Berezin–Toeplitz Quantization and the Least Unsharpness Principle

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We show that compatible almost-complex structures on symplectic manifolds correspond to optimal positive quantizations.

1   Introduction

A Riemannian metric on a symplectic manifold (M, ω) is ω-compatible if it can be written as

$$g_{\omega,J}(\cdot, \cdot) := \omega(\cdot, J\cdot) \,, \tag{1}$$

where J is an almost complex structure on M. Vice versa, an almost complex structure J is ω-compatible if the bilinear form (1) is a Riemannian metric. Compatible geometric structures were introduced as an effective auxiliary tool for detecting rigidity phenomena on symplectic manifolds [19]. In the present paper, we show that these structures naturally arise from the perspective of mathematical physics. Loosely speaking, they correspond to positive quantizations minimizing a natural physical quantity called *unsharpness*, which is one of the main characters of this paper (see Section 3 below).

Quantization is a mathematical recipe behind the quantum-classical correspondence, a fundamental physical principle stating that quantum mechanics contains

classical mechanics in the limiting regime when the Planck constant $\hbar$ tends to zero [17]. There exist two different, albeit related mathematical models of this principle. Assume that the classical phase space is represented by a closed (i.e., compact without boundary) symplectic manifold (M, ω). The 1st model, *deformation quantization*, is a formal associative deformation

$$f * g = fg + \hbar c_1(f, g) + \hbar^2 c_2(f, g) + \cdots$$

of the multiplication on the space $C^\infty(M)$ of smooth real functions on M such that $f * g - g * f = i\hbar\{f, g\} + \mathcal{O}(\hbar^2)$, where $\{\cdot, \cdot\}$ stands for the Poisson bracket [4]. The operation $*$ is called *the star product*, and the Planck constant $\hbar$ plays the role of a formal deformation parameter.

The 2nd model, *geometric quantization*, is described as a linear correspondence $f \mapsto T_\hbar(f)$ between classical observables, that is, real functions f on the phase space M, and quantum observables, that is, Hermitian operators on a complex Hilbert space. This correspondence is assumed to respect, in the leading order as $\hbar \to 0$, a number of basic operations. In the present paper, we focus on *Berezin–Toeplitz quantizations* [5, 9, 10, 14, 20, 30, 36], whose distinctive feature is to be *positive*, that is, to send non-negative functions to non-negative operators (see Section 2). The known models of Berezin–Toeplitz quantization on closed symplectic manifolds (see the discussion at the end of Section 2) determine a deformation quantization [9, 20, 35] and are provided by certain auxiliary data involving in particular an almost complex structure J compatible with the symplectic form on the phase space. While deformation quantizations of closed symplectic manifolds are completely classified up to a natural equivalence, the classification of Berezin–Toeplitz quantizations is not yet completely understood (see however [28] for the relation between the two).

The main finding of the present paper is that conversely, any Berezin–Toeplitz quantization, defined through natural axioms presented in Section 2, gives rise in a canonical way to a Riemannian metric on the phase space. Specifically, we make the natural assumption that there exists a complex-valued bi-differential operator $c : C^\infty(M) \times C^\infty(M) \to C^\infty(M, \mathbb{C})$ such that the $\mathbb{C}$-linear extension of $T_\hbar$ satisfies

$$T_\hbar(f)T_\hbar(g) = T_\hbar(fg) + \hbar\, T_\hbar(c(f, g)) + \mathcal{O}(\hbar^2)\,, \tag{2}$$

for all $f, g \in C^\infty(M)$ as $\hbar \to 0$, and we show that this induces a Riemannian metric G on M by the formula

$$c(f, g) = -\frac{1}{2}G(\operatorname{sgrad} f,\ \operatorname{sgrad} g)\,,$$

where sgrad f stands for the Hamiltonian vector field of a function f on M (see Theorem 4.1 (I) below).

Leaving precise definitions for Section 4, let us discuss the above-mentioned results informally and present a motivation coming from physics. To this end recall that it is classically known, starting from the Groenewold–van Hove theorem, that a Berezin–Toeplitz correspondence cannot be a genuine morphism between the Lie algebras of functions and operators. We focus on yet another constraint on the precision of Berezin–Toeplitz quantizations, which we call *unsharpness*, and which is governed by the Riemannian metric G defined above. The notion of unsharpness is closely related to the Heisenberg uncertainty principle. It comes from an analogy between quantization and measurement based on the formalism of positive operator valued measures (POVMs), which serves both subjects, and which we briefly recall in Section 2. The unsharpness metric is a particular instance of the noise operator [12] describing, loosely speaking, the increment of variances in the process of quantization (see the discussion, p. 13).

In this language, we propose *the least unsharpness principle*, a variational principle selecting quantizations whose unsharpness metric has minimal possible volume on phase space. It turns out that the least unsharpness volume equals the symplectic volume (see Theorem 4.1 (II) below), so that a quantization satisfying the least unsharpness principle determines a compatible almost complex structure J on (M, ω), in the sense that its unsharpness metric satisfies $G = g_{\omega, J}$ as in (1). We refer to Section 3 for basic properties of unsharpness, while existence of the unsharpness metric and the least unsharpness principle are stated in Section 4 and proved in Section 5.

The unsharpness metric is a natural geometric invariant of a Berezin–Toeplitz quantization and can be seen as a 1st step towards classification. As a case study, we show in Section 6 that for $SU(2)$-equivariant quantizations of the two-dimensional sphere, the unsharpness metric completely determines the quantization up to conjugation and up to 2nd order as $\hbar \to 0$. Further comments on classification can be found in Section 8.3.

Some historical remarks are in order. A canonical appearance of Riemannian geometry in quantization was discussed on a number of occasions in physical literature. Anandan and Aharonov [1] and Ashtekar and Schilling [3] developed a geometric approach to quantum mechanics based on the Fubini–Study metric on the projective space of pure quantum states. Klauder (see, e.g., [29]) studied a model of a path-integral quantization where the role of a metric was to define a Brownian motion on the phase space. The idea of selecting optimal quantizations as those possessing the least uncertainty goes back to Gerhenstaber [18]. He deals with quantizations

that do not necessarily preserve positivity, and his least uncertainty principle implies that unsharpness identically vanishes on some restricted class of observables (see Section 8.1 for further discussion). Finally, while classification of equivariant quantizations is known in the context of deformation quantization [2, 7], our setting, including the notion of equivalence, is substantially different. The case of $SU(2)$-equivariant Berezin–Toeplitz quantizations of the sphere that we settle in Section 6 is, to the best of our knowledge, the 1st one where a complete classification is currently available.

2 Berezin–Toeplitz Quantization

Given a finite-dimensional complex Hilbert space H, we write $\mathcal{L}(H)$ for the space of all Hermitian operators (representing quantum observables), and $\mathcal{S}(H) \subset \mathcal{L}(H)$ for the subset of all non-negative trace-one Hermitian operators (representing quantum states).

Definition 2.1.   An $\mathcal{L}(H)$-valued POVM on a set M equipped with a σ-algebra $\mathcal{C}$ is a map $\Gamma : \mathcal{C} \to \mathcal{L}(H)$ that satisfies the following conditions:

- $\Gamma(\emptyset) = 0$ and $\Gamma(M) = \mathbb{1}$;
- $\Gamma(X) \geq 0$ for every $X \in \mathcal{C}$;
- (σ-additivity) $\Gamma\left(\bigsqcup_{i\in\mathbb{N}} X_i\right) = \sum_{i\in\mathbb{N}} \Gamma(X_i)$ for any sequence of pair-wise disjoint subsets $\{X_i \in \mathcal{C}\}_{i\in\mathbb{N}}$.

According to [16], for every $\mathcal{L}(H)$-valued POVM measure Γ, there exist a probability measure α on $(M, \mathcal{C})$ and a measurable function $F : M \to \mathcal{S}(H)$ such that

$$d\Gamma = n \, F \, d\alpha \, , \tag{3}$$

where $n = \dim_{\mathbb{C}} H$.

Remark 2.2.   In the context of quantization, the state $F_x \in \mathcal{S}(H)$ is called the *coherent state* associated with $x \in M$ and describes the quantization of a classical particle sitting at $x \in M$ in phase space.

For any real classical observable $f \in L^1(M, \alpha)$, we define the *quantization $T(f)$* as the integral

$$T(f) := \int_M f \, d\Gamma \in \mathcal{L}(H) \, . \tag{4}$$

The dual map $T^* : \mathcal{L}(H) \to L^\infty(M)$ with respect to the scalar product on $\mathcal{L}(H)$ defined by $((A, B)) := \mathrm{tr}(AB), A, B \in \mathcal{L}(H)$, satisfies $T^*(A)(x) = n\, \mathrm{tr}(AF(x))$, for any $x \in M$ and $A \in \mathcal{L}(H)$.

Remark 2.3. For a quantum observable A, the function $T^*(A) \in L^\infty(M)$ has a natural interpretation as the classical observable whose value at $x \in M$ is the expectation value of A at the associated coherent state F_x. Thus, we call $T^*(A)$ the *dequantization* of the quantum observable $A \in \mathcal{L}(H)$.

Definition 2.4. The composition

$$\mathcal{B} := \frac{1}{n} T^* T : L^1(M, \alpha) \longrightarrow L^\infty(M)\,,$$

$$f(x) \longmapsto \mathrm{tr}\left(T(f)F_x\right)$$

(5)

is called the *Berezin transform* associated to the POVM Γ.

Remark 2.5. The Berezin transform can be interpreted as quantization followed by dequantization. It is a measure of the blurring induced by quantization.

