_p} \rho_{h^p} \, \langle \Pi_{h^p} s_j, s_j \rangle_{L^2(h^p)} = \sum_{j=1}^{n_p} |s_j|_{h^p}^2 \,. \tag{2.32}$$

This gives the characterization of the Rawnsley function as a *density of states*, which in turn coincides with the *Bergman kernel* with respect to $d\nu_h$ along the diagonal, as described in [27, §4.1.9]. The following Theorem describes the semi-classical behavior of the Rawnsley function as $p \to +\infty$, extending [9,47].

Theorem 2.9. [10, Th. 1.3] *There exist functions $b_h^{(r)} \in \mathscr{C}^\infty(X,\mathbb{R})$ for all $r \in \mathbb{N}$ such that for any m, $k \in \mathbb{N}$, there exists $C_{m,k} > 0$ such that for all $p \in \mathbb{N}^*$ big enough,*

$$\left| \rho_{h^p} - p^n \sum_{r=0}^{k-1} \frac{1}{p^r} b_h^{(r)} \right|_{\mathscr{C}^m} \leqslant C_{m,k} p^{n-k} , \tag{2.33}$$

where $b_h^{(0)} \in \mathscr{C}^\infty(X,\mathbb{R})$ is given by the identity $b_h^{(0)}\, d\nu_h = \omega_h^n / n!$.

Furthermore, the functions $b_h^{(r)} \in \mathscr{C}^\infty(X,\mathbb{R})$ for all $r \in \mathbb{N}$ depend smoothly on $h \in \mathrm{Met}^+(L)$ and its successive derivatives, and for each m, $k \in \mathbb{N}$, there exists $l \in \mathbb{N}$ such that the constant $C_{m,k} > 0$ can be chosen uniformly for $h \in \mathrm{Met}^+(L)$ in a bounded subset in $\mathscr{C}^l$-norm.

The concepts introduced in Definition 2.8 induce a *coherent state quantization*, described via the following fundamental tools.

Definition 2.10. For any $p \in \mathbb{N}^*$, the *Berezin-Toeplitz quantization map* is the linear map $T_{h^p} : \mathscr{C}^\infty(X,\mathbb{R}) \to \mathscr{L}(\mathscr{H}_p)$ defined for any $f \in \mathscr{C}^\infty(X,\mathbb{R})$ by the formula

$$T_{h^p}(f) := \int_X f(x)\, \Pi_{h^p}(x)\, \rho_{h^p}(x)\, d\nu_h(x) . \tag{2.34}$$

The *Berezin symbol* is the linear map $\sigma_{h^p} : \mathscr{L}(\mathscr{H}_p) \to \mathscr{C}^\infty(X,\mathbb{R})$ defined for any $A \in \mathscr{L}(\mathscr{H}_p)$ and $x \in X$ by the formula

$$\sigma_{h^p}(A)(x) := \mathrm{Tr}[\Pi_{h^p}(x) A] . \tag{2.35}$$

Using formula (2.31), we get for any $f \in \mathscr{C}^\infty(X,\mathbb{R})$ and any s_1, $s_2 \in \mathscr{H}_p$,

$$\langle T_{h^p}(f) s_1, s_2 \rangle_{L^2(h^p)} = \int_X f(x)\, \langle s_1(x), s_2(x) \rangle_{h^p}\, d\nu_h(x) , \tag{2.36}$$

recovering from Definition 2.10 the usual definition of Berezin-Toeplitz quantization associated with the volume form $d\nu_h$, as described in [27, Chap. 7]. The following fundamental Theorem describes the semi-classical behavior of the Berezin-Toeplitz quantization as $p \to +\infty$. We write $\|\cdot\|_{op}$ for the operator norm on endomorphisms of $\mathscr{H}_p$.

Theorem 2.11. [28, Th. 1.1] *For any f, $g \in \mathscr{C}^\infty(X,\mathbb{R})$, there exist bi-differential operators $C_h^{(r)}$ for all $r \in \mathbb{N}$ such that for any $k \in \mathbb{N}$, there exists $C_k > 0$ such that for all $p \in \mathbb{N}^*$ big enough,*

$$\left\| T_{h^p}(f) T_{h^p}(g) - \sum_{r=0}^{k-1} \frac{1}{p^r} T_{h^p}(C_h^{(r)}(f,g)) \right\|_{op} \leqslant \frac{C_k}{p^k} , \tag{2.37}$$

where $C_h^{(0)}(f,g) \in \mathscr{C}^\infty(X,\mathbb{R})$ *is given by* $C_h^{(0)}(f,g) = fg$.

Furthermore, the bi-differential operators $C_h^{(r)}$ *depend smoothly on* $h \in \mathrm{Met}^+(L)$ *and its successive derivatives for all* $r \in \mathbb{N}$, *and for each* $k \in \mathbb{N}$, *there exists* $l \in \mathbb{N}$ *such that the constant* $C_k > 0$ *can be chosen uniformly for* $f, g \in \mathscr{C}^\infty(X,\mathbb{R})$ *and* $h \in \mathrm{Met}^+(L)$ *in a bounded subset in* $\mathscr{C}^l$*-norm.*

In the context of quantization, the Berezin symbol (2.35) of a quantum observable $A \in \mathscr{L}(\mathscr{H}_p)$ is interpreted as the classical observable given by the expectation value of A at coherent states. The following result shows that this operator is dual to the Berezin-Toeplitz quantization with respect to the trace norm on $\mathscr{L}(\mathscr{H}_p)$ and the L^2-norm on $\mathscr{C}^\infty(X,\mathbb{R})$ induced by the density $\rho_{h^p}\, d\nu_h$.

Proposition 2.12. *For any* $A \in \mathscr{L}(\mathscr{H}_p)$ *and* $f \in \mathscr{C}^\infty(X,\mathbb{R})$, *we have*

$$\mathrm{Tr}[T_{h^p}(f)\, A] = \int_X f\, \sigma_{h^p}(A)\, \rho_{h^p}\, d\nu_h \,. \tag{2.38}$$

Furthermore, we have $T_{h^p}(1) = \mathrm{Id}_{\mathscr{H}_p}$ *and* $\sigma_{h^p}(\mathrm{Id}_{\mathscr{H}_p}) = 1$.

Proof. Formula (2.38) is an immediate consequence of Definition 2.10. On the other hand, by Definition 2.8 we have

$$\int_X \Pi_{h^p}(x)\, \rho_{h^p}(x)\, d\nu_h(x) = \mathrm{Id}_{\mathscr{H}_p} \,. \tag{2.39}$$

This implies the identity $T_{h^p}(1) = \mathrm{Id}_{\mathscr{H}_p}$, while the second identity is a consequence of the fact that Π_{h^p} is a rank-1 projector, so that $\mathrm{Tr}[\Pi_{h^p}] = 1$. □

This gives rise to the following concept, which will be the main technical tool of this paper.

Definition 2.13. The *Berezin transform* is the linear operator

$$\begin{aligned} \mathscr{B}_{h^p} : \mathscr{C}^\infty(X,\mathbb{R}) &\longrightarrow \mathscr{C}^\infty(X,\mathbb{R}) \,, \\ f &\longmapsto \sigma_{h^p}\left(T_{h^p}(f)\right) . \end{aligned} \tag{2.40}$$

As explained in details in [22, §2], the Berezin transform is a *Markov operator* with stationary measure $\rho_{h^p}\, d\nu_h$, and measures the delocalisation of a classical observable after quantization. From this point of view, the following semi-classical result can be thought as a quantitative refinement of the celebrated *Heisenberg's uncertainty principle*, and refines a semi-classical expansion due to Karabegov and Schlichenmaier [24].

Theorem 2.14. [27, Lem. 7.2.4], [22, Prop. 4.8] *For any $f \in \mathscr{C}^\infty(X,\mathbb{R})$, there exist differential operators $D_h^{(r)} \in \mathscr{C}^\infty(X,\mathbb{R})$ for all $r \in \mathbb{N}$ such that for any k, $m \in \mathbb{N}$, there exists a constant $C_{m,k} > 0$ such that for all $p \in \mathbb{N}^*$ big enough,*

$$\left| \mathscr{B}_{h^p}(f) - \sum_{r=0}^{k-1} p^{-r}\, D_h^{(r)}(f) \right|_{\mathscr{C}^m} \leqslant \frac{C_{m,k}}{p^k}\,, \tag{2.41}$$

with $D_h^{(0)}(f) = f$ and $D_h^{(1)}(f) = \frac{1}{4\pi}\Delta_h f$, where Δ_h is the Riemannian Laplacian of (X, g_h^{TX}).

Furthermore, the differential operators $D_h^{(r)}$ depend smoothly on $h \in \mathrm{Met}^+(L)$ and its successive derivatives for all $r \in \mathbb{N}$, and for every m, $k \in \mathbb{N}$, there exists $l \in \mathbb{N}$ such that the constant $C_{m,k} > 0$ can be chosen uniformly for $f \in \mathscr{C}^\infty(X,\mathbb{R})$ and $h \in \mathrm{Met}^+(L)$ in a bounded subset in $\mathscr{C}^l$-norm.

3. Quantum action of the automorphism group

One basic property of any reasonable quantization is its compatibility with symmetries, represented here by the action of the automorphism group $\mathrm{Aut}(X)$. In this Section, we will establish this fact through the use of Berezin-Toeplitz quantization, and establish fundamental properties of the quantized Futaki invariant (1.7) as a holomorphic invariant for this action. In the whole Section, we fix $h \in \mathrm{Met}^+(L)$ and consider the setting of Section 2.3.

3.1. Quantization of the holomorphy potentials

Let $K \subset \mathrm{Aut}_0(X)$ be the identity component of the group of holomorphic isometries of (X, g_h^{TX}). In the notations of Section 2.3, the action of K on X lifts to a unitary action on $\mathscr{H}_p$, and any $\xi \in \sqrt{-1}\,\mathrm{Lie}\, T$ induces a Hermitian operator $L_\xi \in \mathscr{L}(\mathscr{H}_p)$ defined via formula (2.4). On the other hand, Remark 2.3 implies that $\theta_h(\xi) \in \mathscr{C}^\infty(X,\mathbb{R})$ generates a *Hamiltonian flow* $\phi_{tJ\xi} \in K$, for all $t \in \mathbb{R}$. From general principles, the Hermitian operator $L_\xi \in \mathscr{L}(\mathscr{H}_p)$ can thus be seen as the appropriate quantization of the classical observable $\theta_h(\xi) \in \mathscr{C}^\infty(X,\mathbb{R})$, and the following result illustrates this principle via Berezin-Toeplitz quantization. Recall that we consider Berezin-Toeplitz quantization with respect to the anticanonical volume form (1.4) instead of the usual Liouville form.

Proposition 3.1. *For any $\xi \in \mathrm{Lie}\,\mathrm{Aut}(X)$, the induced operator $L_\xi \in \mathrm{End}(\mathscr{H}_p)$ satisfies*

$$L_\xi = (p+1)\, T_{h^p}\left(\theta_h(\xi)\right)\,. \tag{3.1}$$

In particular, there exists a constant $C > 0$, independent of $h^p \in \mathrm{Met}^+(L^p)$ and $\xi \in \mathrm{Lie}\,\mathrm{Aut}(X)$, such that for any $p \in \mathbb{N}^*$, we have

$$\|L_\xi\|_{op} < C\, p\, |\xi|\,. \tag{3.2}$$

Proof. This is version of an argument due to Tuynman [45], adapted to the anticanonical volume form (1.4). Recall that the Chern connection ∇^h induces the holomorphic structure of L, and recall the variation formulas (2.3) and (2.5) for the anticanonical volume form. Then by Proposition 2.2, for any holomorphic sections s_1, $s_2 \in H^0(X, L^p)$ we get

$$\begin{aligned}\int_X \langle \nabla^{h^p}_\xi s_1, s_2\rangle_{h^p}\, d\nu_h &= \int_X \langle \nabla^{h^p}_{\xi^{1,0}} s_1, s_2\rangle_{h^p}\, d\nu_h = \int_X L_{\xi^{1,0}} \langle s_1, s_2\rangle_{h^p}\, d\nu_h \\ &= -\int_X \langle s_1, s_2\rangle_{h^p}\, L_{\xi^{1,0}} d\nu_h = \int_X \theta_h(\xi)\, \langle s_1, s_2\rangle_{h^p}\, d\nu_h\end{aligned} \tag{3.3}$$

On the other hand, Definition 2.1 implies that for any $s \in \mathscr{H}_p$, we have

$$p\,\theta_h(\xi)\, s = L_\xi\, s - \nabla^{h^p}_\xi s\,. \tag{3.4}$$

Then using formula (2.36) for the Berezin-Toeplitz quantization map, for any s_1, $s_2 \in \mathscr{H}_p$ we get

$$\begin{aligned}\langle L_\xi\, s_1, s_2\rangle_{L^2(h^p)} &= p\int_X \theta_h(\xi)\langle s_1, s_2\rangle_{h^p}\, d\nu_h + \int_X \langle \nabla^{h^p}_\xi s_1, s_2\rangle_{h^p}\, d\nu_h \\ &= (p+1)\ \langle T_p(\theta_h(\xi))\, s_1, s_2\rangle_{L^2(h^p)}\,.\end{aligned} \tag{3.5}$$

This gives the identity (3.1), which implies in turn the inequality (3.2) via formula (2.36). □

Using Theorem 2.11, we can establish the following result, which is an exponentiation of Proposition 3.1.

Proposition 3.2. *There exist functions $\theta^{(j)}_\xi \in \mathscr{C}^\infty(X, \mathbb{C})$ for all $j \in \mathbb{N}$, depending smoothly on $\xi \in \mathrm{Lie\,Aut}(X)$, such that for any $k \in \mathbb{N}$, the exponential $e^{L_\xi/p} \in \mathrm{End}(\mathscr{H}_p)$ satisfies the following expansion in the sense of the operator norm as $p \to +\infty$,*

$$e^{L_\xi/p} = T_{h^p}\left(e^{\theta_h(\xi)}\right) + \sum_{j=1}^{k-1} p^{-j}\, T_{h^p}(\theta^{(j)}_\xi) + O(p^{-k})\,. \tag{3.6}$$

In particular, there exists a constant $C > 0$ such that for all $p \in \mathbb{N}^*$ big enough,

$$C^{-1} p^n \leqslant \mathrm{Tr}\left[e^{L_\xi/p}\right] \leqslant C p^n\,. \tag{3.7}$$

These estimates are uniform for $\xi \in \mathrm{Lie\,Aut}(X)$ and $h \in \mathrm{Met}^+(L)$ in any bounded set in $\mathscr{C}^l$-norm.

Proof. In this proof, the notation $O(p^{-k})$ for some $k \in \mathbb{N}$ is taken in the sense of the operator norm as $p \to +\infty$, uniformly in the $\mathscr{C}^l$-norm of $\theta_h(\xi) \in \mathscr{C}^\infty(X, \mathbb{R})$ and $h \in \mathrm{Met}^+(L)$, for some $l \in \mathbb{N}$ only depending on $k \in \mathbb{N}$ and for all $t \in [0,1]$.

First note that Proposition 3.1 implies the following ordinary differential equation in $t \in [0,1]$,

$$\begin{cases} \frac{\partial}{\partial t} e^{tL_\xi/p} = \left(1+\frac{1}{p}\right) T_{h^p}(\theta_h(\xi)) e^{tL_\xi/p} \\ e^{tL_\xi/p}\big|_{t=0} = \mathrm{Id}_{\mathscr{H}_p}. \end{cases} \tag{3.8}$$

On the other hand, Theorem 2.11 implies that for all $t \in [0,1]$, we have

$$\frac{\partial}{\partial t} T_{h^p}(e^{t\theta_h(\xi)}) = T_{h^p}(\theta_h(\xi)) T_{h^p}(e^{t\theta_h(\xi)}) + O(p^{-1}). \tag{3.9}$$

Using the fact from Definition 2.10 and formula (2.39) that $T_{h^p}(1) = \mathrm{Id}_{\mathscr{H}_p}$, we can then apply Grönwall's lemma to the difference of (3.8) and (3.9) to get the expansion (3.6) for $k = 1$.

Assume now by induction on $k \geqslant 2$ that there exist a function $g_{k,t} \in \mathscr{C}^\infty(X, \mathbb{C})$ and functions $f_{j,t} \in \mathscr{C}^\infty(X, \mathbb{C})$ for all $1 \leqslant j \leqslant k-1$, all smooth in $t \in [0,1]$, such that

$$\left(1+\frac{1}{p}\right) T_{h^p}(\theta_h(\xi)) T_{h^p}\left(\sum_{j=0}^{k-1} p^{-j} f_{j,t}\right) = \frac{\partial}{\partial t} T_{h^p}\left(\sum_{j=0}^{k-1} p^{-j} f_{j,t}\right) + p^{-k} T_{h^p}(g_{k,t}) + O(p^{-k+1}). \tag{3.10}$$

Using Theorem 2.11, we then see that (3.10) holds for k replaced by $k+1$, with the function $f_{k,t} \in \mathscr{C}^\infty(X, \mathbb{R})$ defined as the solution of the ordinary differential equation

$$\begin{cases} \frac{\partial}{\partial t} f_{k,t} = f_{k,t}\, \theta_h(\xi) + g_{k,t} \\ f_{k,0} = 0. \end{cases} \tag{3.11}$$

This shows by induction that (3.10) holds for all $k \in \mathbb{N}$. Applying Grönwall's lemma to the difference of (3.8) and (3.10) as above, this gives the result taking $\theta_\xi^{(j)} := f_{j,1}$ for all $j \in \mathbb{N}$. The smooth dependance on $\xi \in \mathrm{Lie\,Aut}(X)$ is then clear from the ordinary differential equation (3.11).

Recall on the other hand that the coherent state projector of Definition 2.8 is a rank-1 projector, so that $\mathrm{Tr}[\Pi_p(x)] = 1$ for all $x \in X$. Using Theorem 2.9 and formula (2.39), we then get a constant $C > 0$ such that the dimension of $\mathscr{H}_p$ satisfies $n_p < Cp^n$ for all $p \in \mathbb{N}^*$, which implies $\mathrm{Tr}\,[A] \leqslant Cp^n \,\|A\|_{op}$ for all $A \in \mathrm{End}(\mathscr{H}_p)$ and $p \in \mathbb{N}^*$ by

Cauchy-Schwartz inequality. The inequality (3.7) then follows by applying this estimate to $A = O(p^{-1})$ in the expansion (3.6) for $k = 1$, and by Definition 2.10 of the Berezin-Toeplitz quantization map. $\square$

3.2. Quantized Futaki invariants

In this Section, we study the quantized Futaki invariants Section 3.2, which play a central role in this paper. Recall that for any $\xi \in \mathrm{Lie}\,\mathrm{Aut}(X)$ and any $p \in \mathbb{N}^*$, we have an operator $L_\xi \in \mathrm{End}(H^0(X, L^p))$ defined by formula (2.4).

Recall the decomposition (2.8) for the Lie algebra of the complexification $K_\mathbb{C} \subset \mathrm{Aut}_0(X)$ of any compact subgroup $K \subset \mathrm{Aut}_0(X)$. We will establish a quantized version of Proposition 2.4 of Tian and Zhu, whose first part is given by the following result. It can also be found in [41, §3.1], but we give here a short proof using Berezin-Toeplitz quantization.

Lemma 3.3. *Let $T \subset \mathrm{Aut}(X)$ be a compact torus. Then for any $\in \mathbb{N}$, the functional $F_p : \sqrt{-1}\,\mathrm{Lie}\,T \to \mathbb{R}$ defined by the formula*

$$F_p(\xi) := p\,\mathrm{Tr}[e^{L_\xi/p}]\,, \tag{3.12}$$

is strictly convex and proper.

Proof. Let $T \subset \mathrm{Aut}(X)$ be a compact torus, and consider the setting of Section 2.3 with a T-invariant positive Hermitian metric $h \in \mathrm{Met}^+(L)^T$, which can always be constructed by average over T. Then for any $\xi \in \sqrt{-1}\,\mathrm{Lie}\,T$, Proposition 2.2 implies that there exists $x \in X$ such that $\theta_h(\xi)(x) > 0$. Let now $p \in \mathbb{N}^*$ be large enough so that the Kodaira map (2.28) is well-defined, and let $s_p \in \mathscr{H}_p$ with $\|s_p\|_{L^2(h^p)} = 1$ be in the image of the coherent state projector $\Pi_{h^p}(x) \in \mathscr{L}(\mathscr{H}_p)$ of Definition 2.8. Then from Theorem 2.14 and Proposition 3.1, as $p \to +\infty$ we get

$$\langle L_\xi s_x, s_x\rangle_{L^2(h^p)} = \mathrm{Tr}[L_\xi \Pi_{h^p}(x)] = \mathscr{B}_{h^p}(\theta_h(\xi))(x) = \theta_h(\xi)(x) + O(p^{-1})\,, \tag{3.13}$$

uniformly in the $\mathscr{C}^l$-norm of $\theta_h(\xi)$ for some $l \in \mathbb{N}$. Thus there exists $p_0 \in \mathbb{N}$ independent of $\xi \in \mathrm{Lie}\,\mathrm{Aut}(X)$ such that $L_\xi \in \mathscr{L}(\mathscr{H}_p)$ is non-negative for all $p \geqslant p_0$. For any $p \in \mathbb{N}^*$, write $\mathrm{Spec}_p(T) \subset (\mathrm{Lie}\,T)^*$ for the joint spectrum of $L_\xi \in \mathscr{L}(\mathscr{H}_p)$ for all $\xi \in \sqrt{-1}\,\mathrm{Lie}\,T$. Taking $p \geqslant p_0$, we then get that for any $\xi \in \sqrt{-1}\,\mathrm{Lie}\,T$, there exists $\chi_0 \in \mathrm{Spec}_p(T)$ such that $(\chi_0, \xi) > 0$. Thus for any $\eta \in \sqrt{-1}\,\mathrm{Lie}\,T$, we have

$$\frac{d^2}{dt^2} F_p(\eta + t\xi) = p^{-1} \sum_{\chi \in \mathrm{Spec}_p(T)} (\chi, \xi)^2 e^{(\chi, \eta + t\xi)/p} > 0 \tag{3.14}$$

and

$$F_p(\eta + t\xi) = p \sum_{\chi \in \mathrm{Spec}_p(T)} e^{\langle \chi, \eta + t\xi \rangle / p} \geqslant p\, e^{\langle \chi_0, \eta \rangle / p} e^{t \langle \chi_0, \xi \rangle / p} \xrightarrow{t \to +\infty} +\infty \,. \tag{3.15}$$

This proves the result. □

We then have the following corollary, which is a central property of the quantized Futaki invariants (1.7) and which is the quantization of the second part of Proposition 2.4. It can also be found in [41, §3.2].

Corollary 3.4. *Let $K \subset \mathrm{Aut}_0(X)$ be a connected compact subgroup, and let $T \subset K$ be the identity component of its center. Then for any $p \in \mathbb{N}^*$ big enough, there exists a unique $\xi_p \in \sqrt{-1}\,\mathrm{Lie}\, K$ such that the associated quantized Futaki invariant $\mathrm{Fut}_p^{\xi_p} : \sqrt{-1}\,\mathrm{Lie}\, K \to \mathbb{C}$ vanishes identically, which is given by the unique minimizer of the functional $F_p : \sqrt{-1}\,\mathrm{Lie}\, T \to \mathbb{R}$ of Lemma 3.3.*

Proof. Fix $p \in \mathbb{N}^*$, and note that for any compact torus $T \subset \mathrm{Aut}_0(X)$ and any $\xi, \eta \in \sqrt{-1}\ \mathrm{Lie}\, T$, we have

$$\mathrm{Fut}_p^{\xi}(\eta) = \frac{\partial}{\partial t}\Big|_{t=0} F_p(\xi + t\eta) \,. \tag{3.16}$$

Now Lemma 3.3 implies that $F_p : \sqrt{-1}\,\mathrm{Lie}\, T \to \mathbb{R}$ admits a unique minimizer, which is the unique $\xi_p \in \sqrt{-1}\,\mathrm{Lie}\, T$ such that $\mathrm{Fut}_p^{\xi}(\eta) = 0$ for all $\eta \in \sqrt{-1}\,\mathrm{Lie}\, T$.

