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Geometric quantization of Hamiltonian flows and the Gutzwiller trace formula

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Abstract
We use the theory of Berezin–Toeplitz operators of Ma and Marinescu to study the quantum Hamiltonian dynamics associated with classical Hamiltonian flows over closed prequantized symplectic manifolds in the context of geometric quantization of Kostant and Souriau. We express the associated evolution operators via parallel transport in the quantum spaces over the induced path of almost complex structures, and we establish various semi-classical estimates. In particular, we establish a Gutzwiller trace formula for the Kostant–Souriau operator and compute explicitly the leading term. We then describe a potential application to contact topology.

Keywords Geometric quantization · Berezin-Toeplitz operators · Hamiltonian flows · Gutzwiller trace formula · Contact topology

Mathematics Subject Classification 53D50 · 37C27 · 32A25 · 57R17 · 58J20

1 Introduction

Given a classical phase space X, the goal of quantization is to produce a Hilbert space $\mathscr{H}$ of quantum states, such that the classical dynamics over X, described as flows of diffeomorphisms, are mapped in a natural way to the associated quantum dynamics of $\mathscr{H}$, described as 1-parameter families of unitary operators. In the context of *geometric quantization*, introduced independently by Kostant [23] and Souriau [34], the classical phase space is represented by a $2n$-dimensional symplectic manifold (X, ω) without boundary, endowed with a Hermitian line bundle (L, h^L) together with a Hermitian connection ∇^L with curvature R^L satisfying the following *prequantization condition*,

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$$\omega = \frac{\sqrt{-1}}{2\pi} R^L. \tag{1.1}$$

The construction of an associated Hilbert space of quantum states depends on the extra data of a *polarization*, and the best choice of such a polarization usually depends on the physical situation at hand. In particular, the classical dynamics of a symplectic manifold (X, ω) is entirely determined by a *time-dependent Hamiltonian* $F \in \mathscr{C}^\infty(\mathbb{R} \times X, \mathbb{R})$, and the corresponding quantum dynamics are sometimes much easier to infer for one specific choice of polarization. For the general theory as well as numerous examples, we refer to the classical book of Woodhouse [35, Chap. 5, Chap. 9].

In this paper, we focus our attention on *compact* symplectic manifolds, and consider the polarization induced by an almost complex structure $J \in \mathrm{End}(TX)$ compatible with ω, which always exists. The associated Hilbert space $\mathscr{H}_p$ of quantum states will depend of an integer $p \in \mathbb{N}$, representing a *quantum number*, and the goal of this paper is to study the behavior of the quantum dynamics associated with the classical *Hamiltonian flow* of $F \in \mathscr{C}^\infty(\mathbb{R} \times X, \mathbb{R})$ as p tends to infinity. This limit is called the *semi-classical limit*, when the scale gets so large that we recover the laws of classical mechanics as an approximation of the laws of quantum mechanics. We show, in particular, that the quantum dynamics approximate the corresponding classical dynamics as $p \to +\infty$.

The construction we present in this paper holds for any compact prequantized symplectic manifold and coincides with the *holomorphic quantization* of Kostant and Souriau in the particular case when the almost complex structure $J \in \mathrm{End}(TX)$ is *integrable*, making (X, J, ω) into a *Kähler manifold*. Then (L, h^L) admits a natural holomorphic structure for which ∇^L is its *Chern connection*, and the space $\mathscr{H}_p$ of quantum states coincides with the associated space of *holomorphic sections* of the pth tensor power $L^p := L^{\otimes p}$, for all $p \in \mathbb{N}$ big enough. The natural L^2*-Hermitian product* (2.3) on the space $\mathscr{C}^\infty(X, L^p)$ of smooth sections of L^p then gives $\mathscr{H}_p$ the structure of a Hilbert space. In the very restrictive case when the Hamiltonian flow acts by biholomorphisms on (X, L), the quantum dynamics is simply given by the induced action on the space of holomorphic sections $\mathscr{H}_p$ for all $p \in \mathbb{N}$. In contrast, our results apply to the holomorphic quantization of general Hamiltonian flows.

In Sect. 2, we consider a general almost complex structure $J \in \mathrm{End}(TX)$ compatible with ω, and for all $p \in \mathbb{N}$, we define in (2.17) the space $\mathscr{H}_p$ of quantum states as the direct sum of eigenspaces associated with the small eigenvalues of the following *renormalized Bochner Laplacian*

$$\Delta_p := \Delta^{L^p} - 2\pi n p\,, \tag{1.2}$$

acting on the smooth sections $\mathscr{C}^\infty(X, L^p)$ of L^p, where Δ^{L^p} is the usual Bochner Laplacian of (L^p, h^{L^p}) associated with the Riemannian metric $g^{TX} := \omega(\cdot, J\cdot)$. This follows an idea of Guillemin and Uribe [17], and we consider here a more general construction due to Ma and Marinescu [27], where L^p is replaced by $E \otimes L^p$ for all $p \in \mathbb{N}$, for any Hermitian vector bundle with connection (E, h^E, ∇^E). Note that both constructions admit an extension to the case of a general J-invariant metric