To study the quantum-classical correspondence, we need to introduce a parameter $\hbar$ in the above story, which can be thought as the *Planck constant*, and from which we recover the laws of classical mechanics as $\hbar \to 0$. This is given a precise meaning via the following definition.

Definition 2.6. Let (M, ω) be a closed connected symplectic manifold of dimension $2d$ and $\mathcal{C}$ be the σ-algebra of its Borel sets in M. A *Berezin–Toeplitz quantization* of M is the following data:

- a subset $\Lambda \subset \mathbb{R}_{>0}$ having 0 as limit point ;
- a finite-dimensional complex Hilbert space $H_\hbar$ for each $\hbar \in \Lambda$;
- an $\mathcal{L}(H_\hbar)$-valued POVMs $\Gamma_\hbar$ on M for each $\hbar \in \Lambda$,

such that the *Toeplitz map $T_\hbar$* $: C^\infty(M, \mathbb{C}) \to \mathrm{End}(H_\hbar)$ induced for all $\hbar \in \Lambda$ by the quantization map (4) is surjective and satisfies the following estimates, *uniformly* in the C^N-norms of $f, g \in C^\infty(M)$ for some $N \in \mathbb{N}$:

- (P1) **(norm correspondence)**

$$\|f\| - \mathcal{O}(\hbar) \leq \|T_\hbar(f)\|_{\mathrm{op}} \leq \|f\|\,,$$

where $\|\cdot\|_{op}$ is the operator norm and $\|f\| := \max_{x \in M} |f(x)|$;

(P2)   **(bracket correspondence)**

$$\left\| -\frac{i}{\hbar}[T_\hbar(f), T_\hbar(g)] - T_\hbar(\{f, g\}) \right\|_{\mathrm{op}} = \mathcal{O}(\hbar) \, ,$$

where $[\cdot, \cdot]$ stands for the commutator and $\{\cdot, \cdot\}$ for the Poisson bracket (our convention for the Poisson bracket is $\{f, g\} := -\omega(\mathrm{sgrad}f, \mathrm{sgrad}g)$ for all $f, g \in C^\infty(M)$, where $\mathrm{sgrad}f$ is the Hamiltonian vector field of f defined by $\iota_{\mathrm{sgrad}f}\omega + df = 0$);

(P3)   **(quasi-multiplicativity)** There exists a bi-differential operator $c : C^\infty(M) \times C^\infty(M) \to C^\infty(M, \mathbb{C})$ such that

$$\|T_\hbar(f)T_\hbar(g) - T_\hbar(fg) - \hbar T_\hbar(c(f, g))\|_{op} = \mathcal{O}(\hbar^2);$$

(P4)   **(trace correspondence)**

$$\mathrm{tr}(T_\hbar(f)) = (2\pi\hbar)^{-d} \int_M f \, R_\hbar \, \mathrm{d}\mu \, ,$$

where $R_\hbar \in C^\infty(M)$ satisfies $R_\hbar = 1 + \mathcal{O}(\hbar)$, and $d\mu = \frac{\omega^d}{d!}$ is the symplectic volume on M;

(P5)   **(reversibility)** The maps $\mathcal{B}_\hbar : C^\infty(M) \to C^\infty(M)$ induced by the Berezin transform (5) satisfy

$$\mathcal{B}_\hbar(f) = f + \mathcal{O}(\hbar) \, .$$

By uniformly in the C^N-norms of $f, g \in C^\infty(M)$, we mean that there exists a constant $C > 0$ such that, in axioms (P2) and (P3) and for $k = 1, 2$ respectively, the remainders satisfy

$$\left| \mathcal{O}(\hbar^k) \right| \le C\hbar^k \|f\|_{C^N} \|g\|_{C^N} \, , \tag{6}$$

while in axioms (P1) and (P5), the remainders satisfy $|\mathcal{O}(\hbar)| \le C\hbar \|f\|_{C^N}$. Writing the density (3) associated to $\Gamma_\hbar$ in the form

$$d\Gamma_\hbar(x) = n_\hbar \, F_{\hbar,x} \, d\alpha_\hbar(x) \, , \tag{7}$$

the trace correspondence (P4) implies

$$n_\hbar = \frac{\mathrm{Vol}(M, \omega) + \mathcal{O}(\hbar)}{(2\pi\hbar)^d} \, ,$$

and

$$d\alpha_\hbar = \frac{1 + \mathcal{O}(\hbar)}{\mathrm{Vol}(M,\omega)} d\mu \ , \tag{8}$$

where $\mathrm{Vol}(M,\omega) > 0$ denotes the symplectic volume of (M,ω).

The existence of a Berezin–Toeplitz quantization is a highly non-trivial result. To discuss it, recall that an almost complex structure J on M is *ω-compatible* if the form $G_J := \omega(\cdot, J\cdot)$ is a Riemannian metric on M. We refer to (M,ω,J,G_J) as an *almost-Kähler structure* on M. Let us assume additionally that (M,ω) *quantizable*, that is, that the cohomology class $[\omega]/(2\pi)$ is integral, and that the almost complex structure J is integrable, so that (M,ω,J,G_J) is a *Kähler manifold*. Then there is a canonical construction of a Berezin–Toeplitz quantization, where $\Lambda = \{1/k\}_{k\in\mathbb{N}}$ and where the Hilbert spaces $H_\hbar$ consist of the global *holomorphic sections* of a holomorphic Hermitian line bundle with Chern curvature equal to $-ik\omega$, and the associated Toeplitz map $T_\hbar$ sends $f \in C^\infty(M)$ to the multiplication by f followed by the orthogonal L_2-projection on holomorphic sections. The axioms of Definition 2.6 for this construction have been established by Bordemann, Meinrenken, and Schlichenmaier in [9], using the theory of Toeplitz structures developed by Boutet de Monvel and Guillemin in [11]. The extension of this theory to general ω-compatible almost complex structures was established in a series of papers by Guillemin [20], Borthwick and Uribe [10], Schiffman and Zelditch [37], Ma and Marinescu [30], Charles [14], and the 1st-named author, Lu, Ma, and Marinescu [23]. The dependence of the remainders in terms of the derivatives of the functions is discussed in [15].

3 Unsharpness Cocycle

In this section, we study general properties of the bi-differential operator $c : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ from the quasi-multiplicativity property (P3) of a Berezin–Toeplitz quantization. First note that norm correspondence (P1) implies that, if an asymptotic expansion such as the one appearing in (P3) holds, then it is unique, and in particular, the bi-differential operator c is uniquely defined. Then the associativity of the composition of operators implies that c is a *Hochschild cocycle*, meaning that for all $f_1, f_2, f_3 \in C^\infty(M)$, we have

$$f_1 c(f_2, f_3) - c(f_1 f_2, f_3) + c(f_1, f_2 f_3) - c(f_1, f_2) f_3 = 0 \ . \tag{9}$$

Denote by c_- and c_+ its anti-symmetric and symmetric parts, respectively:

$$c_-(f,g) := \frac{c(f,g) - c(g,f)}{2} \quad \text{and} \quad c_+(f,g) := \frac{c(f,g) + c(g,f)}{2} .$$

By the bracket correspondence (P2), we have $T_\hbar(2c_-(f,g) - i\{f,g\}) = \mathcal{O}(\hbar)$, and hence by the norm correspondence (P1), we get the formula

$$c_-(f,g) = \frac{i}{2}\{f,g\} . \tag{10}$$

Thus, the anti-symmetric part c_- (responsible for the non-commutativity of quantum observables) does not depend on a choice of a quantization. In contrast, the symmetric part c_+ does depend on a choice of a quantization. By the quasi-multiplicativity property (P3), the cocycle c_+ associated to a Berezin–Toeplitz quantization measures its failure of being a multiplicative morphism on Poisson-commutative subspaces of $C^\infty(M)$. From formula (10) and basic properties of the Poisson bracket, we know that c^-, hence also c^+, satisfy formula (9) for a Hochschild cocycle.

Definition 3.1.   We say that c_+ is the *unsharpness cocycle* of a quantization or simply its *unsharpness*.

Note that by formula (4), the operator $T_\hbar(f) \in \text{End}(H_\hbar)$ is Hermitian if and only if $f \in C^\infty(M,\mathbb{C})$ is real valued, and as the square of a Hermitian operator is Hermitian, the quasi-multiplicativity property (P3) then shows that $c^+ : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ is a real-valued bi-differential operator. It is also a symmetric Hochschild cocycle, so that as explained for example, in [33, Th. 2.15], it is a differential Hochschild *coboundary*. This means that there exists a real-valued differential operator $a : C^\infty(M) \to C^\infty(M)$ such that

$$c_+(f,g) = a(fg) - f\,a(g) - g\,a(f) . \tag{11}$$

for all $f, g \in C^\infty(M)$. Since $T_\hbar(1) = \mathbb{1}$, we have that $c_+(1,1) = 0$, and therefore $a(1) = 0$. Note that a is determined up to its 1st-order part. The following result shows that the positivity preserving property imposes a strong condition on c_+.

Theorem 3.2.   The bi-differential operator c_+ is of order $(1,1)$.

The proof is given in Section 7 below. Theorem 3.2 sheds light on the differential operator a appearing in the coboundary formula (11). In fact, let us choose some

Darboux coordinates $U \subset M$, and take $f, g \in C^\infty(M)$ with compact support in U. In these coordinates, we can write

$$c_+(f,g) = \sum_{j,k=1}^{2d} a_{jk} \, \partial_j f \, \partial_k g \, , \tag{12}$$

with smooth $a_{jk} = a_{kj} \in C^\infty(U)$ for each $1 \le j, k \le 2d$. Then one can choose the differential operator

$$a := \frac{1}{2} \sum_{j,k=1}^{2d} \partial_j \left(a_{jk} \partial_k \right) \, . \tag{13}$$

in the coboundary formula (11). Using integration by parts, we see that (13) is symmetric with respect to the canonical L^2-scalar product on $C^\infty(M)$ associated to the symplectic volume, and as the differential operator a is determined up to its 1st-order part, it is the unique such choice.