Let now $K \subset \mathrm{Aut}_0(X)$ be a connected compact subgroup, and let $T \subset K$ be the identity component of its center. Then for any $\xi_1, \xi_2 \in \mathrm{Lie}\, K_{\mathbb{C}}$, we have

$$\mathrm{Tr}[L_{[\xi_1,\xi_2]} e^{L_{\xi_p}/p}] = \mathrm{Tr}\left[[L_{\xi_1}, L_{\xi_2} e^{L_{\xi_p}/p}] \right] = 0 \,, \tag{3.17}$$

so that $\mathrm{Fut}_p^{\xi_p}(\eta) = 0$ for all $\eta \in [\mathrm{Lie}\, K_{\mathbb{C}}, \mathrm{Lie}\, K_{\mathbb{C}}] \subset \mathrm{Lie}\, K_{\mathbb{C}}$. Using the classical decomposition $\mathrm{Lie}\, K_{\mathbb{C}} = \mathrm{Lie}\, T_{\mathbb{C}} \oplus [\mathrm{Lie}\, K_{\mathbb{C}}, \mathrm{Lie}\, K_{\mathbb{C}}]$, we then get that $\mathrm{Fut}_p^{\xi_p}(\eta) = 0$ for all $\eta \in \mathrm{Lie}\, K_{\mathbb{C}}$.

Assume now that $\widetilde{\xi}_p \in \sqrt{-1}\,\mathrm{Lie}\, K$ is such that $\mathrm{Fut}_p^{\widetilde{\xi}_p}(\eta) = 0$ for all $\eta \in \mathrm{Lie}\, K_{\mathbb{C}}$, and let now $\widetilde{T} \subset K$ be a maximal compact torus such that $\widetilde{\xi}_p \in \sqrt{-1}\,\mathrm{Lie}\, \widetilde{T}$. Then we have $\mathrm{Lie}\, T_{\mathbb{C}} \subset \mathrm{Lie}\, \widetilde{T}_{\mathbb{C}}$ by definition of a maximal torus, and we have $\widetilde{\xi}_p = \xi_p$ by uniqueness of the minimizer of $F_p : \sqrt{-1}\,\mathrm{Lie}\, \widetilde{T} \to \mathbb{R}$ given by Lemma 3.3. This gives the result. □

3.3. Asymptotic expansion of the quantized Futaki invariant

In this Section, we fix a connected compact subgroup $K \subset \mathrm{Aut}_0(X)$ and write $T \subset K$ for the identity component of its center. The following result describes the semi-classical behavior of the quantized Futaki invariant (1.7) and the functional (3.12) as $p \to +\infty$, recovering the modified Futaki invariant (2.17) and the functional (2.16) as the highest order coefficient respectively. As explained for instance in [3, §4.4], this is essentially a consequence of the equivariant Riemann-Roch formula, but we give here a short proof using Berezin-Toeplitz quantization.

Proposition 3.5. *There exist smooth maps* $F_p^{(j)} : \sqrt{-1}\,\mathrm{Lie}\,K \to \mathbb{C}$ *for all* $j \in \mathbb{N}$ *such that for any* $k \in \mathbb{N}^*$*, we have the following asymptotic expansion as* $p \to +\infty$*,*

$$\frac{F_p(\eta)}{p^{n+1}} = F(\eta) + \sum_{j=1}^{k-1} p^{-j}\, F_p^{(j)}(\eta) + O(p^{-k})\,. \tag{3.18}$$

Furthermore, there exist linear maps $\mathrm{Fut}_\xi^{(j)} : \mathrm{Lie}\,\mathrm{Aut}(X) \to \mathbb{C}$ *for all* $j \in \mathbb{N}$*, depending smoothly on* $\xi \in \mathrm{Lie}\,\mathrm{Aut}(X)$*, such that for any* $k \in \mathbb{N}^*$*, we have the following asymptotic expansion as* $p \to +\infty$*,*

$$\frac{\mathrm{Fut}_p^{\xi}(\eta)}{p^{n+1}} = \mathrm{Fut}_\xi(\eta) + \sum_{j=1}^{k-1} p^{-j}\; \mathrm{Fut}_\xi^{(j)}(\eta) + O(p^{-k})\,. \tag{3.19}$$

These estimates are uniform for ξ*,* $\eta \in \mathrm{Lie}\,\mathrm{Aut}(X)$ *in any compact set.*

Proof. Fix $h \in \mathrm{Met}^+(L)$. Using Proposition 2.12 and Definition 2.13 and by Theorems 2.9 and 2.14, we get differential operators D_j for any $j \in \mathbb{N}$ such that for any f, $g \in \mathscr{C}^\infty(X,\mathbb{C})$, any $k \in \mathbb{N}$ and as $p \to +\infty$,

$$\begin{aligned} \mathrm{Tr}[T_{h^p}(f)T_{h^p}(g)] &= \int_X f\, \mathscr{B}_{h^p}(g)\, \rho_p\, d\nu_h \\ &= p^n \int_X f\, g\, \frac{\omega_h^n}{n!} + \sum_{j=1}^{k-1} p^{n-j} \int_X f\, D_j(g)\, d\nu_h + O(p^{n-k})\,, \end{aligned} \tag{3.20}$$

uniformly in the derivatives of f, g up to order $l \in \mathbb{N}$ only depending on $k \in \mathbb{N}$.

Now using Propositions 3.1 and 3.2, for any $k \in \mathbb{N}$ and as $p \to +\infty$, the quantized Futaki invariant (1.7) and the functional (3.12) satisfy

$$\begin{aligned} \frac{\mathrm{Fut}_p^{\xi}(\eta)}{p+1} &= \mathrm{Tr}\left[T_{h^p}(\theta_h(\eta))T_{h^p}(e^{\theta_h(\xi)})\right] \\ &\quad + \sum_{j=1}^{k-1} p^{-j}\; \mathrm{Tr}\left[T_{h^p}(\theta_h(\eta))T_{h^p}(\theta_\xi^{(j)})\right] + O(p^{-k}) \quad \text{and} \end{aligned}$$

$$\frac{F_p(\xi)}{p} = \mathrm{Tr}\left[T_{h^p}(e^{\theta_h(\xi)})\right] + \sum_{j=1}^{k-1} p^{-j}\; \mathrm{Tr}\left[T_{h^p}(\theta_\xi^{(j)})\right] + O(p^{-k})\,, \tag{3.21}$$

uniformly for ξ, $\eta \in \mathrm{Lie}\,\mathrm{Aut}(X)$ in any compact set. Using the fact from Proposition 2.12 that $\mathrm{Id}_{\mathscr{H}_p} = T_{h^p}(1)$, we can then apply the identity (3.20) to the expansions (3.21) and compare with Proposition 2.4 for the first coefficients to get the result. □

The following result describes the semi-classical behavior of the sequence of vector fields produced by Corollary 3.4 as $p \to +\infty$, recovering the vector field of Proposition 2.4 as the highest coefficient. This is an anticanonical analogue of the result of Sano and Tipler in [39, Lem. 4.5], and the proof closely follows their strategy.

Proposition 3.6. *There exist $\xi^{(j)} \in \sqrt{-1}\,\mathrm{Lie}\,T$ for all $j \in \mathbb{N}$ such that for any $k \in \mathbb{N}$, the sequence $\{\xi_p \in \sqrt{-1}\,\mathrm{Lie}\,T\}_{p\in\mathbb{N}^*}$ of Corollary 3.4 satisfies the following expansion as $p \to +\infty$,*

$$\xi_p = \xi_\infty + \sum_{j=1}^{k-1} p^{-j}\,\xi^{(j)} + O(p^{-k})\,, \tag{3.22}$$

where $\xi_\infty \in \sqrt{-1}\,\mathrm{Lie}\,T$ is the unique minimizer of the functional (2.16).

Proof. Let $h \in \mathrm{Met}^+(L)^T$ be a T-invariant metric, which always exists by average over T. Using Proposition 3.5, we know that for all $\eta \in \sqrt{-1}\,\mathrm{Lie}\,T$, the functionals (2.16) and (3.12) satisfy $F_p(\eta) \xrightarrow{p\to+\infty} F(\eta)$. As these functionals are strictly convex and proper, this implies that their unique minimizers satisfy

$$\xi_p \xrightarrow{p\to+\infty} \xi_\infty\,. \tag{3.23}$$

From Proposition 2.4, we know that $\mathrm{Fut}_{\xi_\infty}(\eta) = 0$ for all $\eta \in \mathrm{Lie}\,K_{\mathbb{C}}$. Using Proposition 3.5 and formula (2.16) for the first coefficient, we can take the Taylor expansion as $p \to +\infty$ of $\mathrm{Fut}_p^{\xi_\infty+p^{-1}\xi^{(1)}}(\eta)$ for any $\xi^{(1)} \in \sqrt{-1}\,\mathrm{Lie}\,T$ to get

$$\frac{\mathrm{Fut}_p^{\xi_\infty+p^{-1}\xi^{(1)}}(\eta)}{p^{n+1}} = p^{-1}\left(\mathrm{Fut}^{(1)}_{\xi_\infty}(\eta) + \int_X \theta_h(\eta)\,\theta_h(\xi^{(1)})\,e^{\theta_h(\xi_\infty)}\,\frac{\omega_h^n}{n!}\right) + O(p^{-2})\,. \tag{3.24}$$

Recall now the linear embedding $\theta_h : \mathrm{Lie}\,T_{\mathbb{C}} \to \mathscr{C}^\infty(X,\mathbb{C})$ induced by Proposition 2.2, and restrict the scalar product $L^2(h_\infty,\xi_\infty)$ defined in Lemma 2.6 for $K = T$ to the subspace $\langle \theta_h(\xi)\,|\,\xi \in \sqrt{-1}\,\mathrm{Lie}\,T\rangle \subset \mathscr{C}^\infty(X,\mathbb{R})^T$. Then by non-degeneracy, for any linear form $G : \mathrm{Lie}\,T_{\mathbb{C}} \to \mathbb{C}$, there exists a unique $\xi_G \in \sqrt{-1}\,\mathrm{Lie}\,T$ such that for all $\eta \in \mathrm{Lie}\,T_{\mathbb{C}}$,

$$G(\eta) + \int_X \theta_h(\eta)\,\theta_h(\xi_G)\,e^{\theta_h(\xi_\infty)}\,\frac{\omega_h^n}{n!} = 0\,. \tag{3.25}$$

Setting $\xi^{(1)} := \xi_G$ for $G = \mathrm{Fut}^{(1)}_{\xi_\infty}$, the first term of the right hand side of (3.24) vanishes. Now for any $\xi^{(2)} \in \sqrt{-1}\,\mathrm{Lie}\,T$, this together with Proposition 3.5 and formula (3.24) gives a linear form $G^{(2)} : \mathrm{Lie}\,T_{\mathbb{C}} \to \mathbb{C}$ such that for any $\eta \in \mathrm{Lie}\,T_{\mathbb{C}}$ we have as $p \to +\infty$,

$$\begin{aligned}\frac{\mathrm{Fut}_p^{\xi_\infty + p^{-1}\xi^{(1)} + p^{-2}\xi^{(2)}}(\eta)}{p^{n+1}}&\\ &= p^{-2}\left(G^{(2)}(\eta) + \int_X \theta_h(\eta)\,\theta_h(\xi^{(2)})\,e^{\theta_h(\xi_\infty)}\,\frac{\omega_h^n}{n!}\right) + O(p^{-3})\,. \end{aligned} \tag{3.26}$$

Taking $\xi^{(2)} := \xi_{G^{(2)}}$ as in (3.25), the first term in the right hand side of the expansion (3.26) vanishes. Repeating this reasoning, we then construct by induction a sequence $\xi^{(j)} \in \sqrt{-1}\,\mathrm{Lie}\,T$, $j \in \mathbb{N}$, such that for all $k \in \mathbb{N}$ and $\eta \in \mathrm{Lie}\,T_{\mathbb{C}}$, we have as $p \to +\infty$,

$$\mathrm{Fut}_p^{\xi_\infty + \sum_{j=1}^k p^{-j}\xi^{(j)}}(\eta) = O(p^{n-k})\,. \tag{3.27}$$

For all p, $k \in \mathbb{N}$, set $\xi_p^{(k)} := \xi_\infty + \sum_{j=1}^k p^{-j}\xi^{(j)}$. Then using Corollary 3.4 and formula (3.27), we know that for all $k \in \mathbb{N}$ and as $p \to +\infty$,

$$\int_0^1 \frac{\partial}{\partial t}\,\mathrm{Fut}_p^{t\xi_p^{(k)} + (1-t)\xi_p}(\eta)\,dt = \mathrm{Fut}_p^{\xi_p^{(k)}}(\eta) - \mathrm{Fut}_p^{\xi_p}(\eta) = O(p^{n-k})\,, \tag{3.28}$$

uniformly for $\eta \in \mathrm{Lie}\,T_{\mathbb{C}}$ in any compact set. On the other hand, recall from (3.23) that $t\xi_p^{(k)} + (1-t)\xi_p \to \xi_\infty$ uniformly in $t \in [0,1]$ as $p \to +\infty$. Together with Theorem 2.11 and Proposition 3.1, this implies the existence of constants $l \in \mathbb{N}$, C, c, $\varepsilon > 0$ such that for all $p \in \mathbb{N}^*$ big enough,

$$\begin{aligned}\int_0^1 \frac{\partial}{\partial t}\,\mathrm{Fut}_{t\xi_p^{(k)} + (1-t)\xi_p}(\xi_p^{(k)} - \xi_p)\,dt&\\ &= p^{-1}\int_0^1 \mathrm{Tr}\left[\left(L_{\xi_p^{(k)} - \xi_p}\right)^2 \exp\left(p^{-1}\left(tL_{\xi_p^{(k)}} + (1-t)L_{\xi_p}\right)\right)\right] dt\\ &\geqslant \varepsilon p^{-1}(p+1)^2\,\mathrm{Tr}\left[T_{h^p}\left(\theta_h(\xi_p^{(k)} - \xi_p)\right)^2\right]\\ &\geqslant p^{n+1}\left(c - Cp^{-1}\right)\left|\theta_h(\xi_p^{(k)} - \xi_p)\right|^2_{\mathscr{C}^l}\,. \end{aligned} \tag{3.29}$$

Combining the estimates (3.28) and (3.29), we get as $p \to +\infty$,

$$\left|\theta_h(\xi_p^{(k)} - \xi_p)\right|^2_{\mathscr{C}^m} = O(p^{-k-1})\,, \tag{3.30}$$

which in turn implies that $|\xi_p^{(k)} - \xi_p|^2 = O(p^{-k-1})$ by equivalence of norms on the finite dimensional space $\sqrt{-1}\,\mathrm{Lie}\,T$, as $\theta_h : \sqrt{-1}\,\mathrm{Lie}\,T \to \mathscr{C}^\infty(X,\mathbb{R})$ is an embedding by Proposition 2.2. This proves the result. □

4. Balanced metrics

In this Section, we study the notion (1.5) of an anticanonically balanced metric relative to $\xi \in \mathrm{Lie}\,\mathrm{Aut}(X)$, and using the quantized Futaki invariant (1.7) as an obstruction for their existence, we show that the vector field ξ is determined by the complex geometry of X.

In the whole Section, we consider $p \in \mathbb{N}^*$ big enough so that the Kodaira map (2.28) is well-defined and an embedding, and fix a compact torus $T \subset \mathrm{Aut}_0(X)$. We will use freely the decomposition (2.8) for $K = T$.

4.1. Fubini-Study metrics

Let H be a T-invariant Hermitian inner product on $H^0(X,L^p)$, which always exists by average over T. For all $\xi \in \sqrt{-1}\,\mathrm{Lie}\,T$, the operators $L_\xi \in \mathrm{End}(H^0(X,L^p))$ induced by formula (2.4) are then Hermitian with respect to H, so that they admit a joint spectrum $\mathrm{Spec}_p(T) \subset (\mathrm{Lie}\,T)^*$ not depending on H. For any $\chi \in \mathrm{Spec}_p(T)$, write

$$H^0(X,L^p)_\chi := \{s \in H^0(X,L^p) \mid L_\xi\, s = \langle\chi,\xi\rangle\, s \ \text{ for all } \ \xi \in \mathrm{Lie}\,T\}\,. \tag{4.1}$$

Write $\mathscr{B}(H^0(X,L^p)_\chi)$ for the space of bases of $H^0(X,L^p)_\chi$, and set

$$\mathscr{B}(H^0(X,L^p))^T := \prod_{\chi\in\mathrm{Spec}_p(T)} \mathscr{B}(H^0(X,L^p)_\chi)\,. \tag{4.2}$$

For any $\chi \in \mathrm{Spec}_p(T)$, write $n_p(\chi) := \dim H^0(X,L^p)_\chi$. The space $\mathscr{B}(H^0(X,L^p))^T$ admits a free and transitive action of the group

$$\mathrm{GL}(\mathbb{C}^{n_p})^T := \bigotimes_{\chi\in\mathrm{Spec}_p(T)} \mathrm{GL}(\mathbb{C}^{n_p(\chi)})\,, \tag{4.3}$$

acting component by component by the canonical action of $\mathrm{GL}(\mathbb{C}^{n_p(\chi)})$ on bases of $H^0(X,L^p)_\chi$, for all $\chi \in \mathrm{Spec}_p(T)$. Note that for any $\xi \in \mathrm{Lie}\,T_{\mathbb{C}}$, we have $e^{L_\xi} \in \mathrm{GL}(\mathbb{C}^{n_p})^T$ acting in a canonical way as a scalar on each component. For any $n \in \mathbb{N}$, we write $U(n) \subset \mathrm{GL}(\mathbb{C}^n)$ for the subgroup of unitary matrices acting on $\mathbb{C}^n$, and we set

$$U(n_p)^T := \bigotimes_{\chi\in\mathrm{Spec}_p(T)} U(n_p(\chi)) \subset \mathrm{GL}(\mathbb{C}^{n_p})^T\,. \tag{4.4}$$

To any $\mathbf{s} \in \mathscr{B}(H^0(X,L^p))^T$, we can associate a basis $\{s_j\}_{j=1}^{n_p}$ of $H^0(X,L^p)$, uniquely determined up to reordering by the condition that it restricts to the corresponding basis of $H^0(X, L^p)_\chi$ for each $\chi \in \mathrm{Spec}_p(T)$. We write $H_{\mathbf{s}}$ for the unique T-invariant inner product on $H^0(X, L^p)$ for which $\{s_j\}_{j=1}^{n_p}$ is orthonormal. Conversely, we say that $\mathbf{s} \in \mathscr{B}(H^0(X, L^p))^T$ is *orthonormal* with respect to a Hermitian inner product H on $H^0(X, L^p)$ if $\{s_j\}_{j=1}^{n_p}$ is, so that $H_{\mathbf{s}} = H$.

We are now ready to introduce the main definition of the Section.

Definition 4.1. For any $\mathbf{s} \in \mathscr{B}(H^0(X, L^p))^T$, the associated *Fubini-Study metric* $h^p_{\mathbf{s}} \in \mathrm{Met}^+(L^p)$ is characterized for any s_1, $s_2 \in H^0(X, L^p)$ and $x \in X$ by the formula

$$\langle s_1(x), s_2(x)\rangle_{h^p_{\mathbf{s}}} := \langle \Pi_{\mathbf{s}}(x)\, s_1,\ s_2\rangle_{H_{\mathbf{s}}}\,, \tag{4.5}$$

where $\Pi_{\mathbf{s}}(x)$ is the unique orthogonal projector with respect to $H_{\mathbf{s}}$ satisfying

$$\mathrm{Ker}\,\Pi_{\mathbf{s}}(x) = \{\, s \in H^0(X, L^p) \mid s(x) = 0\,\}\,. \tag{4.6}$$

That formula (4.5) defines a positive Hermitian metric is a consequence of the fact that the Kodaira map (2.28) is an embedding.

For any T-invariant Hermitian product H, write $\mathscr{L}(H^0(X, L^p), H)^T$ for the space of Hermitian operators with respect to H commuting with the action of T. Via the action of $\mathrm{GL}(\mathbb{C}^{n_p})^T$ on $\mathscr{B}(H^0(X, L^p))^T$, any given $\mathbf{s} \in \mathscr{B}(H^0(X, L^p))^T$ induces an identification

$$\mathscr{L}(H^0(X, L^p), H_{\mathbf{s}})^T \simeq \mathrm{Herm}(\mathbb{C}^{n_p})^T := \bigoplus_{\chi \in \mathrm{Spec}_p(T)} \mathrm{Herm}(\mathbb{C}^{n_p(\chi)})\,. \tag{4.7}$$

In particular, for any $\xi \in \sqrt{-1}\,\mathrm{Lie}\,T$, we have $L_\xi \in \mathrm{Herm}(\mathbb{C}^{n_p})^T$ not depending on $\mathbf{s} \in \mathscr{B}(H^0(X, L^p))^T$. We then have the following basic variation formula for Fubini-Study metrics.

Proposition 4.2. *For any $\mathbf{s} \in \mathscr{B}(H^0(X, L^p))^T$ and $A \in \mathscr{L}(H^0(X, L^p), H_{\mathbf{s}})^T$, set*

$$\sigma_{\mathbf{s}}(A) := \mathrm{Tr}[A\Pi_{\mathbf{s}}] \in \mathscr{C}^\infty(X, \mathbb{R})\,. \tag{4.8}$$

Then for any $B \in \mathrm{Herm}(\mathbb{C}^{n_p})^T$, we have

$$\sigma_{\mathbf{s}}(e^{2B})\, h^p_{e^B\mathbf{s}} = h^p_{\mathbf{s}}\,, \tag{4.9}$$

in the identification (4.7) induced by $\mathbf{s} \in \mathscr{B}(H^0(X, L^p))^T$.