Example 3.3.   The following result refers to the Berezin–Toeplitz quantization described at the end of the previous section. In this fundamental example, the unsharpness cocycle can be computed explicitly, and formula (14) below can be found for instance in [40, p. 257] for the Kähler case and in [22, 23] for the almost-Kähler case.

Theorem 3.4.   Assume that (M, ω) is quantizable, so that the cohomology class $[\omega]/(2\pi)$ is integral. Then for every ω-compatible almost complex structure $J \in \text{End}(TM)$, there exists a Berezin–Toeplitz quantization of (M, ω) whose unsharpness cocycle satisfies

$$c_+(f,g) = -\frac{1}{2}(\nabla f, \nabla g) \, , \tag{14}$$

for all $f, g \in C^\infty(M, \mathbb{R})$, where the gradient and the product are defined with respect to the Riemannian metric $G_J := \omega(\cdot, J\cdot)$.

Using that

$$\Delta(fg) + 2(\nabla f, \nabla g) = f\Delta g + g\Delta f \, , \tag{15}$$

where Δ is the (positive) Laplace–Beltrami operator associated with G_J, the differential operator in the coboundary formula (11) can then be chosen to be $a = \Delta/4$, and by basic properties of Δ, it is the unique L^2-symmetric choice with respect to the symplectic volume form, as it coincides with the Riemannian volume form of G_J. Using the J-invariance of the metric and the relation $J \, \text{sgrad} f = -\nabla f$ between Hamiltonian vector

field and gradient of a function $f \in C^\infty(M)$ for an ω-compatible metric, formula (14) translates into

$$c_+(f,g) = -\frac{1}{2}G_J(\text{sgrad}\, f, \text{sgrad}\, g) \,. \tag{16}$$

Example 3.5.   We now give an example of a Berezin–Toeplitz quantization whose unsharpness cocycle c_+ is not of the form (16) for some almost-Kähler structure on (M, ω). This example serves as a paradigm for the construction presented in the proof of one of our main results, Theorem 4.1(III) below. Assume (M, ω) quantizable and equipped with an almost-Kähler structure (M, ω, J, G_J), and consider the induced Berezin–Toeplitz quantization of Theorem 3.4. Fix $t > 0$, and using the notations of Example 3.3, consider for any $\hbar \in \Lambda = \{1/k\}_{k \in \mathbb{N}}$ the map $T_\hbar^{(t)} : C^\infty(M) \to \mathcal{L}(H_\hbar)$ defined for any $f \in C^\infty(M)$ by

$$T_\hbar^{(t)}(f) := T_\hbar(e^{-t\hbar\Delta}f) \,.$$

Observe that the heat flow preserves positivity, so that $T_\hbar^{(t)}$ is in fact the quantization map (4) induced by a POVM construction. Then from the classical small time asymptotic expansion of the heat kernel (see e.g., [6, Th. 2.29, (2.8)]), as $\hbar \to 0$, we have

$$e^{-t\hbar\Delta}f = f - t\hbar\Delta f + \mathcal{O}(\hbar^2)\,\|f\|_{C^4} \,, \tag{17}$$

and this implies in particular that all the axioms of Definition 2.6 hold. Let us now calculate the associated unsharpness cocycle, denoted by $c_+^{(t)}$. For any $\hbar \in \Lambda$ and $A, B \in \text{End}(H_\hbar)$, put $A \bullet B := \frac{1}{2}(AB + BA)$, and recall formula (15) for the Laplace–Beltrami operator. Then as $\hbar \to 0$, we have

$$
\begin{aligned}
T_\hbar^{(t)}(f) \bullet T_\hbar^{(t)}(g) &= T_\hbar(f) \bullet T_\hbar(g) - t\hbar T_\hbar(f\Delta g + g\Delta f) + \mathcal{O}(\hbar^2) \\
&= T_\hbar(fg) - \hbar T_\hbar\left(\frac{1}{2}(\nabla f, \nabla g) + t\left(\Delta(fg) + 2(\nabla f, \nabla g)\right)\right) + \mathcal{O}(\hbar^2) \\
&= T_\hbar(fg) + \hbar T_\hbar^{(t)}\left(-\left(\frac{1}{2} + 2t\right)(\nabla f, \nabla g)\right) + \mathcal{O}(\hbar^2) \,,
\end{aligned}
\tag{18}
$$

so that, recalling that the quasi-multiplicativity property (P3) determines the unsharpness cocycle uniquely via norm correspondence (P1), we get

$$
\begin{aligned}
c_+^{(t)}(f,g) &= -\left(\frac{1}{2} + 2t\right)(\nabla f, \nabla g) \\
&= -\frac{1}{2}(1 + 4t)\, G_J(\text{sgrad}\, f, \text{sgrad}\, g) \,.
\end{aligned}
\tag{19}
$$

In particular, we see that $c_+^{(t)}$ is of the form (16) for the Riemannian metric $G^{(t)} := (1 + 4t)\, G_J$ on M, whose volume is strictly bigger than the volume of the almost-Kähler metric G_J. As the volume of an almost-Kähler metric is always equal to the symplectic volume of (M, ω), we see that $c_+^{(t)}$ is not the unsharpness cocycle of a Berezin–Toeplitz quantization coming from Theorem 3.4.

4 The Least Unsharpness Principle

In this section, we state the main theorem on unsharpness of Berezin–Toeplitz quantizations, which we call the *least unsharpness principle*, and discuss its physical meaning.

Recall from Theorem 3.2 that the unsharpness cocycle c_+ of a Berezin–Toeplitz quantization is a bi-differential operator of order $(1, 1)$, so that there exists a bilinear symmetric form G on TM such that

$$c_+(f, g) =: -\frac{1}{2} G(\text{sgrad}\, f, \text{sgrad}\, g)\,, \tag{20}$$

where $\text{sgrad} f$, $\text{sgrad} g$ denote the Hamiltonian vector fields of f, $g \in C^\infty(M, \mathbb{R})$. Our main result provides a description of this bilinear form G.

Theorem 4.1. Let (M, ω) be a closed symplectic manifold.

   (I)   For every Berezin–Toeplitz quantization of M, the form G is a Riemannian metric on M, which can be written as the sum

$$G = \omega(\cdot, J\cdot) + \rho(\cdot, \cdot)\,, \tag{21}$$

         where $J \in \text{End}(TM)$ is a compatible almost complex structure on (M, ω) and ρ is a non-negative symmetric bilinear form on TM.

  (II)   We have $\text{Vol}(M, G) \geq \text{Vol}(M, \omega)$, with equality if and only if $\rho \equiv 0$.

 (III)   If (M, ω) is quantizable, then every Riemannian metric of the form (21) arises from some Berezin–Toeplitz quantization.

The proof is given in Section 5. Let us mention that the proof of item (III) of the theorem is modeled on Example 3.5 above and is constructive. We produce the desired Berezin–Toeplitz quantization with the unsharpness metric given by (21) as the composition of the almost-Kähler quantization associated to (ω, J) and an explicit, albeit non-canonical, Markov operator depending on all the data including ρ.

Remark 4.2.   For a given metric G on M, the decomposition (21) is in general not unique. However, as the proof of Theorem 4.1 (I) will show, there exists a unique ω-compatible almost complex structure J, which additionally is G-orthogonal, that is, $G(J\xi, J\eta) = G(\xi, \eta)$ for all $\xi, \eta \in TM$. Furthermore, for such a G, the symmetric bilinear form $\rho(\xi, \eta) = G(\xi, \eta) - \omega(\xi, J\eta)$ is non-negative, thus providing decomposition (21).

Before going to the proof of Theorem 4.1 in the next section, let us first discuss the physical meaning of the unsharpness cocyle c^+ associated with a Berezin–Toeplitz operator, which shows from general principles that it is at least non-negative. With every quantum state $\theta \in \mathcal{S}(H_\hbar)$, one associates a classical state (called *the Husimi measure*), which is the probability measure μ_θ on M such that

$$\int_M f \, d\mu_\theta = \mathrm{tr}(T_\hbar(f)\theta) \, , f \in C^\infty(M) \, . \tag{22}$$

This equality can be interpreted as follows: the expectation of any classical observable f in the classical state μ_θ coincides with the expectation of the corresponding quantum observable $T_\hbar(f)$ in the state θ. What happens with variances? It turns out that the quantum variance is in general bigger than the classical one. More precisely, we have that

$$\mathbb{V}ar(f, \mu_\theta) = \int_M f^2 d\mu_\theta - \left(\int f d\mu_\theta \right)^2 \, ,$$

$$\mathbb{V}ar(T_\hbar(f), \theta) = \mathrm{tr}(T_\hbar(f)^2\theta) - \left(\mathrm{tr}(T_\hbar(f)\theta) \right)^2 \, ,$$

and hence

$$\mathbb{V}ar(f, \mu_\theta) = \mathbb{V}ar(T_\hbar(f), \theta) + \mathrm{tr}(\Delta_\hbar(f)\theta) \, , \tag{23}$$

where

$$\Delta_\hbar(f) := T_\hbar(f^2) - T_\hbar(f)^2 \, . \tag{24}$$

The operator $\Delta_\hbar(f)$ is called the *noise operator* (see e.g., [12]), whose main property is that it is a *non-negative operator*. It describes the increase of variances, which can be interpreted as the unsharpness of the quantization. Then by the quasi-multiplicativity property (P3), we have

$$\Delta_\hbar(f) = -\hbar \, T_\hbar \left(c_+(f, f) \right) + \mathcal{O}(\hbar^2) \, . \tag{25}$$

Look at the expectation of $\Delta_h(f)$ at the coherent state $F_{h,x}$ of Remark 2.2 associated to $\Gamma_{h'}$,

$$\operatorname{tr}\left(\Delta_h(f)F_{h,x}\right) = -\hbar \operatorname{tr}\left(T_h\left(c_+(f,f)\right)F_{h,x}\right) + \mathcal{O}(\hbar^2)$$

$$= -\hbar\,\mathcal{B}_h\left(c_+(f,f)\right)(x) + \mathcal{O}(\hbar^2) \tag{26}$$

$$= -\hbar\,c_+(f,f)(x) + \mathcal{O}(\hbar^2)\,.$$

This explains the name of unsharpness cocycle for c^+. Since the noise operator is non-negative, we get the following fundamental property of the unsharpness cocycle,

$$- c_+(f,f)(x) \geq 0 \quad \text{for all} \quad x \in M\,. \tag{27}$$

This shows that the symmetric bilinear form G defined in equation (20) is at least semi-positive. This property is the 1st step towards the proof of Theorems 4.1, showing that G is in fact a Riemannian metric, called the *unsharpness metric* of the quantization. Note that this property is also at the basis of the proof of Theorem 3.2.