Proof. First note that for any $\mathbf{s} \in \mathscr{B}(H^0(X, L^p))^T$, writing $\{s_j\}_{j=1}^{n_p}$ for an induced basis of $H^0(X, L^p)$, Definition 4.1 implies

$$\sum_{j=1}^{n_p} |s_j|^2_{h^p_{\mathbf{s}}} = \sum_{j=1}^{n_p} \langle \Pi_{H_{\mathbf{s}}} s_j, s_j\rangle_{H_{\mathbf{s}}} = \mathrm{Tr}[\Pi_{H_{\mathbf{s}}}] = 1\,, \tag{4.10}$$

and this formula characterizes $h^p_{\mathbf{s}} \in \mathrm{Met}^+(L^p)^T$. On the other hand, for any $B \in \mathscr{L}(H^0(X, L^p), H_{\mathbf{s}})$, we have

$$\begin{aligned}\sigma_{H_{\mathbf{s}}}(e^{2B}) &= \mathrm{Tr}[e^B \Pi_{H_{\mathbf{s}}} e^B] \\ &= \sum_{j=1}^{n_p} \langle \Pi_{H_{\mathbf{s}}} e^B s_j, e^B s_j \rangle_{H_{\mathbf{s}}} = \sum_{j=1}^{n_p} \left| e^B s_j \right|^2_{h^p_{\mathbf{s}}} ,\end{aligned} \tag{4.11}$$

which gives the result by the characterization (4.10) applied to both $h_{\mathbf{s}}$ and $h_{e^B\mathbf{s}}$. □

Remark 4.3. Let K be a compact Lie group containing T in its center, and let $h^p \in \mathrm{Met}^+(L^p)^K$ be a K-invariant positive Hermitian metric. Then the associated L^2-Hermitian product $L^2(h^p)$ given by formula (2.26) is also K-invariant, and there exists $\mathbf{s}_p \in \mathscr{B}(H^0(X, L^p))^T$ orthonormal with respect to $L^2(h^p)$. Furthermore, the orthogonal projector $\Pi_{\mathbf{s}_p}(x)$ of Definition 4.1 coincides with the coherent state projector of Definition 2.8 at $x \in X$, so that

$$h^p = \rho_{h^p}\, h^p_{\mathbf{s}_p} , \tag{4.12}$$

and the function $\sigma_{\mathbf{s}_p}(A) \in \mathscr{C}^\infty(X, \mathbb{R})$ defined by formula (4.8) for any $A \in \mathscr{L}(\mathscr{H}_p)$ commuting with the action of T, coincides with its Berezin symbol as in Definition 2.10.

Let us end this section by the following variation formula for the Berezin symbol with respect to a change of basis

Proposition 4.4. *For any $\mathbf{s} \in \mathscr{B}(H^0(X, L^p))^T$, any $B \in \mathrm{Herm}(\mathbb{C}^{n_p})^T$ and any $A \in \mathscr{L}(H^0(X, L^p), H_{e^B\mathbf{s}})^T$, in the identification (4.7) induced by $\mathbf{s}$, we have*

$$\sigma_{e^B\mathbf{s}}(A)\sigma_{\mathbf{s}}(e^{2B}) = \sigma_{\mathbf{s}}(e^B A e^B) . \tag{4.13}$$

In particular, for any $\xi \in \sqrt{-1}\,\mathrm{Lie}\, T$, we have $\sigma_{\mathbf{s}}(e^{2L_\xi} A) = \sigma_{e^{L_\xi}\mathbf{s}}(A)\sigma_{\mathbf{s}}(e^{2L_\xi})$.

Proof. Let $\mathbf{s} \in \mathscr{B}(H^0(X, L^p))^T$ be given, and let $\{s_j\}_{j=1}^{n_p}$ be an induced basis of $H^0(X, L^p)$. Then by Definition 4.1, the projector $\Pi_{\mathbf{s}}(x)$ at $x \in X$ of can be written in this basis as

$$\Pi_{\mathbf{s}}(x) = \Big(\langle s_j(x), s_k(x) \rangle_{h^p_{\mathbf{s}}} \Big)_{j,k=1}^{n_p} . \tag{4.14}$$

Take now $B \in \mathrm{Herm}(\mathbb{C}^{n_p})^T$, seen as a Hermitian operator with respect to $H_{\mathbf{s}}$ via the identification (4.7), and take $A \in \mathscr{L}(H^0(X, L^p), H_{e^B\mathbf{s}})^T$. Writing them in the basis $\{s_j\}_{j=1}^{n_p}$, using Definition 4.1 and Proposition 4.2, we then get

$$\begin{aligned}\sigma_{e^B\mathbf{s}}(A) &= \sum_{j,k=1}^{n_p} \langle Ae^B s_k, e^B s_j\rangle_{h^p_{e^B\mathbf{s}}} \\ &= \sigma_{\mathbf{s}}(e^{2B})^{-1} \sum_{l,m=1}^{n_p} \left(e^B A e^B\right)_{ml} \langle s_l, s_m\rangle_{h^p_{\mathbf{s}}} = \sigma_{\mathbf{s}}(e^{2B})^{-1}\,\sigma_{\mathbf{s}}(e^B A e^B)\,.\end{aligned} \tag{4.15}$$

This gives the result. □

4.2. Relative balanced metrics

In this Section, we introduce the notion of *relatively balanced metrics*, and we exhibit their role as a quantized version of Kähler-Ricci solitons. In particular, we will establish a quantized version of Proposition 2.5 of Tian and Zhu for the quantized Futaki invariant.

Recall that we write $h^p_{\mathbf{s}} \in \mathrm{Met}^+(L^p)^T$ for the Fubini-Study metric of Definition 4.1 associated with any $\mathbf{s} \in \mathscr{B}(H^0(X,L^p))^T$, and recall that for any $\xi \in \mathrm{Lie\,Aut}(X)$, we write $\phi_\xi \in \mathrm{Aut}(X)$ for its exponentiation. We will need the following equivariance property.

Proposition 4.5. *For any $\mathbf{s} \in \mathscr{B}(H^0(X,L^p))^T$ and any $\eta \in \mathrm{Lie\,Aut}(X)$ such that $L_\eta \in \mathscr{L}(H^0(X,L^p),H_{\mathbf{s}})^T$, we have*

$$\phi_\eta^*\, h_{\mathbf{s}} = h_{e^{L_\eta}\mathbf{s}} \quad \textit{and} \quad \phi_\eta^*\,\sigma_{\mathbf{s}}(A) = \sigma_{e^{L_\eta}\mathbf{s}}(e^{-L_\eta} A e^{L_\eta})\,, \tag{4.16}$$

for any $A \in \mathscr{L}(H^0(X,L^p),H_{\mathbf{s}})^T$.

Furthermore, if $H_{\mathbf{s}}$ is preserved by a connected subgroup $K \subset \mathrm{Aut}_0(X)$, the Fubini-Study metric $h^p_{\mathbf{s}} \in \mathrm{Met}^+(L^p)$ is K-invariant.

Proof. Let $\mathbf{s} \in \mathscr{B}(H^0(X,L^p))^T$ be given, and let $\eta \in \mathrm{Lie\,Aut}(X)$ be such that $L_\eta \in \mathscr{L}(H^0(X,L^p),H_{\mathbf{s}})^T$. For any $s_1,\, s_2 \in H^0(X,L^p)$, we have by definition that $\langle s_1,s_2\rangle_{H_{e^{L_\eta}\mathbf{s}}} = \langle e^{-L_\eta}s_1, e^{-L_\eta}s_2\rangle_{H_{\mathbf{s}}}$, and for any $x \in X$, we have the identity

$$\Pi_{e^{L_\eta}\mathbf{s}}(x) = e^{L_\eta}\Pi_{\mathbf{s}}(\phi_\eta(x))e^{-L_\eta}\,, \tag{4.17}$$

which follows from the fact that both sides are orthogonal projectors with respect to $H_{e^{L_\eta}\mathbf{s}}$, and have common kernel by formula (4.6). Plugging these two identities in formula (4.5) for $h^p_{e^{L_\eta}\mathbf{s}}$ and using formula (2.4) for L_η, we get

$$\begin{aligned}\langle s_1(x), s_2(x)\rangle_{h^p_{e^{L_\eta}\mathbf{s}}} &= \langle \Pi_{\mathbf{s}}(\phi_\eta(x))e^{-L_\eta}s_1, e^{-L_\eta}s_2\rangle_{H_{\mathbf{s}}} \\ &= \langle e^{-L_\eta}s_1(\phi_\eta(x)), e^{-L_\eta}s_2(\phi_\eta(x))\rangle_{h^p_{\mathbf{s}}}\,,\end{aligned} \tag{4.18}$$

which gives the first identity of (4.16) by definition of the pullback of a Hermitian metric. On the other hand, from formula (4.17) we get

$$\sigma_{\mathbf{s}}(A)(\phi_\eta(x)) = \mathrm{Tr}[e^{-L_\eta}\Pi_{e^{L_\eta}\mathbf{s}}(x)e^{L_\eta}A] = \sigma_{e^{L_\eta}\mathbf{s}}(e^{-L_\eta}Ae^{L_\eta})(x)\,. \tag{4.19}$$

This gives the second identity of (4.16) and concludes the proof. Finally, the fact that $h^p_{\mathbf{s}}$ is K-invariant when $K \subset \mathrm{Aut}_0(X)$ preserves $H_{\mathbf{s}}$ then follows from Definition 4.1, as $\Pi_{\mathbf{s}}$ only depends on $H_{\mathbf{s}}$. $\square$

To simplify notations, let us write $\omega_{\mathbf{s}} := \omega_{h_{\mathbf{s}}}$ for the Kähler form associated with the Fubini-Study metric induced by $\mathbf{s} \in \mathscr{B}(H^0(X, L^p))^T$. From Proposition 4.5, a positive Hermitian metric $h^p \in \mathrm{Met}^+(L^p)^T$ is anticanonically balanced relative to $\xi \in \mathrm{Lie}\,\mathrm{Aut}(X)$ in the sense of formula (1.5) if $\xi \in \sqrt{-1}\,\mathrm{Lie}\,T$ and if for any $\mathbf{s}_p \in \mathscr{B}(H^0(X, L^p))^T$ orthonormal with respect to $L^2(h^p)$, we have

$$\omega_{h^p} = \omega_{e^{L_\xi/2p}\mathbf{s}_p}\,. \tag{4.20}$$

We then have the following useful alternative characterization of relatively balanced metrics.

Proposition 4.6. *A positive Hermitian metric $h^p \in \mathrm{Met}^+(L^p)^T$ is anticanonically balanced relative to $\xi \in \sqrt{-1}\,\mathrm{Lie}\,T$ if and only if the associated Rawnsley function satisfies*

$$\sigma_{h^p}(e^{L_\xi/p})\rho_{h^p} = \frac{\mathrm{Tr}[e^{L_\xi/p}]}{\mathrm{Vol}(d\nu_h)}\,. \tag{4.21}$$

Proof. Consider $h^p \in \mathrm{Met}^+(L^p)^T$, and let $s_p \in \mathscr{B}(H^0(X, L^p))^T$ be orthonormal with respect to $L^2(h^p)$ as in Remark 4.3. Using also Proposition 4.2, we have

$$h^p = \rho_{h^p}\, h^p_{\mathbf{s}_p} = \rho_{h^p}\, \sigma_{h^p}(e^{L_\xi/p})\, h^p_{e^{L_\xi/2p}\mathbf{s}_p}\,. \tag{4.22}$$

Using (1.5), we then get that $h^p \in \mathrm{Met}^+(L^p)^T$ is anticanonically balanced relative to $\xi \in \sqrt{-1}\,\mathrm{Lie}\,T$ if and only if the function $\rho_{h^p}\,\sigma_{h^p}(e^{L_\xi/p}) \in \mathscr{C}^\infty(X, \mathbb{R})$ is constant over X. To compute this constant, it suffices to note that Proposition 2.12 implies

$$\int_X \rho_{h^p}\, \sigma_{h^p}(e^{L_\xi/p})\, d\nu_h = \mathrm{Tr}[e^{L_\xi/p}]\,. \tag{4.23}$$

This gives the result. $\square$

Using this characterization of relative anticanonically balanced metrics together with the tools of Sections 2.3 and 3.1, we can now give a short proof of the following key fact.

Proposition 4.7. *If there exists an anticanonically balanced metric $h^p \in \mathrm{Met}^+(L^p)$ relative to $\xi \in \mathrm{Lie}\,\mathrm{Aut}(X)$, then the quantized Futaki invariant $\mathrm{Fut}^\xi_p : \mathrm{Lie}\,\mathrm{Aut}(X) \to \mathbb{C}$ vanishes identically.*

Proof. Let $h^p \in \mathrm{Met}^+(L^p)^T$ be anticanonically balanced metric relative to $\xi \in \sqrt{-1}\,\mathrm{Lie}\,T$. Then by definition (1.7) of the quantized Futaki invariant relative to ξ, using Propositions 2.2, 2.12, 3.1 and 4.6, for any $\eta \in \mathrm{Lie}\,\mathrm{Aut}(X)$ we get

$$\begin{aligned}\frac{\mathrm{Fut}_p^{\xi}(\eta)}{p+1} = \mathrm{Tr}[T_{h^p}(\theta_h(\eta))e^{L_\xi/p}] &= \int_X \theta_h(\eta)\,\sigma_{h^p}(e^{L_\xi/p})\,\rho_{h^p}\,d\nu_h \\ &= \frac{\mathrm{Tr}[e^{L_\xi/p}]}{\mathrm{Vol}(d\nu_h)}\int_X \theta_h(\eta)\,d\nu_h = 0\,.\end{aligned} \tag{4.24}$$

This shows the result. □

4.3. *Relative moment maps*

In this section, we give the finite dimensional characterization of relative balanced metrics, using a relative version of Donaldson's moment map picture in [13]. For any $\mathbf{s} \in \mathscr{B}(H^0(X, L^p))^T$, let us write $d\nu_{\mathbf{s}} := d\nu_{h_{\mathbf{s}}}$ for the anticanonical volume form (1.4) induced by the associated Fubini-Study metric.

Definition 4.8. The *anticanonical moment map* relative to $\xi \in \sqrt{-1}\,\mathrm{Lie}\,T$ is the map $\mu_\xi : \mathscr{B}(H^0(X, L^p))^T \to \mathrm{Herm}(\mathbb{C}^{n_p})^T$ defined for all $\mathbf{s} \in \mathscr{B}(H^0(X, L^p))^T$ by the formula

$$\mu_\xi(\mathbf{s}) := \left(\int_X \langle s_j(x), s_k(x)\rangle_{e^{L_\xi/2p}\mathbf{s}}\, d\nu_{e^{L_\xi/2p}\mathbf{s}}(x)\right)_{j,\,k=1}^{n_p} - \frac{\mathrm{Vol}(d\nu_{\mathbf{s}})}{\mathrm{Tr}[e^{L_\xi/p}]}\mathrm{Id}\,, \tag{4.25}$$

where $\{s_j\}_{j=1}^{n_p}$ is an induced basis of $H^0(X, L^p)$.

Note that we wrote formula (4.25) as an element of $\mathrm{Herm}(\mathbb{C}^{n_p})$ instead of $\mathrm{Herm}(\mathbb{C}^{n_p})^T$. However, Proposition 4.5 shows that $L^2(h_{e^{L_\xi/2p}\mathbf{s}})$ is T-invariant, so that the right hand side of (4.25) splits into blocks corresponding to the eigenspaces (4.1) of the action of T on $H^0(X, L^p)$, giving an element of $\mathrm{Herm}(\mathbb{C}^{n_p})^T$ as in formula (4.7) depending only on $\mathbf{s} \in \mathscr{B}(H^0(X, L^p))^T$. This identification will always be implicitly understood in the sequel.

Note that we do not claim that Definition 4.8 defines an actual relative moment map in the usual sense, and we will consequently not use any moment map property as such anywhere in this paper. We will however stick to this name, as it is the anticanonical analogue of the relative moment map considered by Sano and Tipler in [39, §3.3]. Its relevance in the context of balanced metrics comes from the following basic result, which follows immediately from the definition.

Proposition 4.9. *For any $\xi \in \sqrt{-1}\,\mathrm{Lie}\,T$ and $\mathbf{s} \in \mathscr{B}(H^0(X,L^p))^T$, we have*

$$\mu_\xi(\mathbf{s}) = 0 \tag{4.26}$$

if and only if there exists an anticanonically balanced metric $h^p \in \mathrm{Met}^+(L^p)^T$ relative to ξ for which $\mathbf{s}$ is orthonormal with respect to $L^2(h^p)$.

Proof. Writing

$$h^p := \frac{\mathrm{Vol}(d\nu_{\mathbf{s}})}{\mathrm{Tr}[e^{L_\xi/p}]}\, h^p_{e^{L_\xi/p}\mathbf{s}}\,, \tag{4.27}$$

Definition 4.8 shows that $\mathbf{s}$ is orthonormal with respect to $L^2(h^p)$ if and only if $\mu_\xi(\mathbf{s}) = 0$. Hence formula (4.27) for $h^p \in \mathrm{Met}^+(L^p)^T$ implies formula (4.20) for an anticanonically balanced metric with respect to ξ. This gives the result. □

For any $\xi \in \sqrt{-1}\,\mathrm{Lie}\,T$, consider the scalar product $\langle\cdot,\cdot\rangle_\xi$ defined on any A, $B \in \mathrm{Herm}(\mathbb{C}^{n_p})^T$ by

$$\langle A, B\rangle_\xi := \mathrm{Tr}[e^{L_\xi/p}AB]\,. \tag{4.28}$$

We then have the following important obstruction result, which is compatible with Proposition 4.7 via Proposition 4.9.

Proposition 4.10. *For any $\xi \in \sqrt{-1}\,\mathrm{Lie}\,T$ and $\mathbf{s} \in \mathscr{B}(H^0(X,L^p))^T$, we have the identity $\langle \mathrm{Id}, \mu_\xi(\mathbf{s})\rangle_\xi = 0$, and for any $\eta \in \sqrt{-1}\,\mathrm{Lie}\,T$, we have the identity*

$$p\langle L_\eta, \mu_\xi(\mathbf{s})\rangle_\xi = -\frac{\mathrm{Vol}(d\nu_{\mathbf{s}})}{\mathrm{Tr}[e^{L_\xi/p}]}\,\mathrm{Fut}_p^\xi(\eta)\,. \tag{4.29}$$

Furthermore, for any connected compact subgroup $K \subset \mathrm{Aut}_0(X)$ preserving $H_{\mathbf{s}}$ and containing $T \subset K$ in its center, we have that $\mu_\xi(\mathbf{s}) \in \mathrm{Herm}(\mathbb{C}^{n_p})^T$ commutes with the action of K on $H^0(X,L^p)$ via the identification (4.7).

Proof. Let $\xi \in \sqrt{-1}\,\mathrm{Lie}\,T$ and $\mathbf{s} \in \mathscr{B}(H^0(X,L^p))^T$ be given, and let $\{s_j\}_{j=1}^{n_p}$ be an induced basis of $H^0(X,L^p)$. Using formula (4.14) for the basis $\{e^{L_\xi/2p}s_j\}_{j=1}^{n_p}$, the coherent state projector $\Pi_{e^{L_\xi/2p}\mathbf{s}}(x)$ at $x \in X$ in the basis $\{s_j\}_{j=1}^{n_p}$ reads

$$e^{-L_\xi/2p}\Pi_{e^{L_\xi/2p}\mathbf{s}}(x)e^{-L_\xi/2p} = \Big(\langle s_j(x), s_k(x)\rangle_{h^p_{e^{L_\xi/2p}\mathbf{s}}}\Big)_{j,\,k=1}^{n_p}. \tag{4.30}$$

Using Proposition 4.5 together with formula (2.5), the fact that $\Pi_{e^{L_\xi/2p}\mathbf{s}}(x)$ is a rank-1 projector implies

$$\mathrm{Tr}[e^{L_\xi/p}\mu_\xi(\mathbf{s})] = \int_X \mathrm{Tr}[\Pi_{e^{L_\xi/2p}\mathbf{s}}(x)]\, d\nu_{e^{L_\xi/2p}\mathbf{s}}(x) - \mathrm{Vol}(d\nu_{\mathbf{s}}) = 0\,. \tag{4.31}$$

This proves the first assertion.

On the other hand, using Propositions 4.2 and 4.5, for all $\eta \in \sqrt{-1}\,\mathrm{Lie}\,T$, formula (2.5) implies

$$\int_X \sigma_{\mathbf{s}}(L_\eta)\, d\nu_{\mathbf{s}} = p\,\frac{\partial}{\partial t}\Big|_{t=0} \int_X \phi_{t\eta}^*\, d\nu_h = 0\,. \tag{4.32}$$

From formulas (1.7) and (4.30), this gives

$$\begin{aligned}\mathrm{Tr}[e^{L_\xi/p}L_\eta\,\mu_\xi(\mathbf{s})] &= \int_X \sigma_{e^{L_\xi/2p}\mathbf{s}}(L_\eta)\, d\nu_{e^{L_\xi/2p}\mathbf{s}} - \frac{\mathrm{Vol}(d\nu_{\mathbf{s}})}{\mathrm{Tr}[e^{L_\xi/p}]}\,\mathrm{Fut}_p^\xi(\eta) \\ &= -\frac{\mathrm{Vol}(d\nu_{\mathbf{s}})}{\mathrm{Tr}[e^{L_\xi/p}]}\,\mathrm{Fut}_p^\xi(\eta)\,.\end{aligned} \tag{4.33}$$

Finally, let $K \subset \mathrm{Aut}_0(X)$ be a connected compact subgroup preserving $H_{\mathbf{s}}$ containing $T \subset K$ in its center, and recall from Proposition 4.5 that $h_{\mathbf{s}} \in \mathrm{Met}^+(L)^K$, and for any $\eta \in \mathrm{Lie}\,K$, we have $\Pi_{e^{L_\eta}\mathbf{s}} = \Pi_{\mathbf{s}}$. Then using formulas (2.5), (4.17) and (4.30), in the identification (4.7) we get

$$\begin{aligned}e^{-L_\eta}\mu_\xi(\mathbf{s})e^{L_\eta} &= \int_X e^{-L_\xi/2p}e^{-L_\eta}\Pi_{e^{L_\xi/2p}\mathbf{s}}(x)e^{L_\eta}e^{-L_\xi/2p}\, d\nu_{\mathbf{s}} - \frac{\mathrm{Vol}(d\nu_{\mathbf{s}})}{\mathrm{Tr}[e^{L_\xi/p}]}\mathrm{Id} \\ &= \int_X e^{-L_\xi/2p}\Pi_{e^{L_\xi/2p}\mathbf{s}}(\phi_\eta(x))e^{-L_\xi/2p}\, d\nu_{\mathbf{s}} - \frac{\mathrm{Vol}(d\nu_{\mathbf{s}})}{\mathrm{Tr}[e^{L_\xi/p}]}\mathrm{Id} \\ &= \mu_\xi(\mathbf{s})\,,\end{aligned} \tag{4.34}$$

where we used a change of variable with respect to $\phi_\eta \in \mathrm{Aut}_0(X)$ to get the last line. This concludes the proof. □

5. Equivariant Berezin-Toeplitz quantization

In this Section, we fix a connected compact subgroup $K \subset \mathrm{Aut}_0(X)$, and write $T \subset K$ for the identity component of its center. We will study the properties of the Berezin-Toeplitz quantization of Section 2.3 with respect to the action of K. Let $p \in \mathbb{N}^*$ be large enough so that the Kodaira map (2.28) is an embedding, and consider the setting or Section 2.3 for a K-invariant positive Hermitian metric $h^p \in \mathrm{Met}^+(L^p)^K$.

5.1. Quantum channel

Recall that we write $\mathscr{C}^\infty(X,\mathbb{R})^K$ for the space of K-invariant functions on X, and $\mathscr{L}(\mathscr{H}_p)^K$ for the space of Hermitian operators commuting with the action of K on $\mathscr{H}_p$.

Lemma 5.1. *The symbol and quantization maps introduced in Definition 2.10 restrict to linear maps* $\sigma_{h^p} : \mathscr{L}(\mathscr{H}_p)^K \to \mathscr{C}^\infty(X,\mathbb{R})^K$ *and* $T_{h^p} : \mathscr{C}^\infty(X,\mathbb{R})^K \to \mathscr{L}(\mathscr{H}_p)^K$*.*

Proof. Following Remark 4.3, let $\mathbf{s}_p \in \mathscr{B}(H^0(X,L^p))^T$ be orthonormal with respect to $L^2(h^p)$. As K preserves $L^2(h^p)$ and as the coherent state projector of Definition 4.1 only depends on $H_{\mathbf{s}_p} = L^2(h^p)$, we have $\Pi_{e^{L_\eta}\mathbf{s}_p} = \Pi_{\mathbf{s}_p} = \Pi_{h^p}$ for all $\eta \in \mathrm{Lie}\,K$. Proposition 4.5 then implies that $\sigma_{h^p}(A) \in \mathscr{C}^\infty(X,\mathbb{R})^K$ for all $A \in \mathscr{L}(\mathscr{H}_p)^K$. This shows that the Berezin symbol restricts to a map $\sigma_{h^p} : \mathscr{L}(\mathscr{H}_p)^K \to \mathscr{C}^\infty(X,\mathbb{R})^K$.

On the other hand, Proposition 4.5 shows that $h^p_{\mathbf{s}_p} \in \mathrm{Met}^+(L^p)$ is K-invariant, and formula (4.12) then implies that $\rho_{h^p} \in \mathscr{C}^\infty(X,\mathbb{R})^K$. Thus for any $\eta \in \mathrm{Lie}\,K$ and $f \in \mathscr{C}^\infty(X,\mathbb{R})^K$, we can use formula (4.17) and a change of variable with respect to $\phi_\xi \in \mathrm{Aut}_0(X)$ to get

$$\begin{aligned} T_{h^p}(f) &= \int_X f(\phi_\eta(x))\,\Pi_{h^p}(\phi_\eta(x))\,\rho_{h^p}(\phi_\eta(x))\,\phi_\eta^*\,d\nu_h(x) \\ &= \int_X f(x)\,e^{L_\eta}\Pi_{h^p}(x)e^{-L_\eta}\,\rho_{h^p}(x)\,d\nu_h(x) = e^{L_\eta}T_{h^p}(f)e^{-L_\eta}\,, \end{aligned} \tag{5.1}$$

so that $T_{h^p}(f) \in \mathscr{L}(\mathscr{H}_p)^K$ for all $f \in \mathscr{C}^\infty(X,\mathbb{R})^K$. This concludes the proof. □

From now on, we fix $\xi \in \sqrt{-1}\,\mathrm{Lie}\,T$. Recall the scalar product (4.28) on the real vector space $\mathscr{L}(\mathscr{H}_p)^K \simeq \mathrm{Herm}(\mathbb{C}^{n_p})^K$, and consider the scalar product $L^2(h,\xi,p)$ defined on any $f,\,g \in \mathscr{C}^\infty(X,\mathbb{R})^K$ by

$$\langle f,g\rangle_{L^2(h,\xi,p)} := \int_X f\,g\,\frac{\sigma_{h^p}(e^{L_\xi/p})\,\rho_{h^p}}{\mathrm{Tr}[e^{L_\xi/p}]}\,d\nu_h\,. \tag{5.2}$$

The following result is the equivariant version of the duality Proposition 2.12 between Berezin symbol and Berezin-Toeplitz quantization.

Proposition 5.2. *For any* $A \in \mathscr{L}(\mathscr{H}_p)^K$ *and* $f \in \mathscr{C}^\infty(X,\mathbb{R})^K$*, we have*

$$\frac{\langle T_{h^p}(f),A\rangle_\xi}{\mathrm{Tr}[e^{L_\xi/p}]} = \langle f, \phi^*_{\xi/2p}\sigma_{h^p}(A)\rangle_{L^2(h,\xi,p)}\,. \tag{5.3}$$

Proof. Following Remark 4.3, let $\mathbf{s}_p \in \mathscr{B}(H^0(X,L^p))^T$ be orthonormal with respect to $L^2(h^p)$. Then using Propositions 2.12, 4.4 and 4.5, for any $A \in \mathscr{L}(\mathscr{H}_p)^K$ and $f \in \mathscr{C}^\infty(X,\mathbb{R})^K$, we get

$$\begin{aligned}\operatorname{Tr}[e^{L_\xi/p}AT_{h^p}(f)] &= \int_X \sigma_{h^p}(e^{L_\xi/p}A)\,\rho_{h^p}\,d\nu_h \\ &= \int_X \sigma_{e^{L_\xi/p}\mathbf{s}_p}(A)\,\sigma_{h^p}(e^{L_\xi/p})\,\rho_{h^p}\,d\nu_h \\ &= \int_X \phi^*_{\xi/2p}\,\sigma_{h^p}(A)\,\sigma_{h^p}(e^{L_\xi/p})\,\rho_{h^p}\,d\nu_h\,.\end{aligned} \tag{5.4}$$

This gives the result. □

We can now introduce the main tool of this Section.

Definition 5.3. For any $\xi \in \sqrt{-1}\,\mathrm{Lie}\,T$, the *Berezin-Toeplitz quantum channel relative to* ξ is the linear map defined by

$$\begin{aligned}\mathscr{E}^\xi_{h^p} : \mathscr{L}(\mathscr{H}_p)^K &\longrightarrow \mathscr{L}(\mathscr{H}_p)^K \\ A &\longmapsto T_{h^p}\left(\phi^*_{\xi/2p}\,\sigma_{h^p}(A)\right).\end{aligned} \tag{5.5}$$

Proposition 5.2 the shows that the quantum channel relative to $\xi \in \sqrt{-1}\,\mathrm{Lie}\,T$ is a positive and self-adjoint operator acting on the real Hilbert space $\mathscr{L}(\mathscr{H}_p)^K$ endowed with the scalar product $\langle\cdot,\cdot\rangle_\xi$ defined by formula (4.28).

5.2. Berezin transform

The goal of this section is to extend the results of [22] on the Berezin transform of Definition 2.13 to the equivariant setting of Section 5. For any $\xi \in \sqrt{-1}\,\mathrm{Lie}\,T$, we consider the linear isomorphisms $\phi^*_\xi : \mathscr{C}^\infty(X,\mathbb{R})^K \to \mathscr{C}^\infty(X,\mathbb{R})^K$ by pullback with respect to $\phi_\xi \in T_\mathbb{C}$. The following basic result draws a link with the quantum channel of Definition 5.3.

Proposition 5.4. *For any* $\xi \in \sqrt{-1}\,\mathrm{Lie}\,T$*, the linear map*

$$\phi^*_{\xi/2p}\,\mathscr{B}_{h^p} : \mathscr{C}^\infty(X,\mathbb{R})^K \longrightarrow \mathscr{C}^\infty(X,\mathbb{R})^K \tag{5.6}$$

is a positive and self-adjoint operator with respect to the scalar product $L^2(h,\xi,p)$ *given by formula* (5.2).

Furthermore, the positive spectrums of $\phi^*_{\xi/2p}\,\mathscr{B}_{h^p}$ *and* $\mathscr{E}^\xi_{h^p}$ *coincide.*

Proof. The fact that $\phi^*_{\xi/2p}\,\mathscr{B}_{h^p}$ is a self-adjoint and positive operator on $\mathscr{C}^\infty(X,\mathbb{R})^K$ with respect to the scalar product $L^2(h,\xi,p)$ is a straightforward consequence of Proposition 5.2. Furthermore, this operator factorizes through the finite dimensional space

$\mathscr{L}(\mathscr{H}_p)^K$, so that in particular, it is a compact operator with smooth kernel. This implies that $\mathrm{Spec}(\phi^*_{\xi/2p}\mathscr{B}_{h^p})$ is discrete and contains a finite number of non-vanishing eigenvalues counted with multiplicity.

Let now $f \in \mathscr{C}^\infty(X,\mathbb{R})^K$ be an eigenfunction of $\phi^*_{\xi/2p}\mathscr{B}_{h^p}$ with eigenvalue $\lambda \neq 0$. Then from Definition 2.13 and Definition 5.3, we have

$$\mathscr{E}^\xi_{h^p}(T_{h^p}(f)) = T_{h^p}\left(\phi^*_{\xi/2p}\mathscr{B}_{h^p}(f)\right) = \lambda\, T_{h^p}(f)\,, \tag{5.7}$$

so that $T_{h^p}(f) \in \mathscr{L}(\mathscr{H}_p)^K$ is a non-vanishing eigenvector of $\mathscr{E}^\xi_{h^p}$ associated with the eigenvalue λ, since by definition $\sigma_{e^{L_\xi}\mathbf{s}_p}(T_{h^p}(f)) = \lambda f \neq 0$. This shows that the positive spectrums of $\phi^*_{\xi/2p}\mathscr{B}_{h^p}$ and $\mathscr{E}^\xi_{h^p}$ coincide. This concludes the proof. □

The following result is the analogue of Theorem 2.14 for the equivariant Berezin transform (5.6), where the role of the Riemannian Laplacian is played by the operator $\Delta_h^{(\xi)}$ of Lemma 2.6.

Proposition 5.5. *For any $\xi \in \sqrt{-1}\,\mathrm{Lie}\,T$ and $m \in \mathbb{N}$, there exists $C_m > 0$ such that for any $f \in \mathscr{C}^\infty(X,\mathbb{C})^K$ and all $p \in \mathbb{N}^*$, we have*

$$\left|\phi^*_{\xi/2p}\mathscr{B}_{h^p}f - f + p^{-1}\Delta_h^{(\xi)}f\right|_{\mathscr{C}^m} \leqslant \frac{C_m}{p^2}|f|_{\mathscr{C}^{m+6}}\,. \tag{5.8}$$

Furthermore, there exists $l \in \mathbb{N}$ such that the constant $C_m > 0$ can be chosen uniformly for $\xi \in \sqrt{-1}\,\mathrm{Lie}\,T$ in a compact set and $h \in \mathrm{Met}^+(L)$ in a bounded subset in $\mathscr{C}^l$-norm.

Proof. Using Theorem 2.14, we get for any $m \in \mathbb{N}$ a constant $C_m > 0$ such that for any $f \in \mathscr{C}^\infty(X,\mathbb{C})$ and all $p \in \mathbb{N}^*$, we have

$$\left|\mathscr{B}_{h^p}f - f + \frac{\Delta_h}{4\pi p}f\right|_{\mathscr{C}^m} \leqslant \frac{C_m}{p^2}|f|_{\mathscr{C}^{m+4}}\,. \tag{5.9}$$

On the other hand, considering the Taylor expansion of $\phi_{\xi/2p}$ in p^{-1}, we get for any $m \in \mathbb{N}$ a constant $C_m > 0$ such that for any $f \in \mathscr{C}^\infty(X,\mathbb{R})$ and all $p \in \mathbb{N}^*$,

$$\left|\phi^*_{\xi/2p}f - f - \frac{df.\xi}{2p}\right|_{\mathscr{C}^m} \leqslant \frac{C_m}{p^2}|f|_{\mathscr{C}^{m+2}}\,. \tag{5.10}$$

By definition (2.21) of the operator $\Delta_h^{(\xi)}$ and the fact that $df.\xi = 2df.\xi^{1,0}$ for $f \in \mathscr{C}^\infty(X,\mathbb{C})^K$ and $\xi \in \sqrt{-1}\,\mathrm{Lie}\,T$, this gives the result. □

Let $\varepsilon_0 > 0$ be smaller than the injectivity radius of (X, g_h^{TX}), fix $x_0 \in X$, and let $Z = (Z_1,...,Z_{2d}) \in \mathbb{R}^{2d}$ with $|Z| < \varepsilon_0$ be geodesic normal coordinates around x_0, where $|\cdot|$ is the Euclidean norm of $\mathbb{R}^{2d}$. For any $K(\cdot,\cdot) \in \mathscr{C}^\infty(X \times X,\mathbb{C})$, we write $K_{x_0}(\cdot,\cdot)$

for its image in these coordinates, and we write $|K_x|_{\mathscr{C}^m(X)}$ for the local $\mathscr{C}^m$-norm with respect to $x \in X$.

Let d^X be the Riemannian distance on (X, g_h^{TX}), and recall as in the proof of Proposition 5.4 that $\mathscr{B}_{h^p}$ admits a smooth Schwartz kernel, for all $p \in \mathbb{N}^*$. The main tool of this Section is the following asymptotic expansion as $p \to +\infty$ of this Schwartz kernel, which follows from [10, Th. 4.18']. Let $|\cdot|_{\mathscr{C}^m}$ denote the local $\mathscr{C}^m$ norm on local sections of $L^p \boxtimes (L^p)^*$. In the following statement, the estimate $O(p^{-\infty})$ means $O(p^{-k})$ in the usual sense as $p \to +\infty$, for all $k \in \mathbb{N}$.

Theorem 5.6. [22, Th. 3.7] *For any $m, k \in \mathbb{N}$, $\varepsilon > 0$, there is $C > 0$ such that for all $p \in \mathbb{N}^*$ and $x, y \in X$ satisfying $d^X(x,y) > \varepsilon$, we have*

$$|\mathscr{B}_{h^p}(x,y)|_{\mathscr{C}^m} \leqslant Cp^{-k} \,. \tag{5.11}$$

For any $m, k \in \mathbb{N}$, there is $N \in \mathbb{N}$, $C > 0$ such that for any $x_0 \in X$, $|Z|, |Z'| < \varepsilon_0$ and for all $p \in \mathbb{N}^*$, we have

$$\begin{aligned}\Big|p^{-d}\mathscr{B}_{h^p,x_0}(Z,Z') - \sum_{r=0}^{k-1} p^{-r/2} J_{r,x_0}(\sqrt{p}Z, \sqrt{p}Z') \exp(-\pi p|Z-Z'|^2)\Big|_{\mathscr{C}^m(X)} \\ \leqslant Cp^{-\frac{k}{2}}(1+\sqrt{p}|Z|+\sqrt{p}|Z'|)^N \exp(-\sqrt{p}|Z-Z'|/C) + O(p^{-\infty}) \,,\end{aligned} \tag{5.12}$$

where $J_{r,x_0}(Z,Z')$ are a family of polynomials in $Z, Z' \in \mathbb{R}^{2n}$ of the same parity as $r \in \mathbb{N}$, depending smoothly on $x_0 \in X$. Furthermore, for any $Z, Z' \in \mathbb{R}^{2n}$ we have

$$J_{0,x_0}(Z,Z') = 1 \quad \textit{and} \quad J_{1,x_0}(Z,Z') = 0 \,. \tag{5.13}$$

Finally, for any $m \in \mathbb{N}$, there exists $l \in \mathbb{N}$ such that the estimate (5.12) is uniform for $h \in \mathrm{Met}^+(L)^K$ in a bounded subset in $\mathscr{C}^l$-norm.

5.3. Spectral asymptotics

Fix $h \in \mathrm{Met}^+(L)^K$, and write $\langle\cdot,\cdot\rangle_{L^2}$ for the associated L^2-Hermitian product on $\mathscr{C}^\infty(X,\mathbb{C})^K$, defined by formula (2.22) for $\xi = 0$. We write $\|\cdot\|_{L_2}$ for the associated norm, and $L^2(X,\mathbb{C})^K$ for the induced completion of $\mathscr{C}^\infty(X,\mathbb{C})^K$. In the notations of Section 2.2, the Riemannian Laplacian Δ_h is then an elliptic self-adjoint operator acting on $L^2(X,\mathbb{C})^K$, and we write

$$0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \dots \leqslant \lambda_k \leqslant \dots \,, \tag{5.14}$$

for the increasing sequence of its eigenvalues. For all $j \in \mathbb{N}$, let $e_j \in \mathscr{C}^\infty(X,\mathbb{C})^K$ be the normalized eigenfunction associated with λ_j, so that $\|e_j\|_{L_2} = 1$ and $\Delta e_j = \lambda_j e_j$. For any $F : \mathbb{R} \to \mathbb{R}$ bounded, we define the bounded operator $F(\Delta)$ acting on $L_2(X, \mathbb{C})^K$ by the formula

$$F(\Delta)f = \sum_{i=0}^{+\infty} F(\lambda_j)\langle f, e_j\rangle_{L_2} e_j \ . \tag{5.15}$$

In particular we can consider its associated *heat operator* $e^{-t\Delta_h}$ acting on $\mathscr{C}^\infty(X, \mathbb{C})^K$, for all $t \geqslant 0$. For any $m \in 2\mathbb{N}$, write $\|\cdot\|_{H^m}$ for the norm defined for all $f \in \mathscr{C}^\infty(X, \mathbb{C})^K$ by

$$\|f\|_{H^m} := \|\Delta_h^{m/2} f\|_{L_2} + \|f\|_{L_2} \ . \tag{5.16}$$

By the classical Sobolev embedding theorem and the elliptic estimates for Δ_h, there exists $k \in \mathbb{N}$ such that for any $m \in 2\mathbb{N}$, there is $C_m > 0$ such that

$$|f|_{\mathscr{C}^m} \leqslant C_m \|f\|_{H^{m+k}} \ , \tag{5.17}$$

for all $f \in \mathscr{C}^\infty(X, \mathbb{C})^K$. Furthermore, there exists $l \in \mathbb{N}$ such that the constant $C_m > 0$ can be chosen uniformly for $h \in \mathrm{Met}^+(L)^K$ in a bounded subset in $\mathscr{C}^l$-norm. By convention, we set $\|f\|_{H^0} := \|f\|_{L_2}$.

We then have the following result, which is the analogue of [22, Prop. 3.9] for the equivariant Berezin transform (5.6).

Proposition 5.7. *For any $m \in 2\mathbb{N}$ and any $\xi \in \sqrt{-1}\,\mathrm{Lie}\,T$, there exists $C_m > 0$ such that for any $f \in \mathscr{C}^\infty(X, \mathbb{C})^K$ and all $p \in \mathbb{N}^*$, we have*

$$\left\| \left(e^{-\frac{\Delta_h}{4\pi p}} - e^{\theta_h(\xi)/2} \phi^*_{\xi/2p} \mathscr{B}_{h^p} e^{-\theta_h(\xi)/2} \right) f \right\|_{H^m} \leqslant \frac{C_m}{p} \|f\|_{H^m} \ . \tag{5.18}$$

Furthermore, there exists $l \in \mathbb{N}$ such that the constant $C_m > 0$ can be chosen uniformly for $\xi \in \sqrt{-1}\,\mathrm{Lie}\,T$ in a compact set and $h \in \mathrm{Met}^+(L)^K$ in a bounded subset in $\mathscr{C}^l$-norm.

Proof. In this proof, we use the notation $O(|W|^k) \in \mathbb{R}^{2n}$ or $\mathbb{R}$ in the usual sense as $W \in \mathbb{R}^{2n}$ goes to 0, for any $k \in \mathbb{N}$, and write $\langle\cdot,\cdot\rangle$ for the Euclidean product of $\mathbb{R}^{2n}$.

First note that the operator

$$\widetilde{\mathscr{B}}_p := e^{\theta_h(\xi)/2} \phi^*_{\xi/2p} \mathscr{B}_{h^p} e^{-\theta_h(\xi)/2} \tag{5.19}$$

has a smooth kernel given for all x, $y \in X$ by

$$\widetilde{\mathscr{B}}_p(x, y) = e^{\theta_h(\xi)/2}(x) e^{-\theta_h(\xi)/2}(y)\, \mathscr{B}_{h^p}(\phi_{\xi/2p}(x), y) \,. \tag{5.20}$$

Recall formula (5.12) for the asymptotic expansion as $p \to +\infty$ of the Berezin transform $\mathscr{B}_{h^p}$ in geodesic coordinates Z, $Z' \in \mathbb{R}^{2n}$ with $|Z|$, $|Z'| < \varepsilon_0$ around $x_0 \in X$. For all $\xi \in \sqrt{-1}\,\mathrm{Lie}\,T$ in a compact set, considering the Taylor expansion of $\phi_{\xi/2p}(Z)$ as $p^{-1} \to 0$ and $|Z| \to 0$, we get

$$\begin{aligned}
&\exp(-\pi p\,|\phi_{\xi/2p}(Z) - Z'|^2) \\
&= \exp(-\pi p|Z-Z'|^2 - \pi\langle \xi_{x_0}, Z - Z'\rangle + \langle O(|Z|) + p^{-1}O(|Z|), Z - Z'\rangle) \\
&= \exp(-\pi p|Z-Z'|^2)\Big(1 - \pi p^{-1/2}\langle \xi_{x_0}, \sqrt{p}(Z-Z')\rangle \\
&\qquad\qquad + p^{-1}\langle O(|\sqrt{p}Z|) + p^{-1}O(|\sqrt{p}Z|), \sqrt{p}(Z-Z')\rangle\Big)\,.
\end{aligned} \tag{5.21}$$

On the other hand, recall from Proposition 2.2 that the imaginary part of the holomorphy potential equation (2.7) gives $2\pi\iota_{J\xi}\omega_h = -d\theta_h(\xi)$. Using definition (2.2) for $\langle\cdot,\cdot\rangle := g^{TX}_{h,x_0}$, we get the following Taylor expansions as $Z \to 0$,

$$\begin{aligned}
&e^{\theta_h(\xi)/2}(Z)e^{-\theta_h(\xi)/2}(Z') \\
&= 1 + d\theta_h(\xi).(Z-Z')/2 + (O(|Z|) + O(|Z'|))^2 \\
&= 1 + \pi\langle \xi, Z - Z'\rangle + (O(|Z|) + O(|Z'|))^2 \\
&= 1 + p^{-1/2}\pi\langle \xi, \sqrt{p}(Z-Z')\rangle + p^{-1}(O(|\sqrt{p}Z|) + O(|\sqrt{p}Z'|))^2\,.
\end{aligned} \tag{5.22}$$

Multiplying the estimates (5.21) and (5.22) gives

$$\begin{aligned}
&e^{\theta_h(\xi)/2}(Z)e^{-\theta_h(\xi)/2}(Z')\exp(-\pi p|\phi_{\xi/2p}(Z) - Z'|^2) \\
&\qquad = \exp(-\pi p|Z-Z'|^2)(1 + p^{-1}O(|\sqrt{p}Z|) + p^{-1}O(|\sqrt{p}Z'|))\,,
\end{aligned} \tag{5.23}$$

so that the coefficient of order $p^{-1/2}$ vanishes. Plugging the expansion (5.12) in formula (5.20) for the Schwartz kernel of $\widetilde{\mathscr{B}}_p$, we see that it also satisfies Theorem 5.6, with first coefficients satisfying (5.13).

Setting now $R_p := e^{\frac{\Delta}{4\pi p}} - \widetilde{\mathscr{B}}_p$, and using the classical small-time asymptotic expansion of the heat kernel, as given for example in [1, Th. 2.29], and by (5.12), we see that its Schwartz kernel $R_p(\cdot,\cdot)$ with respect to dv_X satisfies $R_p(x,y) = O(p^{-\infty})$ for all $x, y \in X$ satisfying $d^X(x,y) > \varepsilon_0$, and we get for any $m \in \mathbb{N}$ a constant $C > 0$ and $N \in \mathbb{N}$ such that

$$\begin{aligned}
&|R_{p,x_0}(Z,Z')|_{\mathscr{C}^m(X)} \\
&\qquad \leqslant Cp^{-1}(1 + \sqrt{p}|Z| + \sqrt{p}|Z'|)^N \exp(-\sqrt{p}|Z-Z'|/C) + O(p^{-\infty})\,.
\end{aligned} \tag{5.24}$$

Following the proof of [22, Prop. 3.9], this readily implies that for any $m \in 2\mathbb{N}$, there is a constant $C_m > 0$ such that for all $f \in \mathscr{C}^\infty(X,\mathbb{R})$,

$$\|R_p(f)\|_{H^m} \leqslant \frac{C_m}{p}\|f\|_{H^m}\,. \tag{5.25}$$

The uniformity of $C_m > 0$ with respect to $h \in \mathrm{Met}^+(L)^K$ comes from the uniformity of the small-time asymptotic expansion of the heat kernel with respect to the Riemannian metric together with the uniformity in Theorem 5.6. This gives the result. □

Let us now write $\|\cdot\|_{L^2(h,\xi)}$ and $\|\cdot\|_{L^2(h,\xi,p)}$ for the norms associated with the L^2-Heermitian products (2.22) and (5.2) respectively. Using Theorem 2.9, Theorem 2.14 and Proposition 3.2, we get a constant $C > 0$, uniform for $\xi \in \sqrt{-1}\,\mathrm{Lie}\,T$ in a compact set and $h \in \mathrm{Met}^+(L)^K$ in a bounded subset in $\mathscr{C}^l$-norm for some $l \in \mathbb{N}$, such that

$$\left(1 - \frac{C}{p}\right)\|\cdot\|_{L^2(h,\xi)} \leqslant \|\cdot\|_{L^2(h,\xi,p)} \leqslant \left(1 + \frac{C}{p}\right)\|\cdot\|_{L^2(h,\xi)} . \tag{5.26}$$

Proposition 5.7 implies the following key lemma, which is an analogue of [22, Lem. 3.10] for the equivariant Berezin transform (5.6).