Define the *total unsharpness* of a Berezin–Toeplitz quantization as the volume of the phase space M with respect to the unsharpness metric. With this language, statement (II) of Theorem 4.1 can be interpreted as the *least unsharpness principle*: the minimal possible total unsharpness equals the symplectic volume, and the least unsharpness metrics are induced by compatible almost-complex structures on M.

Remark 4.3.   Let us assume that the Berezin transform admits an asymptotic expansion up to the 1st order as $\hbar \to 0$ of the following form for all $f \in C^\infty(M)$,

$$\mathcal{B}_h(f) = f + \hbar Df + \mathcal{O}(\hbar^2)\,, \tag{28}$$

where D is a differential operator, strengthening the reversibility property (P5). Then by Definition 2.4 of the Berezin transform, formula (4) for the quantization map and the expansion (8) for α_h, for all $f, g \in C^\infty(M)$, we have

$$\frac{1}{n_h}\operatorname{tr}(T_h(f)T_h(g)) = \int_M \mathcal{B}_h(f)\,g\,\mathrm{d}\alpha_h$$

$$= \int_M fg\,\mathrm{d}\alpha_h + \hbar \int_M (Df)\,g\,\mathrm{d}\mu + \mathcal{O}(\hbar^2)\,. \tag{29}$$

On the other hand, by the quasi-multiplicativity property (P3), using formula (10) and basic properties of the Poisson bracket, we get

$$
\begin{aligned}
\frac{1}{n_\hbar}\mathrm{tr}(T_\hbar(f)T_\hbar(g)) &= \int_M (fg + \hbar c(f,g) + \mathcal{O}(\hbar^2))\,\mathrm{d}\alpha_\hbar \\
&= \int_M fg\,\mathrm{d}\alpha_\hbar + \hbar \int_M c_+(f,g)\,\mathrm{d}\mu + \mathcal{O}(\hbar^2)\,.
\end{aligned}
\tag{30}
$$

Then taking f, $g \in C^\infty(M)$ with compact support in Darboux coordinates, using formulas (12) and (13) and integration by parts, we then get

$$
D = -2a\,,
\tag{31}
$$

where $a : C^\infty(M) \to C^\infty(M)$ is the unique L^2-symmetric differential operator on $C^\infty(M)$ with respect to symplectic volume satisfying the coboundary formula (11). In light of Example 3.3, this fact generalizes the Karabegov–Schlichenmaier expansion [25, 28] for the Berezin–Toeplitz quantizations of Theorem 3.4.

Another consequence of the improvement (28) of the reversibility property (P5) is that "unsharpness equals variance on coherent states". To see this, recall definition (22) of the Husimi measure on the coherent state $F_{h,x} \in \mathcal{S}(H_\hbar)$ of Remark 2.2. Then the discussion above implies

$$
\begin{aligned}
\mathbb{V}ar(f, \mu_{F_{x,h}}) &= \mathcal{B}_\hbar(f^2) - \mathcal{B}_\hbar(f)^2 \\
&= -2\hbar\, c^+(f,f) + \mathcal{O}(\hbar^2)\,.
\end{aligned}
\tag{32}
$$

Thus by formula (23) and (26), we get

$$
\mathbb{V}ar(T_\hbar(f), F_{h,x}) = -\hbar\, c_+(f,f)(x) + \mathcal{O}(\hbar^2)\,,
\tag{33}
$$

so that the variance of a quantized observable at coherent states is equal to its unsharpness.

In their geometric formulation of quantum mechanics, Ashtekar and Shilling [3,§ 3.2.3, (26)] consider the projectivization $\mathbb{P}(H_\hbar)$ as a "quantum phase space": a line $\xi \in \mathbb{P}(H_\hbar)$ is identified with the pure state given by the rank-one projector to ξ. In this setting, they give a physical interpretation of the Fubini–Study metric g^{FS} on $\mathbb{P}(H_\hbar)$ in terms of the variance of a quantum observable $A \in \mathcal{L}(H_\hbar)$ at a pure state $\xi \in \mathbb{P}(H_\hbar)$. Specifically, write v_A for the vector field on $\mathbb{P}(H_\hbar)$ induced by the infinitesimal action of

$iA \in \mathfrak{u}(H_h)$, seen as an element of the Lie algebra of the group of unitary operators $U(H_h)$ acting on $\mathbb{P}(H_h)$. Then the variance of A at ξ is given by

$$\mathbb{V}ar(A,\xi) = \frac{1}{2} g_\xi^{FS}(v_A, v_A) .$$

(34)

Back to the quantization, assume further that the coherent states $F_{h,x} \in \mathcal{S}(H_h)$ are pure for all $x \in M$. Consider the induced map

$$F_h : M \longrightarrow \mathbb{P}(H_h) .$$

(35)

Then equation (33) says that the Fubini–Study length of the vector field v_A induced by the quantum observable $A := T_h(f) \in \mathcal{L}(H_h)$ at the coherent state $F_{h,x} \in \mathbb{P}(H_h)$ approaches, as $h \to 0$, the length of the Hamiltonian vector field sgrad f at a point $x \in M$ with respect to our unsharpness metric. In the case of the Kähler quantizations described at the end of Section 2, the map (35) coincides with the *Kodaira map*. Then the picture described above is closely related to a theorem of Tian [38] showing that the pullback of the Fubini–Study metric by the Kodaira map approaches the Kähler metric as $k \to +\infty$.

5   Proof of the Main Theorem

In this section, we prove Theorem 4.1. To this end, first recall from the previous section that the non-negativity of the noise operator (24) leads to the semi-positivity property (27) for the unsharpness. To establish the stronger statement (i) of Theorem 4.1, we will use a stronger property of noise operators coming from the general theory of POVM-based quantum measurements, called the *noise inequality*. It appears in several sources [31], [21, Theorem 7.5], [32, Theorem 9.4.16], albeit none of them contains the version we need. For the sake of completeness, we present a proof that closely follows [32] and is based on an idea from [26].

Consider a set M equipped with a σ-algebra $\mathcal{C}$ together with a finite-dimensional Hilbert space H, and let F be an $\mathcal{L}(H)$-valued POVM in the sense of Definition 2.1. For a bounded function $u \in L^\infty(M)$, we define the *noise operator*

$$\Delta_F(u) := \int_M u^2 \, dF - \left(\int_M u \, dF \right)^2 .$$

For a pair of bounded functions $u, v \in L^{\infty}(M)$, set

$$U := \int_M u \, \mathrm{d}F, \quad V := \int_M v \, \mathrm{d}F.$$

Lemma 5.1.   For every state $\theta \in \mathcal{S}(H)$, we have the inequality

$$\mathrm{tr}\left(\Delta_F(u)\theta\right) \mathrm{tr}\left(\Delta_F(v)\theta\right) \geq \frac{1}{4} \left|\mathrm{tr}\left([U, V]\theta\right)\right|^2. \tag{36}$$

Proof.   By the *Naimark dilation theorem* (see e.g., [32, Theorem 9.4.6]), there exists an isometric embedding of H into a (possibly infinite-dimensional) Hilbert space H' and an $\mathcal{L}(H')$-valued *projector valued* measure P such that for every subset $X \subset \mathcal{C}$, we have

$$F(X) = \Pi P(X)\Pi^* \in \mathcal{L}(H), \tag{37}$$

where $\Pi : H' \to H$ is the orthogonal projector, so that $\Pi^* : H \to H'$ is the inclusion. Here $\mathcal{L}(H')$ stands for the space of all bounded Hermitian operators on H', and an $\mathcal{L}(H')$-projector valued measure is by definition a map $P : \mathcal{C} \to \mathcal{L}(H')$ satisfying the axioms of Definition 2.1, and such that the operators $P(X)$, $X \in \mathcal{C}$, are pair-wise commuting orthogonal projectors.