Lemma 5.8. *For any fixed $L > 0$, consider sequences $\{f_p \in \mathscr{C}^\infty(X,\mathbb{C})^K\}_{p\in\mathbb{N}^*}$ and $\{\mu_p \in \mathrm{Spec}(\phi^*_{\xi/2p}\mathscr{B}_{h^p})\}_{p\in\mathbb{N}^*}$, such that $p|1-\mu_p| < L$ for all $p \in \mathbb{N}^*$ and*

$$\|f_p\|_{L^2(h,\xi,p)} = 1 \quad \text{and} \quad \phi^*_{\xi/2p}\mathscr{B}_{h^p}(f_p) = \mu_p f_p . \tag{5.27}$$

Then for all $m \in 2\mathbb{N}$, there exists $C_{L,m} > 0$ such that for all $p \in \mathbb{N}^*$, we have

$$\|f_p\|_{H^m} \leqslant C_{L,m} , \tag{5.28}$$

not depending on $\xi \in \sqrt{-1}\,\mathrm{Lie}\,T$ in a compact set and $h \in \mathrm{Met}^+(L)^K$ in a bounded subset in $\mathscr{C}^l$-norm for some $l \in \mathbb{N}$.

Proof. Let $\{f_p \in \mathscr{C}^\infty(X,\mathbb{C})^K\}_{p\in\mathbb{N}^*}$ be a sequence satisfying (5.27) as above, and for all $p \in \mathbb{N}^*$, set

$$\|\cdot\|_p := \|e^{-\theta_h(\xi)/2}\cdot\|_{L^2(h,\xi,p)} \quad \text{and} \quad \widetilde{f}_p := e^{\theta_h(\xi)/2} f_p . \tag{5.29}$$

In particular, we have $\|\widetilde{f}_p\|_p = 1$ and $\widetilde{\mathscr{B}}_p(\widetilde{f}_p) = \mu_p\widetilde{f}_p$ for all $p \in \mathbb{N}^*$, for the operator $\widetilde{\mathscr{B}}_p$ defined by formula (5.19), and the estimate (5.26) implies the estimate (5.28) for $m = 0$.

By induction on $m \in 2\mathbb{N}$, assume now that (5.28) is satisfied for $m-2$. Write

$$\begin{aligned} p(e^{-\frac{\Delta_h}{4\pi p}} - \widetilde{\mathscr{B}}_p)\widetilde{f}_p &= p(1-\mu_p)\widetilde{f}_p - p(\mathbb{1} - e^{-\frac{\Delta_h}{4\pi p}})\widetilde{f}_p \\ &= p(1-\mu_p)\widetilde{f}_p - \Delta_h F(\Delta_h/p)\widetilde{f}_p , \end{aligned} \tag{5.30}$$

where the bounded operator $F(\Delta_h/p)$ acting on $L_2(X,\mathbb{C})^K$ is defined as in (5.15) for the continuous function $F : \mathbb{R} \to \mathbb{R}$ given for any $s \in \mathbb{R}^*$ by $F(s) = 4\pi(1 - e^{-s/4\pi})/s$. As $|p(1-\mu_p)| < L$ for all $p \in \mathbb{N}^*$, by Proposition 5.7 and formula (5.16) for $\|\cdot\|_{H^m}$, this gives a constant $C_m > 0$, uniform $h \in \mathrm{Met}^+(L)^K$ in a bounded subset in $\mathscr{C}^l$-norm for some $l \in \mathbb{N}$, such that

$$\|F(\Delta_h/p)\widetilde{f}_p\|_{H^m} \leqslant C_m \|\widetilde{f}_p\|_{H^{m-2}} \;. \tag{5.31}$$

On the other hand, note that by hypothesis, we have $\mu_p \to 1$ as $p \to +\infty$. Using Proposition 5.7 again, we then get $\varepsilon_m > 0$ and $p_m \in \mathbb{N}^*$, uniform $h \in \mathrm{Met}^+(L)^K$ in a bounded subset in $\mathscr{C}^l$-norm, such that for all $p > p_m$,

$$\begin{aligned}\|F(\Delta_h/p)\widetilde{f}_p\|_{H^{2m}} &\geqslant \|F(\Delta_h/p)\widetilde{f}_p + (\widetilde{\mathscr{B}}_p - e^{-\frac{\Delta_h}{4\pi p}})\widetilde{f}_p\|_{H^m} - \|(\widetilde{\mathscr{B}}_p - e^{-\frac{\Delta_h}{4\pi p}})\widetilde{f}_p\|_{H^m} \\ &\geqslant \inf_{s>0} \{F(s) + \mu_p - e^{-s/4\pi}\} \|\widetilde{f}_p\|_{H^m} - C_m p^{-1} \|\widetilde{f}_p\|_{H^m} \\ &\geqslant \varepsilon_m \|\widetilde{f}_p\|_{H^m} \;.\end{aligned} \tag{5.32}$$

Hence by (5.31), we get a constant $C_{L,m} > 0$, uniform in $h \in \mathrm{Met}^+(L)^K$ in a bounded subset in $\mathscr{C}^l$-norm, such that for all $p \in \mathbb{N}^*$, we have $\|\widetilde{f}_p\|_{H^m} \leqslant C_{L,m}$, which gives (5.28) by (5.29). □

Using Lemma 2.6, write

$$0 = \lambda_0(h,\xi) < \lambda_1(h,\xi) \leqslant \cdots \leqslant \lambda_k(h,\xi) \leqslant \ldots \tag{5.33}$$

for the increasing sequence of eigenvalues of $\Delta_h^{(\xi)}$, and using Proposition 5.4, write

$$\gamma_0(h^p,\xi) \geqslant \gamma_1(h^p,\xi) \geqslant \cdots \geqslant \gamma_1(h^p,\xi) \geqslant \cdots \geqslant 0 \tag{5.34}$$

for the decreasing sequence of eigenvalues of $\phi^*_{\xi/2p}\mathscr{B}_{h^p}$. The following result is the analogue of [22, Th. 3.1] for the equivariant Berezin transform (5.6), and is the analytic basis of our proof of Theorem 1.1.

Theorem 5.9. *For every integer $k \in \mathbb{N}$, there exists a constant $C_k > 0$ such that for any $p \in \mathbb{N}^*$,*

$$\left|1 - \gamma_k(h^p,\xi) - p^{-1}\lambda_k(h,\xi)\right| \leqslant C_k p^{-2} \;. \tag{5.35}$$

Moreover, there exists $l \in \mathbb{N}$ such that the constant $C_k > 0$ can be chosen uniformly for $\xi \in \sqrt{-1}\,\mathrm{Lie}\,T$ in a compact set and $h \in \mathrm{Met}^+(L)^K$ in a bounded subset in $\mathscr{C}^l$-norm.

Proof. By Proposition 5.5 and by the Sobolev estimate (5.17), there is $m \in 2\mathbb{N}$, $l \in \mathbb{N}$ and a constant $C > 0$ such that for any $f \in \mathscr{C}^\infty(X,\mathbb{C})^K$,

$$\left\|p(1 - \phi^*_{\xi/2p}\mathscr{B}_{h^p})f - \Delta_h^{(\xi)} f\right\|_{L_2(h,\xi)} \leqslant C p^{-1} \|f\|_{H^m} \;, \tag{5.36}$$

and the estimate (5.26) shows that equation (5.36) also holds in the norm $\|\cdot\|_{L_2(h,\xi,p)}$. Let now $j \in \mathbb{N}$ be fixed, and let $e_j \in \mathscr{C}^\infty(X,\mathbb{C})$ satisfy $\Delta_h^{(\xi)} e_j = \lambda_j(h,\xi)e_j$ and $\|e_j\|_{L_2} = 1$. We then get $C > 0$ such that for all $p \in \mathbb{N}^*$,

$$\left\| p(1 - \phi^*_{\xi/2p}\mathscr{B}_p)e_j - \lambda_j(h,\xi)e_j \right\|_{L_2(h,\xi,p)} \leqslant Cp^{-1} \,. \tag{5.37}$$

Let $m_j \in \mathbb{N}$ be the multiplicity of $\lambda_j(h,\xi)$ as an eigenvalue of $\Delta_h^{(\xi)}$. Then the estimate (5.37) for all eigenfunctions of $\Delta_h^{(\xi)}$ associated with $\lambda_j(h,\xi)$ gives a constant $C_j > 0$ such that for all $p \in \mathbb{N}^*$,

$$\#\left(\mathrm{Spec}\left(p(1-\phi^*_{\xi/2p}\mathscr{B}_p)\right) \cap \left[\lambda_j(h,\xi) - C_j p^{-1}, \lambda_j(h,\xi) + C_j p^{-1}\right]\right) \geqslant m_j \,. \tag{5.38}$$

Conversely, fix $L > 0$ and let $\{f_p\}_{p\in\mathbb{N}^*}$ be the sequence of normalized eigenfunctions considered in Lemma 5.8. Then by (5.36), we get $C > 0$ such that

$$\left\| p(1-\mu_p)f_p - \Delta_h^{(\xi)} f_p \right\|_{L_2(h,\xi)} \leqslant Cp^{-1} \,. \tag{5.39}$$

In particular, we get that

$$\mathrm{dist}\left(p(1-\mu_p), \mathrm{Spec}\,\Delta_h^{(\xi)}\right) \leqslant Cp^{-1} \,, \tag{5.40}$$

showing that all eigenvalues of $p(1 - \phi^*_{\xi/2p}\mathscr{B}_p)$ bounded by some $L > 0$ have to be included in the left hand side of (5.38).

Let us finally show that (5.38) is an equality for $p \in \mathbb{N}^*$ big enough. Let $l \in \mathbb{N}$ with $l \geqslant m_j$ be such that for all $p \in \mathbb{N}^*$, there exists an orthonormal family $\{f_{k,p}\}_{1\leqslant k\leqslant l}$ of eigenfunctions of $\phi^*_{\xi/2p}\mathscr{B}_p$ for $\|\cdot\|_{l^2(h,\xi,p)}$ with associated eigenvalues $\{\mu_{k,p} \in \mathbb{R}\}_{1\leqslant k\leqslant l}$ satisfying

$$p(1-\mu_{k,p}) \in [\lambda_j(h,\xi) - Cp^{-1}, \lambda_j(h,\xi) + Cp^{-1}] \,, \quad \text{for all} \quad 1 \leqslant k \leqslant l \,. \tag{5.41}$$

By Lemma 5.8 and (5.26), the compact inclusion of the Sobolev space H^4 in H^2 gives a subsequence of $\{f_{k,p}\}_{p\in\mathbb{N}^*}$ converging to a function f_k in H^2-norm, for all $1 \leqslant k \leqslant l$. In particular, using (5.26) again, the family $\{f_k\}_{1\leqslant k\leqslant l}$ is orthonormal in $L_2(h,\xi)$ and satisfies $\Delta_h^{(\xi)} f_k = \lambda_j(h,\xi)f_k$ for all $1 \leqslant k \leqslant l$ by (5.39). By definition of the multiplicity $m_j \in \mathbb{N}$ of $\lambda_j(h,\xi)$, this forces $l = m_j$. We thus get

$$\#\left(\mathrm{Spec}\left(p(1-\phi^*_{\xi/2p}\mathscr{B}_p)\right) \cap \left[\lambda_j(h,\xi) - Cp^{-1}, \lambda_j(h,\xi) + Cp^{-1}\right]\right) = m_j \,. \tag{5.42}$$

By the uniformity of the constants in Proposition 5.7, Lemma 5.8, (5.17), (5.26), and by the smooth dependance of the eigenfunctions of $\phi^*_{\xi/2p}\mathscr{B}_{h^p}$ and $\Delta_h^{(\xi)}$ with respect to the initial data, we get the result. □

6. Proof of the main theorem

In this Section, we use the preliminary results of all previous Sections to establish Theorem 1.1. In Section 6.1, we will show how to use the results of Section 2.2 on the differential operator of Tian and Zhu and the results of Section 3.2 on the quantized Futaki invariants (1.7) to construct approximately balanced metrics from a given Kähler-Ricci soliton. The heart of the proof is in Section 6.2, where we use these approximately balanced metrics to establish existence and convergence in Theorem 1.1, applying the tools of Section 5 on the moment map picture of Section 4.3. Finally, we establish uniqueness in Section 6.3 using Proposition 4.7 and an energy functional for the relative moment map.

Throughout the whole section, we will assume given a positive Hermitian metric $h_\infty \in \mathrm{Met}^+(L)$ such that $\omega_{h_\infty} \in \Omega^2(X,\mathbb{R})$ is a Kähler-Ricci soliton with respect to $\xi_\infty \in \mathrm{Lie}\,\mathrm{Aut}(X)$ in the sense of (1.2). We write $K \subset \mathrm{Aut}_0(X)$ for the identity component of the subgroup of isometries of $(X, g^{TX}_{h_\infty})$, and $T \subset K$ for the identity component of its center, so that $h_\infty \in \mathrm{Met}^+(L)^T$ and $\xi_\infty \in \sqrt{-1}\,\mathrm{Lie}\,T$.

6.1. Approximately balanced metrics

The following semi-classical estimate on the Berezin symbol is inspired by [13, Lem. 24, (35)-(35')]. We provide a proof that does not make use of any moment map construction. Write $\|\cdot\|_{tr}$ for the trace norm on $\mathrm{End}(H^0(X,L^p))$.

Lemma 6.1. *For any $m \in \mathbb{N}$ and $h \in \mathrm{Met}^+(L)^T$, there exists a constant $C_m > 0$ such that for any $A \in \mathscr{L}(\mathscr{H}_p)^T$ and all $p \in \mathbb{N}^*$, we have*

$$|\sigma_{h^p}(A)|_{\mathscr{C}^m} \leqslant C_m\, p^{n+\frac{m}{2}}\, \|A\|_{tr}\,. \tag{6.1}$$

Furthermore, there exists $l \in \mathbb{N}$ such that the constant $C_m > 0$ can be chosen uniformly for $h \in \mathrm{Met}^+(L)^T$ in a bounded subset in $\mathscr{C}^l$-norm.

Proof. Using the Sobolev embedding theorem as in [29, Lem. 2], we get for any $m \in \mathbb{N}$ and $h \in \mathrm{Met}^+(L^p)$ a constant $C_m > 0$ such that for all $p \in \mathbb{N}^*$ and any holomorphic section $s \in H^0(X,L^p)$, we have

$$|s|_{\mathscr{C}^m(h^p)} \leqslant C_m\, p^{\frac{n+m}{2}}\, \|s\|_{L^2(h^p)}\,, \tag{6.2}$$

where $|\cdot|_{\mathscr{C}^m(h^p)}$ denotes the $\mathscr{C}^m$-norm with respect to the Chern connection of (L^p,h^p). Replacing h^p by $e^f h^p$ with $\|f\|_{\mathscr{C}^m} < C$ for some fixed $C > 0$ in (6.2), we readily see that $C_m > 0$ can be chosen uniformly for all $h^p \in \mathrm{Met}^+(L^p)^T$ in a subset bounded in $\mathscr{C}^m$-norm.

Let now $\{s_j\}_{j=1}^{n_p}$ be an orthonormal basis for $L^2(h^p)$, and for $A \in \mathrm{Herm}(\mathbb{C}^{n_p})^T$, write $(A_{jk})_{j,k=1}^{n_p} \in \mathrm{Herm}(\mathbb{C}^{n_p})$ for its matrix in this basis. Using formulas (4.12) and (4.14), we know that

$$\sigma_{h^p}(A) = \rho_{h_p}^{-1} \sum_{j,\,k=1}^{n_p} A_{jk}\langle s_k, s_j\rangle_{h^p} \,. \tag{6.3}$$

On the other hand, using Theorem 2.9, we get $C > 0$ such that

$$|\rho_{h^p}^{-1}|_{\mathscr{C}^0} = \min_{x\in X}\left(p^n \frac{\omega_h^n}{d\nu_h\, n!} + O(p^{n-1})\right)^{-1} \leqslant Cp^{-n} \,. \tag{6.4}$$

Using the Leibniz rule on successive derivatives of $\rho_{h_p}^{-1}$ and by Theorem 2.9 again, this implies the existence of $C_m' > 0$, uniform in the $\mathscr{C}^l$-norm of h^p for some $l \in \mathbb{N}$, such that for all $p \in \mathbb{N}^*$, we have

$$|\rho_{h^p}^{-1}|_{\mathscr{C}^m} \leqslant C_m' p^{-n} \,. \tag{6.5}$$

Then using the estimates (6.2), (6.5) and the fact, following from Theorem 2.9 and formula (2.39), that $n_p \leqslant Cp^n$ as $p \to +\infty$ for some $C > 0$, we can use Cauchy-Schwartz inequality on the trace norm to get for all $m \in \mathbb{N}$ a uniform constant $C' > 0$, such that for all $A \in \mathrm{Herm}(\mathbb{C}^{n_p})^T$ and all $p \in \mathbb{N}^*$,

$$\begin{aligned}
|\sigma_{h^p}(A)|_{\mathscr{C}^m} &\leqslant \left|\rho_{h^p}^{-1}\right|_{\mathscr{C}^m} \sum_{j,\,k=1}^{n_p} |A_{jk}\langle s_k, s_j\rangle_{h^p}|_{\mathscr{C}^m} \\
&\leqslant C_m' p^{-n}\|A\|_{tr}\sqrt{\sum_{j,\,k=1}^{n_p}\sum_{r,l=1}^{m}\binom{m}{r}\binom{m}{l}|s_k|_{\mathscr{C}^r(h^p)}|s_j|_{\mathscr{C}^{m-r}(h^p)}|s_k|_{\mathscr{C}^l(h^p)}|s_j|_{\mathscr{C}^{m-l}(h^p)}} \\
&\leqslant C' p^{-n} p^{n+\frac{m}{2}}\,\|A\|_{tr}\, n_p \leqslant C'Cp^{n+\frac{m}{2}}\,\|A\|_{tr} \,.
\end{aligned} \tag{6.6}$$

This gives the result. □

The following result is an extension of the analogous result of Donaldson [13, Th. 26] to the case of general $\mathrm{Aut}(X)$, and is an anticanonical analogue of the result of Sano and Tipler in [39, Th. 5.5]. The proof closely follows their strategy.

Proposition 6.2. *There exist K-invariant functions $f_r \in \mathscr{C}^\infty(X,\mathbb{R})^T$ for all $r \in \mathbb{N}$, such that for every $k, m \in \mathbb{N}$, there exists a constant $C_{k,m} > 0$ such that all $p \in \mathbb{N}^*$ big enough, the T-invariant positive Hermitian metric*

$$h_k(p) := \exp\left(\sum_{r=1}^{k-1} \frac{1}{p^r} f_r\right) h_\infty \in \mathrm{Met}^+(L^p)^K \,, \tag{6.7}$$

have associated Rawnsley function $\rho_{h^p_k(p)} \in \mathscr{C}^\infty(X,\mathbb{R})$ *satisfying*

$$\left| \rho_{h^p_k(p)}\, \sigma_{h^p_k(p)}(e^{L_{\xi_p}/p}) - \frac{\operatorname{Tr}[e^{L_{\xi_p}/p}]}{\operatorname{Vol}\left(d\nu_{h_k(p)}\right)} \right|_{\mathscr{C}^m} \leqslant C_{k,m} p^{n-k}\,, \tag{6.8}$$

for the sequence $\{\xi_p \in \sqrt{-1}\operatorname{Lie} T\}_{p\in\mathbb{N}^*}$ *of Corollary 3.4.*

Proof. In this proof, the notation $O(p^{-k})$ for some $k \in \mathbb{N}$ is taken to be in its usual sense, uniformly in $\mathscr{C}^m$-norm for all $m \in \mathbb{N}$ and uniform the $\mathscr{C}^l$-norm of $h \in \operatorname{Met}^+(L)$ for some $l \in \mathbb{N}$.

Fix $h^p \in \operatorname{Met}^+(L^p)^K$, and consider the setting of Section 5. Using Proposition 3.1 and the definition of the exponential of an operator, we know that for any $\xi \in \operatorname{Lie}\operatorname{Aut}(X)$, we have $e^{L_\xi/p} = \operatorname{Id} + |\xi|\, O(1)$, so that Proposition 3.6 implies that for any $k \in \mathbb{N}$, we have $e^{L_{\xi_p}/p} = e^{L_{\xi_\infty}/p} \prod_{j=1}^k e^{L_{p^{-j}\xi^{(j)}}/p} + O(p^{-k-1})$. We can then use Proposition 3.2 and Theorem 2.11 to get functions $\eta^{(j)}_\xi \in \mathscr{C}^\infty(X,\mathbb{C})$ for all $j \in \mathbb{N}$, depending smoothly on $\xi \in \sqrt{-1}\operatorname{Lie} T$, such that for any $k \in \mathbb{N}$, we have

$$e^{L_{\xi_p}/p} = T_{h^p}\left(e^{\theta_h(\xi_p)}\right) + \sum_{j=1}^k p^{-j}\, T_{h^p}(\eta^{(j)}_{\xi_p}) + O(p^{-k-1})\,. \tag{6.9}$$

Write $R_k(p) \in \mathscr{L}(\mathscr{H}_p)$ for the remainder in (6.9), so that $\|R_k(p)\|_{op} = O(p^{-k-1})$. Using the fact $n_p = O(p^n)$ by Theorem 2.9 and formula (2.39), Cauchy-Schwartz inequality then implies that $\|R_k(p)\|_{tr} = O(p^{\frac{n}{2}-k-1})$. Using now Theorem 2.14, we get functions $g_j(h) \in \mathscr{C}^\infty(X,\mathbb{R})$ for all $j \in \mathbb{N}$, depending smoothly in the successive derivatives of $h \in \operatorname{Met}^+(L)$, such that for any $k_0 \in \mathbb{N}$, Lemma 6.1 applied to the expansion (6.9) for $k > k_0 + (3n+m)/2$ gives

$$\begin{aligned} \sigma_{h^p}(e^{L_{\xi_p}/p}) &= \mathscr{B}_{h^p}(e^{\theta_h(\xi_\infty)}) + \sum_{j=1}^k p^{-j}\, \mathscr{B}_{h^p}(\eta^{(j)}_{\xi_p}) + \sigma_{h^p}(R_k(p)) \\ &= e^{\theta_h(\xi_\infty)} + \sum_{j=1}^{k_0} p^{-j}\, g_j(h) + O(p^{-k_0-1})\,. \end{aligned} \tag{6.10}$$

Comparing with Theorem 2.9, and using Proposition 4.5, we get K-invariant functions $f_j(h) \in \mathscr{C}^\infty(X,\mathbb{R})^K$ for all $j \in \mathbb{N}$, depending smoothly in the successive derivatives of $h \in \operatorname{Met}^+(L)$, such that for any $k \in \mathbb{N}$, we have

$$p^{-n}\, \sigma_{h^p}(e^{L_{\xi_p}/p})\, \rho_{h^p} = \frac{e^{\theta_h(\xi_\infty)}\omega_h^n}{n!\, d\nu_h} + \sum_{j=1}^k p^{-j}\, f_j(h) + O(p^{-k-1})\,. \tag{6.11}$$

Via the characterization (2.20), we see that the first coefficient of (6.11) is constant if and only if $h \in \mathrm{Met}^+(L)$ is a Kähler-Ricci soliton with respect to ξ_∞. Integrating both sides against $d\nu_h$ and using formula (4.23), this implies the result for $k = 1$.