Define a pairing

$$q : \mathcal{L}(H') \times \mathcal{L}(H') \to \mathrm{End}(H),$$

$$q(S, T) := \Pi S(1 - \Pi^*\Pi)T\Pi^*.$$

We claim that for every state $\theta \in \mathcal{S}(H)$ and all $S, T \in \mathcal{L}(H')$, we have

$$\mathrm{tr}(q(S, S)\theta) \, \mathrm{tr}(q(T, T)\theta) \geq \left|\mathrm{tr}(q(S, T)\theta)\right|^2. \tag{38}$$

To see this, note that $1 - \Pi^*\Pi : H \to H$ is the orthogonal projector on the orthogonal complement of H', so that

$$\mathrm{tr}(q(S, T)\theta) = \mathrm{tr}((1 - \Pi^*\Pi)T \, \Pi^*\theta\Pi \, S(1 - \Pi^*\Pi)).$$

Then (38) follows from Cauchy–Schwarz inequality applied to the semi-norm on the space $\mathrm{End}(H')$ of bounded operators of H' defined by $(A, B)_\theta := \mathrm{tr}(A \, \Pi^*\theta\Pi \, B^*)$, for all $A, B \in \mathrm{End}(H')$.

Set now $S = \int u \, dP$ and $T = \int v \, dP$. Since S and T commute, we have $[\Pi S \Pi^*, \Pi T \Pi^*] = q(T, S) - q(S, T)$. On the other hand, by definition (37), we have $\Pi S \Pi^* = U$ and $\Pi T \Pi^* = V$, while using an approximation by simple functions, one computes that $q(S, S) = \Delta_F(u)$ and $q(T, T) = \Delta_F(v)$. The result then directly follows from (38). ∎

Proof of (I): Applying Lemma 5.1 to the Berezin–Toeplitz POVM Γ_h of Definition 2.6, for every state $\theta \in \mathcal{S}(H_h)$ and observables $u, v \in C^\infty(M)$, we get

$$\operatorname{tr}\left(\Delta_h(u)\theta\right) \operatorname{tr}\left(\Delta_h(v)\theta\right) \geq \frac{1}{4} \left|\operatorname{tr}([T_h(u), T_h(v)]\theta)\right|^2 . \tag{39}$$

Now by Definition 2.4 of the Berezin transform and the expression (24) for the noise operator, we know that for all $u \in C^\infty(M)$,

$$\operatorname{tr}\left(\Delta_h(u)F_{h,x}\right) = -\hbar \, \mathcal{B}(c_+(u, u))(x) + \mathcal{O}(\hbar^2) , \tag{40}$$

and for all $u, v \in C^\infty(M)$,

$$\begin{aligned} -i \operatorname{tr}([T_h(u), T_h(v)]F_{h,x}) &= \hbar \operatorname{tr}(T_h(\{u, v\})F_{h,x}) + \mathcal{O}(\hbar^2) \\ &= \hbar \, B_h(\{u, v\})(x) + \mathcal{O}(\hbar^2) . \end{aligned} \tag{41}$$

Thus, applying the noise inequality (39) with θ being the coherent state $F_{h,x}$, we get that

$$\mathcal{B}_h(c_+(u, u))(x)\mathcal{B}_h(c_+(v, v))(x) \geq \frac{1}{4}|\mathcal{B}_h(\{u, v\})(x)|^2 , \tag{42}$$

so that the reversibility property (P5) yields

$$c_+(u, u)c_+(v, v) \geq \frac{1}{4}|\{u, v\}|^2 . \tag{43}$$

Thus, for all $\xi, \eta \in T_x M$, picking functions $u, v \in C^\infty(M)$ with $\operatorname{sgrad} u(x) = \xi$, $\operatorname{sgrad} v(x) = \eta$ and by definition (20) of the bilinear form G, we get

$$G(\xi, \xi)G(\eta, \eta) \geq |\omega(\xi, \eta)|^2 . \tag{44}$$

Now thanks to the non-negativity of the noise operator, which follows from Lemma 5.1, we already know that G is a semi-positive symmetric bilinear form by formula (27). Inequality (44) then shows that G is in fact positive, so that it defines a Riemannian metric on M.

Let $K \in \mathrm{End}(TM)$ the G-antisymmetric operator defined by

$$G(\cdot,\cdot) = \omega(\cdot,K\cdot). \tag{45}$$

Then there exists an orthonormal basis $\{e_j, f_j\}_{1 \leq j \leq \dim M}$ of TM such that $Ke_j = \alpha_j f_j$ and $Kf_j = -\alpha_j e_j$, for $\alpha_j \geq 0$ for all $1 \leq j \leq \dim M$. Define an almost complex structure $J \in \mathrm{End}(TM)$ by the formula

$$Je_j = f_j \text{ and } Jf_j = -e_j. \tag{46}$$

By definition, this almost complex structure is compatible with ω, and G is J-invariant. Set

$$\rho(\cdot,\cdot) := G(\cdot,\cdot) - \omega(\cdot,J\cdot). \tag{47}$$

We then need to show that for any $\xi \in TM$, we have

$$\rho(\xi,\xi) \geq 0. \tag{48}$$

But using (44), we know that

$$G(\xi,\xi) = G(\xi,\xi)^{1/2}\,G(J\xi,J\xi)^{1/2} \geq \omega(\xi,J\xi), \tag{49}$$

which readily implies (48) by definition (47) of ρ.

PROOF OF (II): Recall that the volume of an ω-compatible metric is always equal to the symplectic volume $\mathrm{Vol}(M,\omega)$. Then the statement (II) follows from the general form of an unsharpness metric G given by formula (21).

PROOF OF (III): The construction of a Berezin–Toeplitz quantization with general unsharpness metric of the form (21) is based on Theorem 3.4 and is a modification of the construction of Example 3.5. Instead of dealing with the heat semigroup, which becomes elusive when the form ρ is degenerate, we construct an explicit family of Markov kernels such that the desired quantization is the composition of the almost-Kähler quantization associated with J from formula (21) with the corresponding Markov operator (In the language of quantum measurement theory, the POVM of the quantization constructed below is a smearing of the Berezin–Toeplitz POVM of Theorem 3.4 by the explicitly constructed Markov operator.). Let us pass to precise arguments.

All the estimates in the proof are meant uniformly in $x_0 \in M$. Let $J \in \mathrm{End}(TM)$ be a compatible almost complex structure on (M,ω) and let ρ be a non-negative symmetric

bilinear form on *TM*. Consider the Riemannian metric g over M defined by the formula

$$g(\cdot, \cdot) = \omega(\cdot, J\cdot). \tag{50}$$

For any $t > 0$, we define a smooth endomorphism of the tangent bundle *TM* by the formula

$$A_t := t\left(-\pi J \rho_g J + t\mathbb{1}\right) \in \mathrm{End}(TM), \tag{51}$$

where $\rho_g \in \mathrm{End}(TM)$ is the non-negative symmetric endomorphism defined by

$$g(\rho_g \cdot, \cdot) = \rho. \tag{52}$$

Then A_t is positive symmetric with respect to g, for all $t > 0$.

Let $\epsilon > 0$ be smaller than the injectivity radius of (M, g). For any $x_0 \in M$, consider an isometric identification $(T_{x_0}M, g) \simeq (\mathbb{R}^{2d}, \langle \cdot, \cdot \rangle)$, where $\langle \cdot, \cdot \rangle$ is the standard Euclidean product of $\mathbb{R}^{2d}$, and let $Z = (Z_1, \cdots Z_{2d}) \in \mathbb{R}^{2d}$ be the induced normal coordinates on the geodesic ball $B(x_0, \epsilon) \subset M$ of radius ϵ centered at x_0. We write dZ for the Lebesgue measure on $\mathbb{R}^{2d}$. Let $\varphi : [0, +\infty) \to [0, 1]$ be a smooth function identically equal to 1 over $[0, \epsilon/2)$ and to 0 over $[\epsilon, +\infty)$. We define an operator K_t^ρ acting on $f \in C^\infty(M, \mathbb{R})$ by the following formula in normal coordinates around $x_0 \in M$,

$$K_t^\rho f(x_0) := \frac{1}{\alpha_t(x_0)} \int_{B(x_0, \epsilon)} \varphi(|Z|) f(Z) e^{-\pi \left\langle A_t^{-1} Z, Z \right\rangle} \, dZ, \tag{53}$$

where $\alpha_t(x_0) := \int_{B(x_0, \epsilon)} \varphi(|Z|) e^{-\pi \left\langle A_t^{-1} Z, Z \right\rangle} \, dZ$ is chosen so that $K_t 1 \equiv 1$ for all $t > 0$. Note that $f \geq 0$ implies $K_t f \geq 0$ for all $t > 0$.