Recall that Δ_h denotes the scalar Riemannian Laplacian of (X, g_h^{TX}). Using the variation formula (2.3) for the anticanonical volume form and a classical formula in Kähler geometry, for any $f \in \mathscr{C}^\infty(X, \mathbb{R})$ we get

$$\frac{\partial}{\partial t}\Big|_{t=0} \frac{\omega^n_{e^{tf}h_\infty}}{d\nu_{e^{tf}h_\infty}} = \left(\frac{1}{4\pi}\Delta_{h_\infty} f - f\right)\frac{\omega^n_{h_\infty}}{d\nu_{h_\infty}}\,. \tag{6.12}$$

For any $\xi \in \sqrt{-1}\,\mathrm{Lie}\,T$ and $h \in \mathrm{Met}^+(L)^K$, recall the operator $\Delta_h^{(\xi)}$ acting on $\mathscr{C}^\infty(X, \mathbb{R})^K$ defined in Lemma 2.6. Using Proposition 2.2 on holomorphy potentials and Theorem 2.9, the expansion (6.11) implies that for all $k \in \mathbb{N}$ and $f \in \mathscr{C}^\infty(X, \mathbb{R})^T$, we have

$$\begin{aligned} &p^{-n}\,\sigma_{(e^{p^{-k}f}h)^p}(e^{L_{\xi_p}/p})\,\rho_{(e^{p^{-k}f}h)^p} \\ &\quad = \frac{e^{\theta_h(\xi_\infty)}\omega_h^n}{n!\,d\nu_h} + \sum_{j=1}^{k-1} p^{-j}\,f_j(h) + p^{-k}\left(f_k(h) + \Delta_{h_\infty}^{(\xi_\infty)} f - f\right) + O(p^{-k-1})\,. \end{aligned} \tag{6.13}$$

On the other hand, using Propositions 2.12 and 3.1 and the definition (1.7) of the quantized Futaki invariant, Corollary 3.4 implies that for any $h \in \mathrm{Met}^+(L)$ and all $\eta \in \mathrm{Lie}\,\mathrm{Aut}(X)$, we have

$$\int_X \theta_h(\eta)\,\sigma_{h^p}(e^{L_{\xi_p}/p})\,\rho_{h^p}\,d\nu_h = \frac{\mathrm{Fut}_p^{\xi_p}(\eta)}{p+1} = 0\,. \tag{6.14}$$

Using Proposition 2.7, this implies that the coefficients in the expansion (6.11) for h_∞ satisfy

$$f_j(h_\infty) \in \left(\mathrm{Ker}\left(\Delta_{h_\infty}^{(\xi_\infty)} - \mathrm{Id}\right)\right)^\perp \quad \text{for all} \quad j \in \mathbb{N}\,, \tag{6.15}$$

for the L^2-scalar product $L^2(h,\xi)$ on $\mathscr{C}^\infty(X, \mathbb{R})^K$, defined by formula (2.22) using the characterization (2.20) of Kähler-Ricci solitons. Thus for all $j \in \mathbb{N}$, there exists a function $f_j \in \mathscr{C}^\infty(X, \mathbb{R})^K$ satisfying $f_j(h_\infty) = f_j - \Delta_{h_\infty}^{(\xi_\infty)} f_j$. Taking $h_1(p) := e^{f_1/p}h_\infty \in \mathrm{Met}^+(L)^K$, the second coefficient of the expansion (6.13) with $k = 2$ vanishes, and integrating both sides against $d\nu_{h_1(p)}$ gives the result for $k = 2$ via formula (4.23) as above.

Let us now assume that for some $k \in \mathbb{N}$, we have positive Hermitian metrics $h_k(p) \in \mathrm{Met}^+(L)^K$ as in (6.7) satisfying

$$\sigma_{h_k(p)}(e^{L_{\xi_p}/p})\rho_{h_k^p(p)} = \frac{e^{\theta_{h_\infty}(\xi_\infty)}\omega^n_{h_\infty}}{n!\,d\nu_{h_\infty}} + O(p^{-k})\,. \tag{6.16}$$

As we have $h_k(p) \to h_\infty$ smoothly as $p \to +\infty$ by hypothesis, we can again apply Theorem 2.9 to get expansion (6.11) for $h_k^p(p)$, and taking the Taylor expansion as $p \to +\infty$ of the coefficients $f_r(h_k(p))$ for all $1 \leqslant r \leqslant k+1$, we then get for any $f \in \mathscr{C}^\infty(X,\mathbb{R})^K$,

$$
\begin{aligned}
p^{-n}\, \sigma_{e^{p^{-k}f}h}(e^{L_{\xi_p}/p})\rho_{(e^{p^{-k}f}h)^p} \\
= \frac{e^{\theta_{h_\infty}(\xi_\infty)}\omega_{h_\infty}^n}{n!\, d\nu_{h_\infty}} + p^{-k}\left(f_k(h_\infty) + \Delta_{h_\infty}^{(\xi_\infty)} f - f\right) + O(p^{-k-1})\,. \qquad (6.17)
\end{aligned}
$$

Taking $h_{k+1}(p) := e^{f_k/p^k} h_k(p) \in \mathrm{Met}^+(L^p)^K$ for all $p \in \mathbb{N}^*$, where $f_k \in \mathscr{C}^\infty(X,\mathbb{R})^K$ satisfies $f_k(h_\infty) = f_k - \Delta_{h_\infty}^{(\xi_\infty)} f_k$ thanks to (6.15), we get the result for $k+1$ via formula (4.23) as above. This gives the result for general $k \in \mathbb{N}$ by induction. □

For any $k \in \mathbb{N}$ and $p \in \mathbb{N}^*$ big enough, consider the positive Hermitian metrics $h_k(p) \in \mathrm{Met}^+(L)^K$ constructed in Proposition 6.2 and let $\mathbf{s}_k(p) \in \mathscr{B}(H^0(X,L^p))^T$ be orthonormal with respect to $L^2(h_k^p(p))$. The following Lemma shows that these metrics indeed approximate the Kähler-Ricci soliton.

Lemma 6.3. *For any k, k_0, $m \in \mathbb{N}$ with $k \geqslant k_0 > n+1+m/2$, there exists $C > 0$ such that for all $p \in \mathbb{N}^*$ and any $B \in \mathrm{Herm}(\mathbb{C}^{n_p})^T$ with $\|B\|_{tr} \leqslant C^{-1}p^{-k_0}$, we have*

$$
\begin{aligned}
\left|\omega_{e^B e^{L_{\xi_p}/2p}\mathbf{s}_k(p)} - \omega_{h_\infty}\right|_{\mathscr{C}^m} \leqslant \frac{C}{p}\,, \\
\left|\frac{d\nu_{e^B e^{L_{\xi_p}/2p}\mathbf{s}_k(p)}}{d\nu_{h_k(p)}} - \frac{\mathrm{Vol}(d\nu_{e^B\mathbf{s}_k(p)})}{\mathrm{Vol}(d\nu_{h_k(p)})}\right|_{\mathscr{C}^0} \leqslant Cp^{-k_0-1}\,,
\end{aligned}
\qquad (6.18)
$$

and $C^{-1} < \mathrm{Vol}(d\nu_{e^B\mathbf{s}_k(p)}) < C$.

Proof. Using Propositions 4.2 and 4.4, we know that for all $k \in \mathbb{N}$, $p \in \mathbb{N}^*$ and $B \in \mathrm{Herm}(\mathbb{C}^{n_p})^T$, we have

$$
\begin{aligned}
h_k^p(p) &= \rho_{h_k^p(p)}\, h_{\mathbf{s}_k(p)} = \rho_{h_k^p(p)}\, \sigma_{h_k^p(p)}(e^{L_{\xi_p}/p})\, h^p_{e^{L_{\xi_p}/2p}\mathbf{s}_k(p)} \\
&= \rho_{h_k^p(p)}\, \sigma_{h_k^p(p)}(e^{L_{\xi_p}/p})\, \sigma_{e^{L_{\xi_p}/p}\mathbf{s}_k(p)}(e^{2B})\, h^p_{e^B e^{L_{\xi_p}/2p}\mathbf{s}_k(p)}\,.
\end{aligned}
\qquad (6.19)
$$

By definition of the Kähler form (1.1), we then get

$$\begin{aligned}\omega_{e^B e^{L_{\xi_p}/p}\mathbf{s}_k(p)} &= \omega_{h_k(p)} - \frac{\sqrt{-1}}{2\pi p}\overline{\partial}\partial\log\sigma_{e^{L_{\xi_p}/p}\mathbf{s}_k(p)}\left(e^{2B}\right)\\ &\quad - \frac{\sqrt{-1}}{2\pi p}\overline{\partial}\partial\log\left(\rho_{h_k^p(p)}\sigma_{h_k^p(p)}(e^{L_{\xi_p}/p})\right)\\ = \omega_{h_k(p)} &- \frac{\sqrt{-1}}{2\pi p}\overline{\partial}\partial\log\left(1+\sigma_{e^{L_{\xi_p}/p}\mathbf{s}_k(p)}\left(e^{2B}-\mathrm{Id}\right)\right)\\ &- \frac{\sqrt{-1}}{2\pi p}\overline{\partial}\partial\log\left(1+\left(\frac{\mathrm{Vol}(d\nu_{h_k(p)})}{\mathrm{Tr}\left[e^{L_{\xi_p}/p}\right]}\rho_{h_k^p(p)}\sigma_{h_k^p(p)}(e^{L_{\xi_p}/p})-1\right)\right).\end{aligned} \tag{6.20}$$

Recall from Propositions 3.2 and 3.6 that $C^{-1}p^n < \mathrm{Tr}\left[e^{L_{\xi_p}/p}\right] < Cp^n$ for some $C>0$, while $\mathrm{Vol}(d\nu_{h_k(p)}) \to \mathrm{Vol}(d\nu_{h_\infty})$ as $p\to+\infty$ by definition (6.7) of $h_k(p)$. Then by Lemma 6.1 and Proposition 6.2, we can take the Taylor expansion as $p\to+\infty$ of formula (6.20) to get that for any k, k_0, $m\in\mathbb{N}$ with $k\geqslant k_0>n+1+m/2$, there exists $C>0$ such for all $B\in\mathrm{Herm}(\mathbb{C}^{n_p})$ with $\|B\|_{\xi_p}\leqslant C^{-1}p^{-k_0}$, we have

$$\left|\omega_{e^B e^{L_{\xi_p}/p}\mathbf{s}_k(p)} - \omega_{h_k(p)}\right|_{\mathscr{C}^{m-2}} \leqslant Cp^{-k_0-1}. \tag{6.21}$$

By formula (6.7) for $h_k(p)$ and the corresponding formula for $\omega_{h_k(p)}$ as in (6.20), this implies the first inequality of Lemma 6.3.

Let us now establish the second inequality of Lemma 6.3. For any $p\in\mathbb{N}^*$ and $B\in\mathrm{Herm}(\mathbb{C}^{n_p})$, formula (2.3) and (6.19) give

$$\begin{aligned}\log\frac{d\nu_{e^B e^{L_{\xi_p}/p}\mathbf{s}_k(p)}}{d\nu_{h_k(p)}} &- \frac{1}{p}\log\frac{\mathrm{Vol}(d\nu_{h_k(p)})}{\mathrm{Tr}\left[e^{L_{\xi_p}/p}\right]}\\ &= -\frac{1}{p}\log\left(\frac{\mathrm{Vol}(d\nu_{h_k(p)})}{\mathrm{Tr}\left[e^{L_{\xi_p}/p}\right]}\rho_{h_k^p(p)}\right) - \frac{1}{p}\log\sigma_{e^{L_{\xi_p}/p}\mathbf{s}_k(p)}(e^{2B}).\end{aligned} \tag{6.22}$$

Taking the Taylor expansion as $p\to+\infty$ of the right hand side of (6.22) in the same way as we did to deduce (6.21) from (6.20), for any k, $k_0\in\mathbb{N}$ with $k\geqslant k_0>n+1+m/2$, we get a constant $C>0$ such that for all $p\in\mathbb{N}^*$ and all $B\in\mathrm{Herm}(\mathbb{C}^{n_p})$ with $\|B\|_{tr}\leqslant C^{-1}p^{-k_0}$, we have

$$\left|\frac{d\nu_{e^B e^{L_{\xi_p}/p}\mathbf{s}_k(p)}}{d\nu_{h_k(p)}} - \frac{1}{p}\log\frac{\mathrm{Vol}(d\nu_{h_k(p)})}{\mathrm{Tr}\left[e^{L_{\xi_p}/p}\right]}\right|_{\mathscr{C}^0} \leqslant Cp^{-k_0-1}. \tag{6.23}$$

Taking the integral of both sides against the probability measure $d\nu_{h_k(p)}/\mathrm{Vol}(d\nu_{h_k(p)})$, we see that there is $C>0$ such that the constants $V_p>0$ for all $p\in\mathbb{N}^*$ satisfy

$$\left| \frac{\mathrm{Vol}\left(d\nu_{e^B e^{L_{\xi_p}/p}\mathbf{s}_k(p)}\right)}{\mathrm{Vol}(d\nu_{h_k(p)})} - \frac{1}{p}\log\frac{\mathrm{Vol}(d\nu_{h_k(p)})}{\mathrm{Tr}\left[e^{L_{\xi_p}/p}\right]} \right| < Cp^{-k_0-1}\,. \tag{6.24}$$

Using the fact from Proposition 4.5 and formula (2.5) that $\mathrm{Vol}(d\nu_{\mathbf{s}}) = \mathrm{Vol}(d\nu_{e^{L_\xi}\mathbf{s}})$ for all $\mathbf{s} \in \mathscr{B}(H^0(X,L^p))^T$ and $\xi \in \sqrt{-1}\,\mathrm{Lie}\,T$, we get the second inequality of (6.18) by combining inequalities (6.23) and (6.24). The last statement then follows from the inequality (6.24), recalling that $C^{-1}p^n < \mathrm{Tr}\left[e^{L_{\xi_p}/p}\right] < Cp^n$ and $\mathrm{Vol}(d\nu_{h_k(p)}) \to \mathrm{Vol}(d\nu_{h_\infty})$ as $p \to +\infty$, so that the second term of the left hand side of (6.24) goes to 0 as $p \to +\infty$. This concludes the proof. □

6.2. Moment map picture and existence

The goal of this Section is to give a proof of existence and convergence in Theorem 1.1. In order to do so, it will be convenient to introduce some extra notations. Recall the setting of Section 4.1, and write $\mathrm{Prod}(H^0(X,L^p))^K$ for the space of K-invariant Hermitian inner products on $H^0(X,L^p)$. Given a fixed basis $\mathbf{s} \in \mathscr{B}(H^0(X,L^p))^T$ inducing $H_{\mathbf{s}} \in \mathrm{Prod}(H^0(X,L^p))^K$, consider the induced identification $H^0(X,L^p) \simeq \mathbb{C}^{n_p}$. Via this identification, write $\mathrm{GL}(\mathbb{C}^{n^p})^K \subset \mathrm{GL}(\mathbb{C}^{n_p})^T$ for the group of invertible endomorphisms commuting with the induced action of K on $\mathbb{C}^{n_p}$, and write $U(n_p)^K \subset \mathrm{GL}(\mathbb{C}^{n^p})^K$ for the subgroup of unitary matrices commuting with the action of K. The action of $G \in \mathrm{GL}(\mathbb{C}^{n^p})^K$ on $\mathbf{s}$ induces again a K-invariant product $H_{G\mathbf{s}} \in \mathrm{Prod}(H^0(X,L^p))^K$, and as for (4.7), we have an identification

$$\mathscr{L}(H^0(X,L^p), H_{G\mathbf{s}})^K \simeq \mathrm{Herm}(\mathbb{C}^{n_p})^K\,, \tag{6.25}$$

where $\mathrm{Herm}(\mathbb{C}^{n_p})^K \subset \mathrm{Herm}(\mathbb{C}^{n_p})^T$ is the space of Hermitian matrices commuting with the induced action of K on $\mathbb{C}^{n_p}$. Note that the second statement of Proposition 4.10 precisely says that $\mu_\xi(G\mathbf{s}) \in \mathrm{Herm}(\mathbb{C}^{n_p})^K$.

Our strategy for the proof of Theorem 1.1 is based on the following fundamental link between the anticanonical moment map of Definition 4.8 and the Berezin-Toeplitz quantum channel of Definition 5.3. Following Remark 4.3, let $h^p \in \mathrm{Met}^+(L^p)^K$ be a K-invariant positive Hermitian metric, let $\mathbf{s}_p \in \mathscr{B}(H^0(X,L^p))^T$ be orthonormal with respect to $L^2(h^p)$, and consider the induced identification (6.25). By Proposition 4.10, for any $\xi \in \sqrt{-1}\,\mathrm{Lie}\,T$ and $A \in \mathrm{Herm}(\mathbb{C}^{n_p})^K$ we have

$$D_{\mathbf{s}_p}\mu_\xi(A) := \frac{\partial}{\partial t}\Big|_{t=0} \mu_\xi(e^{tA}\mathbf{s}_p) \in \mathrm{Herm}(\mathbb{C}^{n_p})^K\,. \tag{6.26}$$

Let $\langle\cdot,\cdot\rangle_\xi$ be the scalar product (4.28) on $\mathrm{Herm}(\mathbb{C}^{n_p})^K$, and write $\|\cdot\|_\xi$ for the associated norm.

Proposition 6.4. *Assume that $h^p \in \mathrm{Met}^+(L^p)^K$ is anticanonically balanced relative to $\xi_p \in \sqrt{-1}\,\mathrm{Lie}\,T$, and let $\mathbf{s}_p \in \mathscr{B}(H^0(X,L)^p)^T$ be orthonormal with respect to $L^2(h^p)$. Then for all $A \in \mathrm{Herm}(\mathbb{C}^{n_p})^K$ satisfying $\langle \mathrm{Id}, A\rangle_{\xi_p} = 0$, we have*

$$\frac{\mathrm{Tr}[e^{L_{\xi_p}/p}]}{2\,\mathrm{Vol}(d\nu_h)}\langle A, D_{\mathbf{s}_p}\mu_{\xi_p}(A)\rangle_{\xi_p} = \|A\|^2_{\xi_p} - \left(1+\frac{1}{p}\right)\langle A, \mathscr{E}^{\xi_p}_{h^p}(A)\rangle_{\xi_p}\,. \tag{6.27}$$

Proof. Let us first compute $D_{\mathbf{s}}\mu_\xi(A) \in \mathrm{Herm}(\mathbb{C}^{n_p})^T$, for general $\xi \in \sqrt{-1}\,\mathrm{Lie}\,T$, $\mathbf{s} \in \mathscr{B}(H^0(X,L^p))^T$ and $A \in \mathrm{Herm}(\mathbb{C}^{n_p})^T$. Using Proposition 4.2, Proposition 4.5 and formula (4.30), in the identification (4.7) we get

$$\begin{aligned} D_{\mathbf{s}}\mu_\xi(A) = &\left(\int_X \frac{\partial}{\partial t}\Big|_{t=0}\langle e^{tA}s_j, e^{tA}s_k\rangle_{h^p_{e^{tA}e^{L_\xi/2p}\mathbf{s}_p}}\, d\nu_{e^{L_\xi/2p}\mathbf{s}}\right)_{j,\,k=1}^{n_p} \\ &+ \left(\int_X \langle s_j, s_k\rangle_{h^p_{e^{L_\xi/2p}\mathbf{s}}}\frac{\partial}{\partial t}\Big|_{t=0} d\nu_{e^{tA}e^{L_\xi/2p}\mathbf{s}}\right)_{j,\,k=1}^{n_p} - \left(\frac{\partial}{\partial t}\Big|_{t=0}\frac{\mathrm{Vol}(d\nu_{e^{tA}\mathbf{s}})}{\mathrm{Tr}[e^{L_\xi/p}]}\right)\mathrm{Id} \\ = &\int_X e^{-L_\xi/2p}(A\Pi_{e^{L_\xi/2p}\mathbf{s}} + \Pi_{e^{L_\xi/2p}\mathbf{s}}A - 2\sigma_{e^{L_\xi/2p}\mathbf{s}}(A)\Pi_{e^{L_\xi/2p}\mathbf{s}})e^{-L_\xi/2p}\, d\nu_{e^{L_\xi/2p}\mathbf{s}} \\ &- \frac{2}{p}\int_X \sigma_{e^{L_\xi/2p}\mathbf{s}}(A)e^{-L_\xi/2p}\Pi_{e^{L_\xi/2p}\mathbf{s}}e^{-L_\xi/2p}\, d\nu_{e^{L_\xi/2p}\mathbf{s}} \\ &+ \frac{\mathrm{Vol}(d\nu_{\mathbf{s}})}{\mathrm{Tr}[e^{L_\xi/p}]}\left(\frac{2}{p}\int_X \sigma_{\mathbf{s}}(A)\, d\nu_{\mathbf{s}}\right)\mathrm{Id}\,. \end{aligned} \tag{6.28}$$

Then by Propositions 4.4 and 4.5, for any $A \in \mathrm{Herm}(\mathbb{C}^{n_p})^T$ with $\mathrm{Tr}[e^{L_\xi/p}A] = 0$, we get

$$\begin{aligned} &\frac{1}{2}\,\mathrm{Tr}[e^{L_\xi/p}A\,D_{\mathbf{s}}\mu_\xi(A)] \\ &\quad = \int_X \sigma_{e^{L_\xi/2p}\mathbf{s}}(A^2)\, d\nu_{e^{L_\xi/2p}\mathbf{s}} - \left(1+\frac{1}{p}\right)\int_X \sigma_{e^{L_\xi/2p}\mathbf{s}}(A)^2\, d\nu_{e^{L_\xi/2p}\mathbf{s}} \end{aligned} \tag{6.29}$$

On the other hand, from Proposition 5.2 and Definition 5.3, for any $h^p \in \mathrm{Met}^+(L^p)^T$ we get

$$\begin{aligned} \mathrm{Tr}[e^{L_\xi/p}A^2] &= \int_X \sigma_{h^p}(A^2)\,\sigma_{h^p}(e^{L_\xi/p})\,\rho_{h^p}\,d\nu_h\,, \\ \mathrm{Tr}\left[e^{L_\xi/p}A\,\mathscr{E}^\xi_{h^p}(A)\right] &= \int_X \sigma_{h^p}(A)^2\sigma_{h^p}(e^{L_\xi/p})\,\rho_{h^p}\,d\nu_h\,. \end{aligned} \tag{6.30}$$

Using Proposition 4.6 and comparing formulas (6.29) and (6.30), this gives the result. □

For any $k \in \mathbb{N}$ and $p \in \mathbb{N}^*$ big enough, consider the positive Hermitian metric $h_k^p(p) \in \mathrm{Met}^+(L)^K$ constructed in Proposition 6.2, and let $\mathbf{s}_k(p) \in \mathscr{B}(H^0(X,L^p))^T$ be orthonormal with respect to $L^2(h_k^p(p))$. Consider the induced identification (6.25), and let $\xi_p \in \sqrt{-1}\,\mathrm{Lie}\,T$ be given by Corollary 3.4. The following result constitutes the heart of our strategy, using the asymptotics of the spectral gap of the Berezin transform given in Theorem 5.9 to give a crucial estimate from above on the Berezin-Toeplitz quantum channel.

Theorem 6.5. *There exists $\varepsilon > 0$ such that for all $k \geqslant n+3$, all $p \in \mathbb{N}^*$ big enough and for any $A \in \mathrm{Herm}(\mathbb{C}^{n_p})^K$ satisfying $\langle \mathrm{Id}, A\rangle_{\xi_p} = \langle L_\eta, A\rangle_{\xi_p} = 0$ for all $\eta \in \sqrt{-1}\,\mathrm{Lie}\,T$, we have*

$$\langle A, \mathscr{E}^{\xi_p}_{h_k^p(p)}(A)\rangle_{\xi_p} \leqslant \big(1-(1+\varepsilon)p^{-1}\big)\,\|A\|^2_{\xi_p}\,. \tag{6.31}$$

Proof. For any $\mathbf{s} \in \mathscr{B}(H^0(X,L^p))^T$ and $\eta \in \sqrt{-1}\,\mathrm{Lie}\,T$, we write $\theta_{\mathbf{s}}(\eta) := \theta_{h_{\mathbf{s}}}(\eta)$ to simplify notations. By Propositions 2.2, 4.2 and 4.5, we have

$$p\,\theta_{\mathbf{s}}(\eta) h^p_{\mathbf{s}} = \frac{\partial}{\partial t}\Big|_{t=0} \phi^*_{t\xi} h^p_{\mathbf{s}} = \sigma_{\mathbf{s}}(L_\eta)\, h^p_{\mathbf{s}}\,. \tag{6.32}$$

Given $h^p \in \mathrm{Met}^+(L^p)^K$ and $\mathbf{s}_p \in \mathscr{B}(H^0(X,L^p))^T$ orthonormal with respect to $L^2(h^p)$, Proposition 3.1 and Definition 5.3 of the quantum channel then imply that for any $\xi, \eta \in \sqrt{-1}\,\mathrm{Lie}\,T$, we have

$$\mathscr{E}^{\xi}_{h^p}(L_\eta) = p\,T_{h^p}(\theta_{e^{L_\xi/2p}\mathbf{s}_p}(\eta))\,. \tag{6.33}$$

Using Propositions 2.2 and 6.2, we get from formula (6.19) a constant $C > 0$ such that $|\theta_{e^{L_{\xi_p}/2p}\mathbf{s}_k(p)}(\eta) - \theta_{h_k(p)}(\eta)|_{\mathscr{C}^0} \leqslant C|\eta|\,p^{n-k}$, for all $\eta \in \sqrt{-1}\,\mathrm{Lie}\,T$ and $p \in \mathbb{N}^*$. Hence by Theorem 2.9, Definition 2.10, and Proposition 3.1 and as $\|\cdot\|_{tr} \leqslant Cp^{n/2}\|\cdot\|_{op}$ for some $C > 0$, formula (6.33) implies

$$\left\| \mathscr{E}^{\xi_p}_{h_k^p(p)}(L_\eta) - \frac{p}{p+1}L_\eta \right\|_{tr} \leqslant Cp^{\frac{3n}{2}+1-k}|\eta|\,. \tag{6.34}$$

On the other hand, recall the notation (5.33) for the increasing sequence of eigenvalues of the Tian-Zhu operator Δ_h^ξ of Lemma 2.6, for any $h \in \mathrm{Met}^+(L)$ and $\xi \in \sqrt{-1}\,\mathrm{Lie}\,T$. Proposition 3.6 and formula (6.7) show that for any $j \in \mathbb{N}$, there exists a constant $C_j > 0$ such that $|\lambda_j(h_k(p),\xi_p) - \lambda_j(h_\infty,\xi_\infty)| \leqslant C_j p^{-1}$, for all $p \in \mathbb{N}^*$. Using Theorem 5.9, we thus get that for any $j \in \mathbb{N}$, there exists a constant $C_j > 0$ such that for all $p \in \mathbb{N}^*$, we have $\big|1-\gamma_j(h_k^p(p),\xi_p) - p^{-1}\lambda_k(h_\infty,\xi_\infty)\big| \leqslant C_j p^{-2}$. Using Proposition 5.4, this shows that there exists ε, $C > 0$ such that for all $p \in \mathbb{N}^*$,

$$\operatorname{Spec}(\mathscr{E}^{\xi_p}_{h^p_k(p)}) \cap [1-(1+\varepsilon)p^{-1}, 1-(1-\varepsilon)p^{-1}] \subset [1-p^{-1}-Cp^{-2}, 1-p^{-1}+Cp^{-2}]. \quad (6.35)$$

Recall from Proposition 2.2 that $\theta_{h_k(p)} : \sqrt{-1}\,\mathrm{Lie}\,T \to \mathscr{C}^\infty(X,\mathbb{R})$ is an embedding. For any η_1, $\eta_2 \in \sqrt{-1}\,\mathrm{Lie}\,T$, Propositions 3.1, 3.2, 5.2, 5.5 and 6.2 imply that for all $p \in \mathbb{N}^*$, we have

$$\frac{\langle L_{\eta_1}, L_{\eta_2}\rangle_{\xi_p}}{p^{n+2}} = \langle \theta_{h_k(p)}(\eta_1), \theta_{h_k(p)}(\eta_2)\rangle_{L^2(h_k(p),\xi_p,p)} + |\eta_1|\,|\eta_2|\,O(p^{-1}), \quad (6.36)$$

so that $\|L_\eta\|_{\xi_p} \geqslant \varepsilon|\eta|p^{\frac{n}{2}+1}$ for some $\varepsilon > 0$ not depending of $p \in \mathbb{N}^*$ big enough. Note also from Propositions 3.1 and 3.6 that there is $C > 0$ such that the norm induced by (4.28) satisfies $C^{-1}\|\cdot\|_{tr} \leqslant \|\cdot\|_{\xi_p} \leqslant C\|\cdot\|_{tr}$ for all $p \in \mathbb{N}^*$.

Set now $k \geqslant n+3$. Using Proposition 3.1 together with an elementary Lemma on quasi-modes (see for instance [23, Lem. 2.1]), formulas (6.34) to (6.36) imply the existence of a constant $C > 0$ and an eigenvector $\widetilde{L_\eta} \in \mathrm{Herm}(\mathbb{C}^{n_p})^K$ of $\mathscr{E}^{\xi_p}_{h^p_k(p)}$ with associated eigenvalue $\lambda \in [1-p^{-1}-Cp^{-2}, 1-p^{-1}+Cp^{-2}]$ and such that

$$\|\widetilde{L_\eta}\|_{\xi_p} = \|L_\eta\|_{\xi_p} \geqslant \varepsilon|\eta|p^{\frac{n}{2}+1} \quad \text{and} \quad \left\|\widetilde{L_\eta} - L_\eta\right\|_{\xi_p} \leqslant Cp^{\frac{n}{2}-1}|\eta|. \quad (6.37)$$

Using Proposition 2.7 and Theorem 5.9 again, we know that the dimension of the sum of eigenspaces associated with the right hand side of (6.35) is equal to $\dim T$, so that by formula (6.36), the operators $\widetilde{L_\eta} \in \mathrm{Herm}(\mathbb{C}^{n_p})^K$ for all $\eta \in \sqrt{-1}\,\mathrm{Lie}\,T$ generate this subspace as soon as $p \in \mathbb{N}^*$ is big enough. As we have $\mathscr{E}^{\xi_p}_{h^p_k(p)}(\mathrm{Id}) = \mathrm{Id}$ by Proposition 2.12, Lemma 2.6, Proposition 5.4 and formula (6.35) imply that (6.31) holds for $A \in \mathrm{Herm}(\mathbb{C}^{n_p})^K$ belonging to the orthogonal of the subspace generated by $\mathrm{Id}_{\mathscr{H}_p}$ and $\widetilde{L_\eta}$ for all $\eta \in \sqrt{-1}\,\mathrm{Lie}\,K$. Now for any $A \in \mathrm{Herm}(\mathbb{C}^{n_p})^K$ satisfying $\langle \mathrm{Id}, A\rangle_{\xi_p} = \langle L_\eta, A\rangle_{\xi_p} = 0$ for all $\eta \in \sqrt{-1}\,\mathrm{Lie}\,T$, formula (6.37) and Cauchy-Schwartz imply the existence of $C > 0$ such that

$$\langle \widetilde{L_\eta}, A\rangle_{\xi_p} \leqslant C|\eta|\,p^{\frac{n}{2}-1}\|A\|_{\xi_p} \leqslant \frac{C}{\varepsilon}\,p^{-2}\|\widetilde{L_\eta}\|_{\xi_p}\|A\|_{\xi_p}. \quad (6.38)$$

It then suffices to consider the splitting of A into the eigenspaces of $\mathscr{E}^{\xi_p}_{h^p_k(p)}$ to get the result. □

Using the relation between the derivative of the moment map and the relative quantum channel given in Proposition 6.4, we can now apply the estimate of Theorem 6.5 to give an estimate from below for the derivative of the moment map at the approximately balanced bases. This lower bound constitutes the core of the proof of Theorem 1.1, and this shows how Berezin-Toeplitz quantization can be used to bypass the delicate geometric argument in the proofs of Donaldson [13] and Phong and Sturm [34] of the analogous result for the original notion of balanced metrics.

Corollary 6.6. *For any k, $k_0 \in \mathbb{N}$ with $k \geqslant k_0 \geqslant n+3$, there exists $\varepsilon > 0$ such that for all $p \in \mathbb{N}^*$ big enough, for all $B \in \mathrm{Herm}(\mathbb{C}^{n_p})^K$ with $\|B\|_{\xi_p} \leqslant \varepsilon p^{-k_0}$ and all $A \in \mathrm{Herm}(\mathbb{C}^{n_p})^K$ satisfying $\langle \mathrm{Id}, A\rangle_{\xi_p} = \langle L_\eta, A\rangle_{\xi_p} = 0$ for all $\eta \in \sqrt{-1}\,\mathrm{Lie}\,K$, we have*

$$\frac{\mathrm{Tr}[e^{L_{\xi_p}/p}]}{\mathrm{Vol}(d\nu_{e^B\mathbf{s}_k(p)})}\langle A, D_{e^B\mathbf{s}_k(p)}\mu_{\xi_p}(A)\rangle_{\xi_p} \geqslant \frac{\varepsilon}{p}\|A\|^2_{\xi_p}\,. \tag{6.39}$$

Proof. The proof uses the fact that Proposition 6.4 is approximately satisfied for approximately balanced metrics. First note that for all $\mathbf{s} \in \mathscr{B}(H^0(X, L^p))^T$ and all $A \in \mathrm{Herm}(\mathbb{C}^{n_p})^K$, Cauchy-Schwartz inequality and the fact that $\Pi_{\mathbf{s}}$ is a rank-1 projector implies that $|\sigma_{\mathbf{s}}(A)|_{\mathscr{C}^0} \leqslant \|A\|_{tr}$ and $|\sigma_{\mathbf{s}}(A^2)|_{\mathscr{C}^0} \leqslant \|A\|^2_{tr}$. Consider the operator S_p acting on $A \in \mathscr{L}(H^0(X, L^p), H_{\mathbf{s}})^K$ by

$$S_p(A) := A - \left(1 + \frac{1}{p}\right)\mathscr{E}^{\xi_p}_{h^p_k(p)}(A)\,. \tag{6.40}$$

Then plugging $\mathbf{s} = e^B\mathbf{s}_k(p)$ into formula (6.29) and comparing with (6.30), we can use Proposition 6.2 and Lemma 6.3 to get a constant $C > 0$ such that for all $p \in \mathbb{N}^*$, for all $B \in \mathrm{Herm}(\mathbb{C}^{n_p})^K$ with $\|B\|_{\xi_p} \leqslant C^{-1}p^{-k_0}$ and for all $A \in \mathrm{Herm}(\mathbb{C}^{n_p})^K$ with $\langle \mathrm{Id}, A\rangle_{\xi_p} = 0$, we get that

$$\left|\frac{\mathrm{Tr}[e^{L_{\xi_p}/p}]}{\mathrm{Vol}(d\nu_{e^B\mathbf{s}_k(p)})}\langle A, D_{e^B\mathbf{s}_k(p)}\,\mu_{\xi_p}(A)\rangle_{\xi_p} - 2\langle A, S_p(A)\rangle_{\xi_p}\right| \leqslant C\,p^{n-k_0}\|A\|^2_{\xi_p}\,. \tag{6.41}$$

We then get the result from Theorem 6.5 by taking $k_0 \geqslant n+3$. □

Thanks to the lower bound of Corollary 6.6, we can now follow the standard strategy of Donaldson in [13], adapted to the case of general $\mathrm{Aut}(X)$. The following result is inspired from the moment map Lemma of Donaldson in [13, Prop. 17]. We provide a proof working in greater generality, as we do not claim that Definition 4.8 defines a moment map of any kind.

Proposition 6.7. *Consider a $\mathscr{C}^1$-map*

$$\mu : \mathscr{B}(H^0(X, L^p))^T \to \mathrm{Herm}(\mathbb{C}^{n_p})^T \tag{6.42}$$

satisfying $\mu(\mathbf{s}) \in \mathrm{Herm}(\mathbb{C}^{n_p})^K$ for all $\mathbf{s} \in \mathscr{B}(H^0(X, L^p))^T$ inducing a K-invariant product $H_{\mathbf{s}} \in \mathrm{Prod}(H^0(X, L^p))^K$, and such that $\mu(U\mathbf{s}) = U\mu(\mathbf{s})\,U^*$ for all $U \in U(n_p)^T$ and $\langle \mathrm{Id}, \mu(\mathbf{s})\rangle_{\xi_p} = \langle L_\eta, \mu(\mathbf{s})\rangle_{\xi_p} = 0$ for all $\eta \in \sqrt{-1}\,\mathrm{Lie}\,T$.

Assume that there exist $\mathbf{s} \in \mathscr{B}(H^0(X, L^p))^T$ inducing $H_{\mathbf{s}} \in \mathrm{Prod}(H^0(X, L^p))^K$ and λ, $\delta > 0$ such that

(1) *$\lambda\,\|\mu_{\xi_p}(\mathbf{s})\|_{\xi_p} < \delta$;*

(2) $\lambda\langle A, D_{e^B\mathbf{s}}\mu_{\xi_p}(A)\rangle_{\xi_p} \geqslant \|A\|^2_{\xi_p}$, *for all* $B \in \mathrm{Herm}(\mathbb{C}^{n_p})^K$ *such that* $\|B\|_{\xi_p} \leqslant \delta$ *and all* $A \in \mathrm{Herm}(\mathbb{C}^{n_p})^K$ *such that* $\langle \mathrm{Id}, A\rangle_{\xi_p} = \langle L_\eta, A\rangle_{\xi_p} = 0$ *for all* $\eta \in \sqrt{-1}\,\mathrm{Lie}\,T$.

Then there exists $B \in \mathrm{Herm}(\mathbb{C}^{n_p})^K$ *with* $\|B\|_{\xi_p} \leqslant \delta$ *and* $\mu(e^B\mathbf{s}) = 0$.

Proof. First note that for any A, $B \in \mathrm{Herm}(\mathbb{C}^{n_p})^T$, $U \in U(n_p)^T$ and $\mathbf{s} \in \mathscr{B}(H^0(X, L^p))^T$, using that $\mu(U\mathbf{s}) = U\mu(\mathbf{s})\,U^*$, we get

$$\begin{aligned}\mathrm{Tr}[A\,D_{U\mathbf{s}}\mu(A)\,e^{L_{\xi_p}/p}] &= \frac{\partial}{\partial t}\Big|_{t=0}\mathrm{Tr}[A\,\mu(e^{tA}U\mathbf{s})\,e^{L_{\xi_p}/p}] \\ &= \mathrm{Tr}[U^*AU\,D_{\mathbf{s}}\mu(U^*AU)\,e^{L_{\xi_p}/p}]\,.\end{aligned} \tag{6.43}$$

Thus assumption (2) is equivalent to

(2′) $\lambda\langle A, D_{Ue^B\mathbf{s}}\mu(A)\rangle_{\xi_p} \geqslant \|A\|^2_{\xi_p}$, for all $B \in \mathrm{Herm}(\mathbb{C}^{n_p})^K$ such that $\|B\|_{\xi_p} \leqslant \delta$, all $U \in U(n_p)^K$ and all $A \in \mathrm{Herm}(\mathbb{C}^{n_p})^K$ such that $\langle \mathrm{Id}, A\rangle_{\xi_p} = \langle L_\eta, A\rangle_{\xi_p} = 0$ for all $\eta \in \sqrt{-1}\,\mathrm{Lie}\,T$.

Let now $\mathbf{s} \in \mathscr{B}(H^0(X, L^p))^T$ inducing $H_{\mathbf{s}} \in \mathrm{Prod}(H^0(X, L^p))^K$ be such that assumptions (1) and (2) are satisfied, and consider the induced identification (6.25). Then the map

$$\begin{aligned}\mathrm{GL}(\mathbb{C}^{n_p})^K &\longrightarrow \mathrm{Prod}(H^0(X, L^p))^K \\ G &\longmapsto H_{G\mathbf{s}}\,,\end{aligned} \tag{6.44}$$

identifies $\mathrm{Prod}(H^0(X, L^p))^K$ with the quotient of $\mathrm{GL}(C^{n_p})^K$ by $U(n_p)^K$. This realizes $\mathrm{Prod}(H^0(X, L^p))^K$ as a symmetric space, whose tangent space at every point is naturally identified with $\mathrm{Herm}(\mathbb{C}^{n_p})^K$. The scalar product $\langle\cdot,\cdot\rangle_{\xi_p}$ then makes this space into a complete Riemannian manifold, whose geodesics are of the form

$$t \longmapsto H_{e^{tB}\mathbf{s}} \in \mathrm{Prod}(H^0(X, L^p))^K,\; t \in \mathbb{R}\,, \tag{6.45}$$

for all $B \in \mathrm{Herm}(\mathbb{C}^{n_p})^K$.

Note on the other hand that the tangent space of the orbit $\mathrm{GL}(\mathbb{C}^{n_p})^K.\,\mathbf{s} \subset \mathscr{B}(H^0(X, L^p))^T$ is naturally identified with the space of endomorphisms commuting with the action of K on $\mathbb{C}^{n_p}$. Then by assumption, the restriction of the map (6.42) to this orbit can be identified with a vector field along this orbit, and we define $\mathbf{s}_t \in \mathrm{GL}(\mathbb{C}^{n_p})^K.\,\mathbf{s}$ for all $t > 0$ as the solution of the ODE

$$\begin{cases}\frac{\partial}{\partial t}\,\mathbf{s}_t = -\mu(\mathbf{s}_t) \quad \text{for all} \quad t \geqslant 0\,, \\ \\ \mathbf{s}_0 = \mathbf{s}\,.\end{cases} \tag{6.46}$$

If $\mu(\mathbf{s}) = 0$, then the result is trivially satisfied, so that we can assume $\mu(\mathbf{s}) \neq 0$, in which case $\mu(\mathbf{s}_t) \neq 0$ for all $t \geqslant 0$. Let $t_0 \geqslant 0$ be such that there exist $U_t \in U(n_p)^K$

and $B_t \in \mathrm{Herm}(\mathbb{C}^{n_p})^K$ with $\|B_t\|_{\xi_p} \leqslant \delta$ such that $\mathbf{s}_t = U_t e^{B_t}\mathbf{s}$ for all $t \in [0, t_0]$. Using assumption $(2')$ with $A := \mu(\mathbf{s}_t)$, for all $t \in [0, t_0]$ we have

$$-\lambda \frac{\partial}{\partial t} \|\mu(\mathbf{s}_t)\|^2_{\xi_p} = 2\lambda \langle \mu(\mathbf{s}_t)\, D_{\mathbf{s}_t}\mu(\mu(\mathbf{s}_t))\rangle_{\xi_p} \geqslant 2\|\mu(\mathbf{s}_t)\|^2_{\xi_p}\,. \tag{6.47}$$

By derivation of the square, this implies $\lambda \frac{\partial}{\partial t}\|\mu(\mathbf{s}_t)\|_{\xi_p} \leqslant -\|\mu(\mathbf{s}_t)\|_{\xi_p}$ for all $t \in [0, t_0]$, so that using Grönwall's lemma with initial condition (1) and as $\mu(\mathbf{s}_t) = U_t \mu(e^{B_t}\mathbf{s})\, U_t^*$, we get

$$\|\mu(e^{B_t}\mathbf{s})\|_{\xi_p} = \|\mu(\mathbf{s}_t)\|_{\xi_p} \leqslant e^{-t/\lambda}\, \|\mu(\mathbf{s})\|_{\xi_p} < \frac{\delta}{\lambda}\, e^{-t/\lambda}\,. \tag{6.48}$$

Then by equation (6.46), the Riemannian length $L(t_0) \geqslant 0$ of the path $\{t \mapsto H_{\mathbf{s}_t}\}_{t\in[0,t_0]} \subset \mathrm{Prod}(H^0(X, L^p))^K$ satisfies

$$L(t_0) = \int_0^{t_0} \|\mu(\mathbf{s}_t)\|_{\xi_p}\, dt < \frac{\delta}{\lambda} \int_0^{+\infty} e^{-t/\lambda}\, dt = \delta\,. \tag{6.49}$$

This means that there exists $\varepsilon > 0$ such that all points of $\{t \mapsto H_{\mathbf{s}_t}\}_{t\in[0,t_0+\varepsilon]}$ can be joined by a geodesic of length strictly less than δ, i.e., that for each $t \in [0, t_0 + \varepsilon]$, there exists $B_t \in \mathrm{Herm}(\mathbb{C}^{n_p})^K$ with $\|B_t\|_{\xi_p} \leqslant \delta$ such that $H_{\mathbf{s}_t} = H_{e^{B_t}\mathbf{s}}$, so that there exists $U_t \in U(n_p)^K$ such that $\mathbf{s}_t = U_t e^{B_t}\mathbf{s}$. Thus $I := \{t_0 \geqslant 0 \,|\, L(t_0) < \delta\}$ is non-empty, open and closed in $[0, +\infty[$, so that $I = [0, +\infty[$. In particular, the path $\{t \mapsto H_{\mathbf{s}_t}\}_{t>0}$ has total Riemannian length strictly less than δ, so that it converges to a limit point $H_{e^{B_\infty}\mathbf{s}} \in \mathrm{Prod}(H^0(X, L^p))^K$ by completeness, with $B_\infty \in \mathrm{Herm}(\mathbb{C}^{n_p})^K$ satisfying $\|B_\infty\|_{\xi_p} \leqslant \delta$. Finally, inequality (6.48) for all $t > 0$ implies

$$\|\mu(e^{B_\infty}\mathbf{s})\|_{\xi_p} = \lim_{t\to+\infty} \|\mu(e^{B_t}\mathbf{s})\|_{\xi_p} = 0\,. \tag{6.50}$$