Fix $x_0 \in M$, and consider the isometric identification $(T_{x_0}M, g) \simeq (\mathbb{R}^{2d}, \langle \cdot, \cdot \rangle)$ in which A_t is diagonal, so that using definition (51), we can write

$$A_{t, x_0} = \mathrm{diag}\left(t(\lambda_1 + t), \cdots, t(\lambda_{2d} + t)\right), \tag{54}$$

where $\{\lambda_j \geq 0\}_{1 \leq j \leq 2d}$ are the eigenvalues of $-\pi J \rho_g J$ over $T_{x_0}M$. Using the multi-index notation $\alpha = (\alpha_1, \cdots, \alpha_{2d}) \in \mathbb{N}^{2d}$, we will use the following Taylor expansion of f up to order 4 as $|Z| \to 0$,

$$f(Z) = \sum_{0 \leq |\alpha| \leq 3} \frac{\partial^{|\alpha|} f}{\partial Z^\alpha}(x_0) \frac{Z^\alpha}{\alpha!} + O(|Z|^4) \|f\|_{C^4}. \tag{55}$$

On the other hand, using the change of variables $Z_j \mapsto Z_j / t^{1/2} (\lambda_j + t)^{1/2}$ for each $1 \leq j \leq 2d$ and the exponential decrease of the Gaussian function, we get a constant $\delta > 0$ for any $\alpha \in \mathbb{N}^{2d}$ such that the following estimate holds as $t \to 0$,

$$
\int_{B(x_0,\epsilon)} \varphi(|Z|) Z^\alpha \, e^{-\pi \left\langle A_t^{-1} Z, Z \right\rangle} \, dZ = \int_{\mathbb{R}^{2d}} Z^\alpha \, e^{-\pi \sum_{j=1}^{2d} \left(t^{-1} (\lambda_j + t)^{-1} Z_j^2 \right)} dZ
$$
$$
- \int_{\mathbb{R}^{2d}} (1 - \varphi(|Z|)) Z^\alpha \, e^{-\pi \sum_{j=1}^{2d} \left(t^{-1} (\lambda_j + t)^{-1} Z_j^2 \right)} \, dZ
\tag{56}
$$
$$
= \prod_{j=1}^{2d} t^{1/2} (\lambda_j + t)^{1/2} \left(t(\lambda_j + t) \right)^{\alpha_j/2} \int_{\mathbb{R}^{2d}} Z^\alpha \, e^{-\pi |Z|^2} dZ + O(e^{-\delta/t}).
$$

Note that we can then explicitly evaluate the integral in the last line of (56) using basic properties of the Gaussian function, and it vanishes as soon as there is an odd monomial inside Z^α. Then considering the Taylor expansion (55) inside the right hand side of equation (53) and using the estimate (56), we get as $t \to 0$,

$$
K_t^\rho f(x_0) = f(x_0) + \frac{\prod_{k=1}^{2d} t^{1/2} (\lambda_k + t)^{1/2}}{\alpha_t(x_0)} \left(\sum_{j=1}^{2d} \frac{t(\lambda_j + t)}{4\pi} \frac{\partial^2 f}{\partial Z_j^2}(x_0) + O(t^2) \|f\|_{C^4} \right).
\tag{57}
$$

On the other hand, it follows from the definition of α_t and the estimate (56) that as $t \to 0$, we have

$$
\alpha_t(x_0) = \prod_{j=1}^{2d} t^{1/2} (\lambda_j + t)^{1/2} (1 + O(e^{-\delta/t})).
\tag{58}
$$

Then we get from equation (57) that as $t \to 0$,

$$
K_t^\rho f(x_0) = f(x_0) + t \sum_{j=1}^{2d} \frac{\lambda_j + t}{4\pi} \frac{\partial^2 f}{\partial Z_j^2}(x_0) + O(t^2) \|f\|_{C^4}
$$
$$
= f(x_0) + t \sum_{j=1}^{2d} \frac{\lambda_j}{4\pi} \frac{\partial^2 f}{\partial Z_j^2}(x_0) + O(t^2) \|f\|_{C^4}.
\tag{59}
$$

Then writing T_h for the Berezin–Toeplitz quantization of (M, ω, J), the quantization T_h^ρ defined for all $f \in C^\infty(M, \mathbb{R})$ by

$$
T_h^\rho(f) := T_h \left(K_h^\rho f \right),
\tag{60}
$$

has unsharpness metric G given by formula (21): in fact, for any $u, v \in C^\infty(M, \mathbb{R})$, writing $\nabla_g u$, $\nabla_g v$ for their gradient with respect to g and in normal coordinates around $x_0 \in M$ as above, we get from the last line of (59) and following the computations of (18) that the unsharpness cocycle c_+^ρ associated with $T_\hbar^\rho$ satisfies

$$
\begin{aligned}
c_+^\rho(u, v)(x_0) &= -\frac{1}{2} \sum_{j=1}^{2d} \left(\partial_j u(x_0)\, \partial_j v(x_0) + \frac{\lambda_j}{\pi}\, \partial_j u(x_0)\, \partial_j v(x_0) \right) \\
&= -\frac{1}{2} \left(g_{x_0}(\nabla_g u, \nabla_g v) - g_{x_0}(J \rho_g J \nabla_g u, \nabla_g v) \right) \\
&= -\frac{1}{2} \left(g_{x_0}(\mathrm{sgrad}\, u, \mathrm{sgrad}\, v) + \rho_{x_0}(\mathrm{sgrad}\, u, \mathrm{sgrad}\, v) \right).
\end{aligned}
\tag{61}
$$

This shows that $G = g + \rho$, as required.

6   Case Study: $SU(2)$–Equivariant Quantizations

Definition 6.1.   Two Berezin–Toeplitz quantizations $T_\hbar$ and $T_\hbar'$ with families of Hilbert spaces $\{H_\hbar\}$ and $\{H_\hbar'\}$, $\hbar \in \Lambda$, respectively, are called *equivalent* if there exists a sequence of unitary operators $U_\hbar : H_\hbar \to H_\hbar'$ such that for all $f \in C^\infty(M)$,

$$
\| U_\hbar T_\hbar(f) U_\hbar^{-1} - T_\hbar'(f) \|_{op} = \mathcal{O}(\hbar^2).
\tag{62}
$$

Observe that if two quantizations are equivalent, their unsharpness metrics coincide. In this section, we prove a converse statement in the context of $SU(2)$-equivariant quantizations of the two-dimensional sphere (see Section 8.3 below for further discussion). We consider the standard Kähler metric on the two-sphere S^2 normalized so that the total area equals 2π. We denote by L the line bundle dual to the tautological one, and by H_k the $k + 1$-dimensional space of holomorphic sections of its k-th tensor power L^k. One can identify H_k with the space of homogeneous polynomials of two variables, so the group $SU(2)$ acts on H_k via an irreducible unitary representation. Furthermore, $SU(2)$ acts on the space of Hermitian operators $\mathcal{L}(H_k)$ by conjugation. On the other hand, the space $C^\infty(S^2)$ carries the natural action of $SU(2)$ by the change of variables. A quantization $Q_\hbar : C^\infty(S^2) \to \mathcal{L}(H_k)$, $\hbar \in \Lambda := \{1/k\}_{k \in \mathbb{N}}$, is called $SU(2)$-*equivariant* if it intertwines the corresponding (real) representations. For instance, the standard Berezin–Toeplitz quantization $T_\hbar$ sending $f \in C^\infty(S^2)$ to the multiplication by f followed by the orthogonal projection to the space of holomorphic sections is $SU(2)$-equivariant, and the same holds true for its images $T_\hbar^{(t)}$ under diffusion as defined in

Example 3.5. Note that the quantizations $T_\hbar^{(t)}$ are pairwise non-equivalent for different values of t as the corresponding unsharpness metrics are different.

Theorem 6.2.   Every $SU(2)$-equivariant quantization of S^2 is equivalent to $T_\hbar^{(t)}$ for some $t \geq 0$.

Proof.   STEP 1 (APPLYING SCHUR LEMMA): Given any $SU(2)$-equivariant quantization $Q_\hbar$, pass to its complexification (denoted by the same letter)

$$Q_\hbar : C^\infty(S^2, \mathbb{C}) \to \mathcal{L}(H_k) \otimes \mathbb{C} = H_k^* \otimes H_k .$$

On the one hand, $C^\infty(S^2, \mathbb{C})$ splits into the direct sum of irreducible summands V_j, $j = 0, 1, \ldots$ corresponding to the eigenspaces of the Laplace–Beltrami operator associated to the Kähler metric with the eigenvalue $2j(j+1)$, with each V_j isomorphic to H_{2j} as an $SU(2)$-representation. On the other hand,

$$H_k^* \otimes H_k = H_{2k} \oplus H_{2k-2} \oplus \cdots \oplus H_0 .$$

By the Schur Lemma, when $\hbar = 1/k$, we have that $Q_\hbar(V_j) \subset H_{2j}$ with respect to this decomposition, and furthermore there exists a constant $\alpha_{\hbar,j} \in \mathbb{C}$ such that, up to conjugation, we have

$$Q_\hbar = (1 + \alpha_{\hbar,j}) T_\hbar \quad \text{on} \quad V_j . \tag{63}$$

STEP 2 (LEGENDRE POLYNOMIALS): In what follows we introduce another parameter, $n \in \mathbb{N}$. We call a sequence $\{b_{\hbar,n}\}_{n \in \mathbb{N}}$ of the class $\mathcal{O}_N(\hbar^m)$ with $m, N \in \mathbb{N}$ if for some $c > 0$ we have $|b_{\hbar,n}| \leq c\,\hbar^m(n+1)^N$ for all n. In the sequel, the dependence on $\hbar$ of such sequences will be made implicit. Denote by $P_n(z)$ the n-th Legendre polynomial considered as a function on the unit sphere $S^2 = \{x^2 + y^2 + z^2 = 1\}$ lying in V_n. We write ∇ for the gradient with respect to the standard metric on S^2 normalized so that the total area equals 2π. The standard formulas for the Legendre polynomials (see e.g., formulas (43) and (44) in [39]) readily yield

$$P_1 P_n = q_n P_{n+1} + r_n P_{n-1}, \quad q_n = \frac{n+1}{2n+1}, \quad r_n = 1 - q_n , \tag{64}$$

and

$$(\nabla P_1, \nabla P_n) = s_n(-P_{n+1} + P_{n-1}), \quad s_n = \frac{2n(n+1)}{2n+1} . \tag{65}$$

We shall use that there exists $c > 0$ such that

$$\forall r \in \mathbb{N} \; \exists R \in \mathbb{N} \; : \|P_n\|_{C^r} \leq c(n+1)^R \; . \tag{66}$$

This (with $R = r$) follows immediately from the general result about the growth of C^r-norms of the Laplace–Beltrami eigenfunctions on Riemannian manifolds, see [8, Corollary 1.1]. Using the fact that $\max_{x \in S^2} P_n = 1$ by [27, Chapter 7, Theorem 17(i)], the norm correspondence property (P1), which holds uniformly in C^N-norm for some $N \in \mathbb{N}$, together with formula (66) implies

$$\|Q_\hbar(P_n)\|_{op} = 1 - \mathcal{O}_N(\hbar), \;\; \|T_\hbar(P_n)\|_{op} = 1 - \mathcal{O}_N(\hbar) \; .$$

Since $Q_\hbar(P_n) = (1 + \alpha_{\hbar,n}) T_\hbar(P_n)$ by (63), it follows that

$$\alpha_{h,n} = \mathcal{O}_N(\hbar) \; . \tag{67}$$

In the course of the proof, we shall increase the value of N according to our needs.