This gives the result. $\square$

Proof of existence and convergence in Theorem 1.1. Let $h^p \in \mathrm{Met}^+(L^p)^K$, let $\mathbf{s}_p \in \mathscr{B}(H^0(X, L^p))^T$ be orthonormal with respect to $L^2(h^p)$, and consider the identification (6.25). For any $\xi \in \sqrt{-1}\,\mathrm{Lie}\, T$, using Proposition 4.4 and formulas (2.39) and (4.30), we get from Definition 4.8 the following inequality, for all $A \in \mathrm{Herm}(\mathbb{C}^{n_p})^K$,

$$\begin{aligned}
&\frac{\mathrm{Tr}[e^{L_\xi/p}]}{\mathrm{Vol}(d\nu_{\mathbf{s}_p})} \langle A, \mu_\xi(\mathbf{s}_p)\rangle_\xi \\
&= \frac{\mathrm{Tr}[e^{L_\xi/p}]}{\mathrm{Vol}(d\nu_{\mathbf{s}_p})} \int_X \sigma_{e^{L_\xi/2p}\mathbf{s}_p}(A)\, d\nu_{e^{L_\xi/2p}\mathbf{s}_p} - \int_X \sigma_{h^p}(e^{L_\xi/p}A)\, \rho_{h^p}\, d\nu_h \\
&= \int_X \sigma_{e^{L_\xi/2p}\mathbf{s}_p}(A) \left(\frac{\mathrm{Tr}[e^{L_\xi/p}]}{\mathrm{Vol}(d\nu_{\mathbf{s}_p})} \frac{d\nu_{e^{L_\xi/2p}\mathbf{s}_p}}{d\nu_h} - \sigma_{h^p}(e^{L_\xi/p})\, \rho_{h^p} \right) d\nu_h\,.
\end{aligned} \tag{6.51}$$

For any $k \in \mathbb{N}$ and $p \in \mathbb{N}^*$ big enough, consider now the approximately balanced metric $h_k(p) \in \mathrm{Met}^+(L)^K$ of Proposition 6.2, let $\mathbf{s}_k(p) \in \mathscr{B}(H^0(X, L^p))^T$ be orthonormal with respect to $L^2(h_k^p(p))$, and let $\{\xi_p \in \sqrt{-1}\,\mathrm{Lie}\,T\}_{p\in\mathbb{N}^*}$ be the sequence of Corollary 3.4. By Proposition 4.10, the relative moment map $\mu_{\xi_p} : \mathscr{B}(H^0(X, L^p))^T \to \mathrm{Herm}(\mathbb{C}^{n_p})^T$ of Definition 4.8 satisfies the basic assumptions of Proposition 6.7, so that it suffices to show that $\mathbf{s}_k(p) \in \mathscr{B}(H^0(X, L^p))^T$ satisfies the assumptions (1) and (2) of Proposition 6.7, for some appropriate λ, $\delta > 0$. Using Proposition 6.2 and Lemma 6.3 and the fact that $|\sigma_{e^{L_{\xi_p}/2p}\mathbf{s}_k(p)}(A)|_{\mathscr{C}^0} \leqslant \|A\|_{tr}$ by Cauchy-Schwartz inequality, we get from formula (6.51) a constant $C > 0$ such that for all $p \in \mathbb{N}^*$ and all $A \in \mathrm{Herm}(\mathbb{C}^{n_p})^K$, we have

$$\frac{\mathrm{Tr}[e^{L_\xi/p}]}{\mathrm{Vol}(d\nu_{\mathbf{s}_k(p)})}\langle A, \mu_{\xi_p}(\mathbf{s}_k(p))\rangle_{\xi_p} \leqslant C\|A\|_{\xi_p}\, p^{n-k}\,, \tag{6.52}$$

which implies $\frac{\mathrm{Tr}[e^{L_\xi/p}]}{\mathrm{Vol}(d\nu_{\mathbf{s}_k(p)})}\|\mu_\xi(\mathbf{s}_k(p))\|_{\xi_p} \leqslant Cp^{n-k}$ for all $p \in \mathbb{N}^*$. Taking $k_0 \geqslant n+3$, we can then choose $k > k_0 + n + 1$, and Corollary 6.6 together with Proposition 3.2 shows that $\mathbf{s}_k(p) \in \mathscr{B}(H^0(X, L^p))^T$ satisfies the assumptions (1) and (2) of Proposition 6.7 for $p \in \mathbb{N}^*$ big enough, with

$$\lambda := \frac{p}{\varepsilon}\frac{\mathrm{Tr}[e^{L_{\xi_p}/p}]}{\mathrm{Vol}(d\nu_{\mathbf{s}_k(p)})} \qquad \text{and} \qquad \delta := \frac{C}{\varepsilon}p^{n+1-k}\,. \tag{6.53}$$

This gives Hermitian endomorphisms $B_p \in \mathrm{Herm}(\mathbb{C}^{n_p})^K$ with $\|B_p\|_{\xi_p} \leqslant \varepsilon p^{-k_0}$ such that $\mu_\xi(e^{B_p}\mathbf{s}_k(p)) = 0$ for all $p \in \mathbb{N}^*$ big enough. By Proposition 4.9, the Hermitian metrics $h_p := h^p_{e^{B_p}e^{L_{\xi_p}/p}\mathbf{s}_k(p)} \in \mathrm{Met}^+(L^p)^K$ are then anticanonically balanced relative to ξ_p for all $p \in \mathbb{N}^*$ big enough. If we also chose $k_0 > n + 1 + m/2$ for some $m \in \mathbb{N}$, Lemma 6.3 shows the $\mathscr{C}^m$-convergence (1.6) to the Kähler-Ricci soliton ω_{h_∞}. Together with Proposition 3.6, this concludes the proof of Theorem 1.1. □

6.3. Energy functional and uniqueness

In this Section, we will establish the uniqueness statement in Theorem 1.1 as a quantization of the analogous argument of Tian and Zhu in [44, Th. 3.2], using our study in Section 3.2 of quantized Futaki invariants as obstructions for relative anticanonically balanced metrics, and convexity results due to Berndtsson [4,5] applied to the energy functional associated with the relative moment map of Definition 4.8.

Recall the setting of Section 4.1. Via the natural action of $\mathrm{GL}(\mathbb{C}^{n_p})^T$ on the space $\mathscr{B}(H^0(X, L^p))^T$, the quotient map

$$\begin{aligned} \mathscr{B}(H^0(X, L^p))^T &\longrightarrow \mathscr{B}(H^0(X, L^p))^T/U(n_p)^T \\ \mathbf{s} &\longmapsto [\mathbf{s}]\,, \end{aligned} \tag{6.54}$$

identifies $\mathscr{B}(H^0(X,L^p))^T/U(n_p)^T$ with the space of T-invariant Hermitian inner products on $H^0(X,L^p)$. The twisted trace product (4.28) makes this space into a complete Riemannian manifold, whose geodesics are of the form

$$t \longmapsto [e^{tA}\mathbf{s}] \in \mathscr{B}(H^0(X,L^p))^T/U(n_p)^T,\ t\in\mathbb{R}\,, \tag{6.55}$$

for all $A \in \mathrm{Herm}(\mathbb{C}^{n_p})^T$. Fixing a base point $\mathbf{s}_0 \in \mathscr{B}(H^0(X,L^p))^T$, the free and transitive action of $\mathrm{GL}(\mathbb{C}^{n_p})^T$ on $\mathscr{B}(H^0(X,L^p))^T$ induces an identification

$$\mathscr{B}(H^0(X,L^p))^T \simeq \mathrm{GL}(\mathbb{C}^{n_p})^T\,, \tag{6.56}$$

and this induces a determinant map

$$\det{}_{\mathbf{s}_0} : \mathscr{B}(H^0(X,L^p))^T/U(n_p)^T \longrightarrow\,]0,+\infty[\,. \tag{6.57}$$

The following energy functional has been introduced in [3, §4.2.2].

Definition 6.8. The *energy functional* $\Psi_\xi : \mathscr{B}(H^0(X,L^p))^T/U(n_p)^T \to \mathbb{R}$ relative to $\xi \in \sqrt{-1}\,\mathrm{Lie}\,T$ is defined for all $H \in \mathrm{Prod}(H^0(X,L^p))$ by

$$\Psi([\mathbf{s}]) = -\log \mathrm{Vol}(d\nu_{\mathbf{s}}) - \frac{2}{p}\frac{\log \det_{\mathbf{s}_0}[e^{L_\xi/p}\mathbf{s}]}{\mathrm{Tr}[e^{L_\xi/p}]}\,. \tag{6.58}$$

For any $\xi \in \sqrt{-1}\,\mathrm{Lie}\,T$, recall the induced scalar product (4.28) on $\mathrm{Herm}(\mathbb{C}^{n_p})^T$. The role of the energy functional of Definition 6.8 comes from the following identity.

Lemma 6.9. *For all* $\mathbf{s} \in \mathscr{B}(H^0(X,L^p))^T$ *and* $A \in \mathrm{Herm}(\mathbb{C}^{n_p})^T$, *we have*

$$\frac{d}{dt}\Big|_{t=0}\Psi_\xi([e^{tA}\mathbf{s}]) = \frac{2}{p\,\mathrm{Vol}(d\nu_{\mathbf{s}})}\langle \mu_\xi(\mathbf{s}), A\rangle_\xi\,. \tag{6.59}$$

Proof. Using Proposition 4.2 and formula (2.3), from Definition 6.8 we compute

$$\frac{d}{dt}\Big|_{t=0}\Psi([e^{tA}\mathbf{s}]) = \frac{2}{p\,\mathrm{Vol}(d\nu_{\mathbf{s}})}\int_X \sigma_{\mathbf{s}}(A)\,d\nu_{\mathbf{s}} - \frac{2}{p}\frac{\mathrm{Tr}[e^{L_\xi/p}A]}{\mathrm{Tr}[e^{L_\xi/p}]}\,. \tag{6.60}$$

On the other hand, using formula (4.30), Definition 4.8 gives

$$\mathrm{Tr}[e^{L_\xi/p}\mu_\xi(\mathbf{s})A] = \int_X \sigma_{e^{L_\xi/2p}\mathbf{s}}(A)\,d\nu_{e^{L_\xi/2p}\mathbf{s}} - \frac{\mathrm{Vol}(d\nu_{\mathbf{s}})}{\mathrm{Tr}[e^{L_\xi/p}]}\mathrm{Tr}[e^{L_\xi/p}A]\,. \tag{6.61}$$

From formula (2.5) and Proposition 4.5, a change of variable with respect to $\phi_{\xi/2p} \in T_{\mathbb{C}}$ gives the result. □

By Proposition 4.9, this implies in particular that $[\mathbf{s}] \in \mathscr{B}(H^0(X,L^p))^T/U(n_p)^T$ is a critical point of $\Psi_\xi : \mathscr{B}(H^0(X,L^p))^T/U(n_p)^T \to \mathbb{R}$ if and only if there exists $h^p \in \mathrm{Met}^+(L^p)^T$ anticanonically balanced relative to ξ with $\mathbf{s}$ orthonormal with respect to $L^2(h^p)$.

The following result is a consequence of the results of [4,5] on positivity of direct images.

Proposition 6.10. *For any $\xi \in \sqrt{-1}\,\mathrm{Lie}\,T$, the energy functional of Definition 6.8 is convex along geodesics of $\mathscr{B}(H^0(X,L^p))^T/U(n_p)^T$, and strictly convex except along geodesics of the form $t \longmapsto [e^{t(L_\eta+c)}\mathbf{s}]$, with $c \in \mathbb{R}$ and $\eta \in \mathrm{Lie}\,\mathrm{Aut}(X)$ such that $L_\eta \in \mathscr{L}(H^0(X,L^p),H_{\mathbf{s}})^T$.*

Proof. By definition (6.55) of the geodesics in $\mathscr{B}(H^0(X,L^p))^T/U(n_p)^T$, the second term of formula (6.58) for Ψ_ξ is clearly affine along geodesics, so that it suffices to establish the convexity of the first term.

Now as explained in the proof of [2, Lem. 7.2], via formula (2.7) the results of [5] imply that the first term of formula (6.58) is convex along geodesics, and strictly convex along geodesics except those generated by $A \in \mathrm{Herm}(\mathbb{C}^{n_p})^T$ such that there exists $c > 0$ and $\eta \in \mathrm{Lie}\,\mathrm{Aut}(X)$ with

$$\frac{\partial}{\partial t}\Big|_{t=0} h_{e^{tA}\mathbf{s}} = (\theta_{h_{\mathbf{s}}}(\eta) + c)\, h_{\mathbf{s}}\,. \tag{6.62}$$

By Proposition 2.2, the fact that $\theta_{h_{\mathbf{s}}}(\eta) \in \mathscr{C}^\infty(X,\mathbb{R})$ implies that $L_{J\eta}h_{\mathbf{s}} = 0$, so that $L_\eta \in \mathscr{L}(H^0(X,L^p),H_{\mathbf{s}})^T$. By Proposition 2.12 and formula (6.32), we then have $\sigma_{\mathbf{s}}(A) = \sigma_{\mathbf{s}}(L_\eta + c)$. On the other hand, the image of the Kodaira map (2.28) is not contained in any proper projective subspace of $\mathbb{P}(H^0(X,L^p)^*)$ by definition, hence following for instance [22, Prop. 4.8], we know that the Berezin symbol $\sigma_{\mathbf{s}} : \mathscr{L}(H^0(X,L^p),H_{\mathbf{s}})^T \to \mathscr{C}^\infty(X,\mathbb{R})$ is injective (see also [19, §3]). This concludes the proof. □

Proof of uniqueness in Theorem 1.1. First note that, if $h_p \in \mathrm{Met}^+(L^p)^T$ is anticanonically balanced relative to $\xi_p \in \mathrm{Lie}\,\mathrm{Aut}(X)$, then for any $\phi \in \mathrm{Aut}_0(X)$, the pullback metric $\phi^* h_p$ is anticanonically balanced relative to $\phi^* \xi_p$. On the other hand, following the argument of Tian-Zhu in the proof of [44, Th. 3.2], if $\xi, \widetilde{\xi} \in \mathrm{Lie}\,\mathrm{Aut}(X)$ are such that $J\xi$, $J\widetilde{\xi}$ are in the Lie algebra of maximal compact subgroups $K, \widetilde{K} \subset \mathrm{Aut}_0(X)$, then there exists $\phi \in \mathrm{Aut}_0(X)$ such that $\phi^* J\widetilde{\xi} \in K$. Using Proposition 4.7, we are then reduced to show uniqueness up to $\mathrm{Aut}_0(X)$ of anticanonically balanced metrics relative to the vector field $\xi_p \in \sqrt{-1}\,\mathrm{Lie}\,K$ of Corollary 3.4, for a fixed maximal compact subgroup $K \subset \mathrm{Aut}_0(X)$.

Let now $h_p, \widetilde{h}_p \in \mathrm{Met}^+(L^p)$ be anticanonically balanced metrics relative to $\xi_p \in \sqrt{-1}\,\mathrm{Lie}\,K$, let $T \subset K$ be the closure of the subgroup generated by $J\xi_p$ and let $\mathbf{s}_p, \widetilde{\mathbf{s}}_p \in \mathscr{B}(H^0(X,L^p))^T$ be orthonormal with respect to $L^2(h^p)$, $L^2(\widetilde{h}^p)$. Proposition 4.9 and Definition 6.8 imply that $[\mathbf{s}_p]$, $[\widetilde{\mathbf{s}}_p] \in \mathscr{B}(H^0(X,L^p))^T/U(n_p)^T$ are both critical points of Ψ_{ξ_p}, so that Ψ_{ξ_p} cannot be strictly convex along the geodesic joining them. Proposition 6.10 then implies that there exists $c > 0$ and $\eta \in \operatorname{Lie}\operatorname{Aut}(X)$ with $\eta \in \mathscr{L}(H^0(X, L^p), H_{\mathbf{s}_p})^T$ such that $[\widetilde{\mathbf{s}}_p] = [ce^{L_\eta}\mathbf{s}_p]$. Then $\phi_\eta^* \xi_p = \xi_p$, and using the characterization (4.20), Proposition 4.5 implies

$$\omega_{\widetilde{h}_p} = \omega_{e^{L_{\xi_p/2p}}\widetilde{\mathbf{s}}_p} = \omega_{ce^{L_\eta}e^{L_{\xi_p/2p}}\mathbf{s}_p} = \phi_\eta^* \omega_{e^{L_{\xi_p/2p}}\mathbf{s}_p} = \phi_\eta^* \omega_{h_p} \,. \tag{6.63}$$

This concludes the proof. □

References

[1] N. Berline, E. Getzler, M. Vergne, Heat Kernels and Dirac Operators, Grundlehren Text Editions, Springer-Verlag, Berlin, 2004, Corrected reprint of the 1992 original.
[2] R.J. Berman, S. Boucksom, V. Guedj, A. Zeriahi, A variational approach to complex Monge-Ampère equations, Publ. Sci. IHES 117 (2) (2013) 179–245.
[3] R.J. Berman, D. Witt Nyström, Complex optimal transport and the pluripotential theory of Kähler-Ricci solitons, arXiv:1401.8264, 2014.
[4] B. Berndtsson, Curvature of vector bundles associated to holomorphic fibrations, Ann. Math. 169 (2) (2009) 531–560.
[5] B. Berndtsson, Positivity of direct image bundles and convexity on the space of Kähler metrics, J. Differ. Geom. 81 (3) (2009) 457–482.
[6] M. Bordemann, E. Meinrenken, M. Schlichenmaier, Toeplitz quantization of Kähler manifolds and gl(N), $N \to \infty$ limits, Commun. Math. Phys. 165 (2) (1994) 281–296.
[7] L. Boutet de Monvel, V. Guillemin, The Spectral Theory of Toeplitz Operators, Annals of Mathematics Studies, vol. 99, Princeton University Press/University of Tokyo Press, Princeton, NJ/Tokyo, 1981.
[8] L. Boutet de Monvel, J. Sjöstrand, Sur la singularité des noyaux de Bergman et de Szegö, Astérisque No. 34–35 (1976) 123–164.
[9] D. Catlin, The Bergman kernel and a theorem of tian, in: Analysis and Geometry in Several Complex Variables, Trends in Mathematics, Birhauser, Boston, 1999, pp. 1–23.
[10] X. Dai, K. Liu, X. Ma, On the asymptotic expansion of Bergman kernel, J. Differ. Geom. 72 (1) (2006) 1–41.
[11] T. Darvas, Y. Rubinstein, Tian's properness conjectures and Finsler geometry of the space of Kähler metrics, J. Am. Math. Soc. 30 (2) (2017) 347–387.
[12] R. Dervan, Uniform stability of twisted constant scalar curvature Kähler metrics, Int. Math. Res. Not. (15) (2016) 4728–4783.
[13] S.K. Donaldson, Scalar curvature and projective embeddings. I, J. Differ. Geom. 59 (3) (2001) 479–522.
[14] S.K. Donaldson, Some numerical results in complex differential geometry, Pure Appl. Math. Q. 5 (2) (2009) 571–618. Special Issue: In honor of Friedrich Hirzebruch. Part 1.
[15] A. Fujiki, On automorphism groups of compact Kähler manifolds, Invent. Math. 44 (3) (1978) 225–258.
[16] A. Futaki, The Ricci curvature of symplectic quotients of Fano manifolds, Tohoku Math. J. (2) 39 (3) (1987) 329–339.
[17] A. Futaki, Asymptotic Chow semi-stability and integral invariants, Int. J. Math. 15 (9) (2004) 967–979.
[18] A. Futaki, T. Mabuchi, Bilinear forms and extremal Kähler vector fields associated with Kähler classes, Math. Ann. 301 (2) (1995) 199–210.
[19] Y. Hashimoto, Mapping properties of the Hilbert and Fubini–Study maps in Kähler geometry, arXiv:1705.11025, 2021.
[20] Y. Hashimoto, Quantisation of extremal Kähler metrics, J. Geom. Anal. 31 (3) (2021) 2970–3028.
[21] L. Ioos, Anticanonically balanced metrics over Fano manifolds, arXiv:2006.05989, 2020.
[22] L. Ioos, V. Kaminker, L. Polterovich, D. Shmoish, Spectral aspects of the Berezin transform, Ann. Henri Lebesgue 3 (2020) 1343–1387.

[23] L. Ioos, D. Kazhdan, L. Polterovich, Almost representations of Lie algebras and quantization, arXiv: 2005.11693, 2020.
[24] A.V. Karabegov, M. Schlichenmaier, Identification of Berezin-Toeplitz deformation quantization, J. Reine Angew. Math. 540 (2001) 49–76.
[25] J. Keller, Twisted balanced metrics, Lie groups: new research, Novapublishers 366 (2009), 15 pp.
[26] A. Lichnerowicz, Isométries et transformations analytiques d'une variété kählérienne compacte, Bull. Soc. Math. Fr. 87 (1959) 427–437.
[27] X. Ma, G. Marinescu, Holomorphic Morse Inequalities and Bergman Kernels, Progress in Mathematics, vol. 254, Birkhäuser Verlag, Basel, 2007.
[28] X. Ma, G. Marinescu, Toeplitz operators on symplectic manifolds, J. Geom. Anal. 18 (2) (2008) 565–611.
[29] X. Ma, G. Marinescu, Exponential estimate for the asymptotics of Bergman kernels, Math. Ann. 362 (2015) 1327–1347.
[30] T. Mabuchi, Stability of extremal Kähler manifolds, Osaka J. Math. 41 (3) (2004) 563–582.
[31] T. Mabuchi, Asymptotic polybalanced kernels on extremal Kähler manifolds, Asian J. Math. 22 (4) (2018) 647–664.
[32] Y. Matsushima, Sur la structure du groupe d'homéomorphismes analytiques d'une certaine variété kählérienne, Nagoya Math. J. 11 (1957) 145–150.
[33] H. Ono, Y. Sano, N. Yotsutani, An example of an asymptotically Chow unstable manifold with constant scalar curvature, Ann. Inst. Fourier (Grenoble) 62 (4) (2012) 1265–1287.
[34] D.H. Phong, J. Sturm, Scalar curvature, moment maps, and the Deligne pairing, Am. J. Math. 126 (3) (2004) 693–712.
[35] L. Polterovich, Quantum unsharpness and symplectic rigidity, Lett. Math. Phys. 102 (2012) 245–264.
[36] L. Polterovich, Symplectic geometry of quantum noise, Commun. Math. Phys. 327 (2014) 481–519.
[37] Y.A. Rubinstein, G. Tian, K. Zhang, Basis divisors and balanced metrics, J. Reine Angew. Math. 778 (2021) 171–218.
[38] S. Saito, R. Takahashi, Stability of anti-canonically balanced metrics, Asian J. Math. 23 (6) (2019) 1041–1058.
[39] Y. Sano, C. Tipler, A moment map picture of relative balanced metrics on extremal Kähler manifolds, J. Geom. Anal. 31 (6) (2021) 5941–5973.
[40] R. Seyyedali, Relative Chow stability and extremal metrics, Adv. Math. 316 (2017) 770–805.
[41] R. Takahashi, Asymptotic stability for Kähler-Ricci solitons, Math. Z. 281 (3–4) (2015) 1021–1034.
[42] R. Takahashi, Geometric quantization of coupled Kähler-Einstein metrics, Anal. PDE 14 (6) (2021) 1817–1849.
[43] G. Tian, X. Zhu, Uniqueness of Kähler-Ricci solitons, Acta Math. 184 (2) (2000) 271–305.
[44] G. Tian, X. Zhu, A new holomorphic invariant and uniqueness of Kähler-Ricci solitons, Comment. Math. Helv. 77 (2) (2002) 297–325.
[45] G.M. Tuynman, Quantization: towards a comparison between methods, J. Math. Phys. 28 (12) (1987) 2829–2840.
[46] X. Wang, Canonical metrics on stable vector bundles, Commun. Anal. Geom. 13 (2) (2005) 253–285.
[47] S. Zelditch, Szegö kernels and a theorem of Tian, Int. Math. Res. Not. 6 (1998) 317–331.