Step 3 (Main calculation): Since $Q_\hbar$ is $SU(2)$-equivariant, the corresponding unsharpness metric equals μ times the standard one, for some constant $\mu \geq 1$. Thus, the quasi-multiplicativity property (P3), which holds uniformly in C^N-norm for some $N \in \mathbb{N}$, together with formula (66) yields

$$Q_\hbar(P_1) Q_\hbar(P_n) = Q_\hbar \left(P_1 P_n - \frac{\mu}{2} \hbar(\nabla P_1, \nabla P_n) + \mathcal{O}_N(\hbar^2) \right) \; . \tag{68}$$

At the same time

$$T_\hbar(P_1) T_\hbar(P_n) = T_\hbar \left(P_1 P_n - \frac{1}{2} \hbar(\nabla P_1, \nabla P_n) + \mathcal{O}_N(\hbar^2) \right) \; , \tag{69}$$

mind that here μ is replaced by 1. By (63) we have

$$Q_\hbar(P_i) = (1 + \alpha_{\hbar,i}) T_\hbar(P_i) \; . \tag{70}$$

Identities (68) and (69) combined with (64), (65), and (70) enable us to express $T_\hbar(P_1) T_\hbar(P_n)$ as a linear combination of $T_\hbar(P_{n+1})$ and $T_\hbar(P_{n-1})$ in two different ways. The calculation is straightforward, and we obtain the result:

$$A_n T_\hbar(P_{n+1}) + B_n T_\hbar(P_{n-1}) = A_n' T_\hbar(P_{n+1}) + B_n' T_\hbar(P_{n-1}) + \mathcal{O}_N(\hbar^2) \; , \tag{71}$$

for some $A_n, A'_n \in \mathbb{C}$ and $B_n, B'_n \in \mathbb{C}$, $n \in \mathbb{N}$, where

$$B_n = (1 + \alpha_{\hbar,1})^{-1}(1 + \alpha_{\hbar,n})^{-1}(1 + \alpha_{\hbar,n-1})(r_n - \hbar\mu s_n/2) \,,$$

$$B'_n = r_n - \hbar s_n/2 \,.$$

Projecting equation (71) to the space H_{2n-2} (which contains $T_\hbar(V_{n-1})$) and using that the operator norm of $T_\hbar(P_{n-1})$ is bounded away from zero (see Step 2), we get that

$$B_n - B'_n = \mathcal{O}_N(\hbar^2) \,.$$

By using (67) and explicit expressions for q_n, r_n, s_n we get

$$\alpha_{\hbar,n-1} - \alpha_{\hbar,n} - \alpha_{\hbar,1} = (n+1)(\mu - 1)\hbar + \mathcal{O}_N(\hbar^2) \,. \tag{72}$$

Substituting $n = 1$ into (72) we get that

$$\alpha_{\hbar,1} = -(\mu - 1)\hbar + \mathcal{O}_N(\hbar^2) \,.$$

Now we get a recursive formula

$$\alpha_{\hbar,n} = \alpha_{\hbar,n-1} - n(\mu - 1)\hbar + \mathcal{O}_N(\hbar^2) \,.$$

Noticing that $(n+1)\,\mathcal{O}_N(\hbar^2) = \mathcal{O}_{N+1}(\hbar^2)$ and redefining $N \mapsto N+1$ we conclude that

$$\alpha_{\hbar,n} = -\frac{n(n+1)}{2}(\mu - 1)\hbar + \mathcal{O}_N(\hbar^2) \,. \tag{73}$$

STEP 4 (FINALE): Recall that $2n(n+1)$ is the eigenvalue of the Laplacian corresponding to the eigenspace V_n. Let $V = \oplus_{n=0}^\infty V_n$ be the space of all finite linear combinations of spherical harmonics. By norm correspondence (P1) and formula (73), for every $\phi_n \in V_n$ we have

$$Q_\hbar(\phi_n) = \left(1 - \frac{n(n+1)}{2}(\mu - 1)\hbar\right)T_\hbar(\phi_n) + \mathcal{O}_N(\hbar^2)\,\|\phi_n\|_{C^N}$$

$$= T_\hbar(e^{-t\hbar\Delta}\phi_n) + \mathcal{O}_N(\hbar^2)\,\|\phi_n\|_{C^N} = T_\hbar^{(t)}(\phi_n) + \mathcal{O}_N(\hbar^2)\,\|\phi_n\|_{C^N}$$

with $t = (\mu - 1)/4$ in Example 3.5.

Take now any $f \in C^\infty(S^2)$, and decompose it by spherical harmonics: $f = \sum_n \phi_n$. Since f is smooth, the C^N-norms $\|\phi_n\|_{C^N}$ decay faster than any power of n as $n \to +\infty$, so that

$$\|Q_h(f) - T_h^{(t)}(f)\|_{op} \leq c \sum_{n \in \mathbb{N}} n^N \|\phi_n\|_{C^N} \hbar^2 \leq c' \hbar^2 \,.$$

This shows that the quantizations Q_h and $T_h^{(t)}$ are equivalent.   ■

7   The Unsharpness Cocycle Is of Order $(1,1)$

In this section, we prove Theorem 3.2.

Proof.   For every $d \in \mathbb{N}$, we use the standard multi-index notation $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{N}^d$, where for any sequence of symbols $x_1, \cdots, x_d$, we write $x_\alpha := x_{\alpha_1} \cdots x_{\alpha_d}$, so that in particular $\alpha! := \alpha_1! \cdots \alpha_d! \in \mathbb{N}$, and write $|\alpha| := \alpha_1 + \cdots + \alpha_d$.

Note that by (11), to show that c_+ is a bi-differential operator of bi-degree $(1,1)$, we need to show that the differential operator a contains only terms of order 1 and 2. Note that $T_h(1) = \mathbb{1}$ implies $c_+(1,1) = 0$, so that a cannot contain terms of order 0. Let us show that a cannot be of order $k > 2$.

Assume by contradiction that a is of order $k > 2$. Let $x_0 \in M$ be the center of local coordinates $(Z_1, \cdots, Z_{2n}) \in U \subset \mathbb{R}^{2n}$ be such that for all $f \in C^\infty(M, \mathbb{R})$,

$$af(x_0) = \sum_{1 \leq |\alpha| \leq k} a_\alpha \frac{\partial^{|\alpha|} f}{\partial Z^\alpha}(x_0) \,, \tag{74}$$

where the sequence $\{a_\alpha \in \mathbb{R}\}_{1 \leq |\alpha| \leq k}$ is such that $a_\beta \neq 0$ for some $\beta = (\beta_1, \cdots, \beta_{2n}) \in \mathbb{N}^{2n}$ of length $|\beta| = k$. Fix $1 \leq j \leq 2n$ such that $\beta_j \neq 0$, and writing $\hat{\beta} = (\beta_1, \cdots, \beta_j - 1, \cdots, \beta_{2n}) \in \mathbb{N}^{2n-1}$, take $f \in C^\infty(M, \mathbb{R})$ satisfying

$$f(Z) = \frac{c}{\hat{\beta}!} Z^{\hat{\beta}} + Z_j \,, \tag{75}$$

for $Z \in U \subset \mathbb{R}^{2n}$ in the coordinates around $x_0 \in M$ considered above and for some $c \in \mathbb{R}$ to be fixed later. Then this function f and all its derivatives vanish at $x_0 \in M$, except for

$$\frac{\partial^{|\hat{\beta}|} f}{\partial Z^{\hat{\beta}}}(x_0) = c \qquad \text{and} \qquad \frac{\partial f}{\partial Z_j}(x_0) = 1 \,. \tag{76}$$

Then by equations (75) and (74), considering the multi-index $\gamma \in \mathbb{N}^{2n}$ of length $|\gamma| = 2$ such that $\gamma_j = 2$, one gets that for any $f \in C^\infty(M, \mathbb{R})$ satisfying (76),

$$
\begin{aligned}
c_+(f,f)(x_0) &= 2a_\beta \, \frac{\partial^{|\hat{\beta}|} f}{\partial Z^{\hat{\beta}}}(x_0) \, \frac{\partial f}{\partial Z_j}(x_0) + 2a_\gamma \, \frac{\partial f}{\partial Z_j}(x_0) \, \frac{\partial f}{\partial Z_j}(x_0) \\
&= 2a_\beta c + 2a_\gamma \,.
\end{aligned}
\tag{77}
$$

Thus, if $f \in C^\infty(M, \mathbb{R})$ satisfies (77) for $c \in \mathbb{R}$ such that $\mathrm{sign}(a_\beta)c > -a_\gamma/|a_\beta|$, we get that $c_+(f,f)(x_0) > 0$. This contradicts the fact that $c_+(f,f) \le 0$ for all $f \in C^\infty(M, \mathbb{R})$, which holds for every Berezin–Toeplitz quantization by the semi-positivity property (27).   ∎

8   Discussion and Questions

8.1   Historical remarks on unsharpness

The unsharpness cocycle appeared in earlier literature, which, to the best of our knowledge, focused on its elimination, of course, by the price of losing the positivity of a quantization. Let us elaborate on this point. Assume (M, ω) is a quantizable symplectic manifold equipped with a compatible almost-Kähler structure. Consider the induced Berezin–Toeplitz quantization of Theorem 3.4. Using the notations of Example 3.5, define for any $f \in C^\infty(M)$ and $\hbar \in \Lambda = \{1/k\}_{k \in \mathbb{N}}$,

$$
Q_\hbar(f) := T_\hbar\left(f + \frac{\hbar}{4}\Delta f\right) \,.
\tag{78}
$$

This gives rise to a collection of maps $Q_\hbar : C^\infty(M) \to \mathcal{L}(H_\hbar)$ parametrized by $\hbar \in \Lambda$ and satisfying the axioms (P1)–(P4) of Definition 2.6, but which does not preserve positivity, so that they do not come from a POVM construction via formula (4). Then following the computation (18) in Example 3.5, we see that the associated unsharpness cocycle c_+^Q, defined from the quasi-multiplicativity property (P3) as in the beginning of the section, satisfies

$$
c_+^Q(f,g) = 0 \,,
\tag{79}
$$

for all $f, g \in C^\infty(M)$. As noted for instance by Charles in [13, § 1.4] (who uses the holomorphic Laplacian, which is half the Laplace–Beltrami operator), the quantization (78) is, up to twisting with a line bundle, the *metaplectic Kostant–Souriau quantization*, which possesses remarkable sub-principal properties, a fact that is explained conceptually by the vanishing unsharpness property (79).

In the flat case $M = \mathbb{C}$ with the standard symplectic form, Gerstenhaber considers in [18] deformation quantizations parametrized by $\lambda \geq 0$, which, up to the 2nd order in $\hbar$, correspond to the quantization maps parametrized by $\hbar > 0$ defined for any smooth function $f : \mathbb{C} \to \mathbb{R}$ of polynomial growth by

$$Q_\hbar^{(\lambda)}(f) := T_\hbar \left(f + \frac{1-\lambda}{2} \hbar \Delta f \right) . \tag{80}$$

Here $T_\hbar$ is the standard Toeplitz quantization of $\mathbb{C}$, sending f to the multiplication by f followed by the orthogonal L_2-projection on the space of holomorphic functions that are square integrable with respect to a Gaussian measure. Gerstenhaber formulates a *least uncertainty principle* for deformation quantization, which implies in particular that unsharpness vanishes on the classical harmonic oscillator. He then shows that the quantization (80) satisfies this least uncertainty principle if $\lambda = 1/2$, which corresponds to the flat version of the quantization (78).

Note that in the flat case $M = \mathbb{C}$, the classical harmonic oscillator is a sum of squares of the coordinate functions. On the other hand, the quasi-multiplicativity property (P3) implies that for all $f \in C^\infty(M)$ as $\hbar \to 0$,

$$T_\hbar(f)^2 - T_\hbar(f^2) = \hbar T_\hbar(c_+(f,f)) + \mathcal{O}(\hbar^2) .$$

We then see that unsharpness measures in particular the deviation of the quantum harmonic oscillator, defined as a sum of squares of the quantum coordinate operators, from the quantization of the classical harmonic oscillator. This explains in particular the standard justification of the metaplectic correction, as giving the "correct" quantum harmonic oscillator on flat space.

8.2   Least unsharpness surfaces and pseudo-holomorphic curves

Let G be the unsharpness metric associated to a Berezin–Toeplitz quantization of a closed symplectic manifold (M, ω) (see Section 4). *A least unsharpness surface $\Sigma \subset M$ is a two-dimensional oriented submanifold with $\mathrm{Area}_G(\Sigma) = \int_\Sigma \omega$.* Repeating the proof of Theorem 4.1, we see that for such surfaces, the restriction of the Riemannian area form coincides with the restriction of the symplectic form. If G has the minimal possible total unsharpness and hence by Theorem 4.1 (II) comes from some compatible almost-complex structure J on M, the least unsharpness surfaces in M are J-holomorphic curves (cf. [34]). For instance, for the complex projective plane $M = \mathbb{C}P^2$, Gromov's theory

of pseudo-holomorphic curves predicts that for every compatible J, through every two distinct points $A, B \in M$ passes unique such curve Σ in the homology class of $[\mathbb{C}P^1]$.

It is enticing to interpret Σ as a worldsheet of the topological string theory describing a path joining constant loops A and B. Note that the metric G on our "spacetime" M is canonically associated to a Berezin–Toeplitz quantization of M, and the "total unsharpness" $\mathrm{Area}_G(\Sigma)$ of a worldsheet Σ is nothing else but the Nambu–Goto action up to a multiplicative constant. If the total unsharpness of (M, G) is minimal possible, that is, coincides with the symplectic volume of M, the least unsharpness surfaces are J-holomorphic curves for a compatible almost complex structure J defining G, and hence represent "worldsheet instantons." Does there exist an interpretation of this picture in physical terms?

8.3   On classification of Berezin–Toeplitz quantizations

We conclude the paper with a discussion on classification of Berezin–Toeplitz quantizations up to equivalence in the sense of Definition 6.1. In Section 6, we classified $SU(2)$-equivariant quantizations of the two-dimensional sphere. It would be interesting to extend this to equivariant quantizations for more general co-adjoint orbits equipped with the canonical symplectic structure. In the general (not necessarily equivariant) case, the problem is widely open.

In fact, establishing (non)-equivalence of quantizations is a non-trivial problem even for the Berezin–Toeplitz quantizations described at the end of Section 2, where the holomorphic line bundles defining the quantization of (M, ω) could be non-isomorphic. For instance, their Chern classes could differ by torsion even though the associated spaces of holomorphic sections have same dimension. Are the corresponding quantizations equivalent?

Another interesting example is as follows. According to Remark 4.2, there exist metrics G on M admitting different decompositions of the form (21). Each such decomposition determines a Berezin–Toeplitz quantization given by almost-Kähler quantization followed by diffusion, as explained in the proof of Theorem 4.1 (III). Are the quantizations corresponding to different decompositions of the same metric equivalent?

Let us address the question about invariants of quantizations with respect to equivalence. In addition to the unsharpness metric, there is another invariant coming from the trace correspondence, see item (P4) in Definition 2.6. Recall that the latter states that

$$\mathrm{tr}(T_\hbar(f)) = (2\pi\hbar)^{-d} \int_M f\, R_\hbar \, \mathrm{d}\mu \,,$$

where $\dim M = 2d$, the function $R_\hbar \in C^\infty(M)$ satisfies $R_\hbar = 1 + \mathcal{O}(\hbar)$, and $d\mu = \frac{\omega^d}{d!}$ is the symplectic volume on M. Roughly speaking, since the trace is invariant under conjugation, the convergence rate of the sequence of differential forms $R_\hbar d\mu$ to the symplectic volume $d\mu$ as $\hbar \to 0$ does not change up to $\mathcal{O}(\hbar^2)$ under equivalence. For the sake of simplicity, let us, until the end of the paper, enhance axiom (P4) by assuming that there exists a function $r \in C^\infty(M)$ such that

$$R_\hbar = 1 + \hbar\, r + \mathcal{O}(\hbar^2)\ .$$

We shall refer to the form $rd\mu$ as *the Rawnsley form*. Thus, *equivalent quantizations possess the same Rawnsley form*. Put

$$\langle r \rangle := \mathrm{Vol}(M)^{-1} \int_M r\mathrm{d}\mu\ .$$

Remark 8.1.   Substituting $f = 1$ into (P4), we get that $\langle r \rangle$ appears in the dimension formula

$$\dim H_\hbar = \mathrm{Vol}(M)(2\pi\hbar)^{-d}\left(1 + \hbar\,\langle r\rangle + \mathcal{O}(\hbar^2)\right)\ .$$

Let us mention that for Kähler quantizations, an alternative asymptotic expression for the dimension of $H_\hbar$ is given by the Hirzebruch–Riemann–Roch theorem. Comparing coefficients at $\hbar$, one gets a simple topological interpretation of $\langle r \rangle$:

$$\langle r \rangle = 2\pi\,\frac{\left\langle [\omega]^{d-1} \cup c_1(TM), [M]\right\rangle}{2(d-1)!\,\mathrm{Vol}(M)}\ ,$$

where $c_1(TM)$ stands for the 1st Chern class of M.

Example 8.2.   Let v be a vector field on the manifold M generating a flow ϕ_t. Given a Berezin–Toeplitz quantization $T_\hbar$ on M, define a new quantization by setting $T_\hbar^{(v)}(f) := T_\hbar(f \circ \phi_{-\hbar})$. A direct calculation based on the expansion $T_\hbar^{(v)}(f) = T_\hbar\left(f - \hbar\,df(v)\right) + \mathcal{O}(\hbar^2)$ shows that this is a Berezin–Toeplitz quantization whose unsharpness metric coincides with the one of $T_\hbar$, and whose Rawnsley form is given by $(r + \mathrm{div}(v))d\mu$, where $\mathrm{div}(v)$ stands for the divergence of v with respect to the symplectic volume. In particular, it follows that by choosing an appropriate vector field v, one can always achieve the Rawnsley form being equal to $\langle r\rangle d\mu$.

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Question 8.3. Consider a pair of quantizations with the Hilbert spaces of the same dimension. Suppose that their unsharpness metrics and the Rawnsley forms coincide. Are these quantizations equivalent?

The answer in the general (not necessarily equivariant) case is at the moment unclear.

We refer to [24] for a discussion on equivalence of quantizations up to conjugation and *an error of the order* $\mathcal{O}(\hbar)$. It turns out that all (not necessarily positive) quantizations of the sphere and the torus in dimension 2 are equivalent in this weaker sense.

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