g^{TX}, and the general construction of [27] also deals with the case when one adds a potential term to Δ_p. In Theorems 2.5 and 2.6, we describe the results of [21] on the dependence of the quantization to the choice of an almost complex structure $J \in \mathrm{End}(TX)$ at the semi-classical limit $p \to +\infty$. Specifically, we introduce the parallel transport operator $\mathcal{T}_{p,t}$ between different quantum spaces $\mathscr{H}_p$ along a path $\{J_t \in \mathrm{End}(TX)\}_{t\in\mathbb{R}}$ of almost complex structures with respect to the L^2-*connection* (2.17), and we describe in Theorem 2.5 how $\mathcal{T}_{p,t}$ behaves like a *Toeplitz operator* as $p \to +\infty$, giving an explicit formula for the highest order coefficient. This is based on the theory of Berezin–Toeplitz operators for symplectic manifolds developed by Ma and Marinescu in [28], and extended to this context in [22].

In Sect. 3, we show how one can use this parallel transport to construct the quantum Hamiltonian dynamics associated with any $F \in \mathscr{C}^\infty(\mathbb{R} \times X, \mathbb{R})$. In fact, the corresponding Hamiltonian flow $\varphi_t : X \to X$ defined by (3.2) for all $t \in \mathbb{R}$ does not preserves any almost complex structure in general, and thus does not induce an action on $\mathscr{H}_p$ for any $p \in \mathbb{N}$. Instead, fix an almost complex structure $J_0 \in \mathrm{End}(TX)$ compatible with ω, and consider the almost complex structure $J_t := d\varphi_t\, J_0\, d\varphi_t^{-1}$, as well as the spaces $\mathscr{H}_{p,t}$ of quantum states associated with J_t, for all $t \in \mathbb{R}$ and $p \in \mathbb{N}$. This flow, together with its lift to L defined by (3.4), induces a unitary isomorphism $\varphi_{t,p}^* : \mathscr{H}_{p,t} \to \mathscr{H}_{p,0}$ by pullback on $\mathscr{C}^\infty(X, L^p)$, and considering the parallel transport $\mathcal{T}_{p,t} : \mathscr{H}_{p,0} \to \mathscr{H}_{p,t}$ along the path $s \mapsto J_s$ for $s \in [0, t]$, the associated *quantum evolution operator* at time $t \in \mathbb{R}$ is given by the unitary operator $\varphi_{t,p}^*\mathcal{T}_{p,t} \in \mathrm{End}(\mathscr{H}_{p,0})$.

In the rest of the Introduction, we assume that the Hamiltonian $F =: f \in \mathscr{C}^\infty(X, \mathbb{R})$ does not depend on time, and write $\xi_f \in \mathscr{C}^\infty(X, TX)$ for the *Hamiltonian vector field* of f, as defined in (3.1). In that case, we show in Lemma 3.1 that

$$\varphi_{t,p}^*\mathcal{T}_{p,t} = \exp(-2\pi\sqrt{-1}tpQ_p(f)), \tag{1.3}$$

for all $t \in \mathbb{R}$ and $p \in \mathbb{N}$, where $Q_p(f)$ is the *Kostant–Souriau operator* associated with f, defined as an operator acting on $\mathscr{C}^\infty(X, L^p)$ by the formula

$$Q_p(f) := P_p\left(f - \frac{\sqrt{-1}}{2\pi p}\nabla^{L^p}_{\xi_f}\right)P_p, \tag{1.4}$$

where f is the operator of pointwise multiplication by f and $P_p : \mathscr{C}^\infty(X, L^p) \to \mathscr{H}_{p,0}$ is the L^2-orthogonal projection. This is the holomorphic version of the *Blattner–Kostant–Sternberg kernel*, as described for example in [35, § 9.7]. Under this form, it was first noticed by Foth and Uribe in [16, § 3.2], who interpreted the trace of $Q_p(f)$ as a moment map for the group of Hamiltonian diffeomorphisms acting on the space of almost complex structures compatible with ω. Using the results described in Sect. 2, we establish the following semi-classical estimate on its Schwartz kernel (2.26), where $\tau_{t,p}$ denotes the parallel transport in L^p along $s \mapsto \varphi_s(x)$ for $s \in [0, t]$. Here and in all the paper, we use the notation $O(p^{-k})$ in the sense of the corresponding Hermitian norm as $p \to +\infty$, and $O(p^{-\infty})$ means $O(p^{-k})$ for all $k \in \mathbb{N}$.

Proposition 1.1 *For any $t \in \mathbb{R}$ and $x, y \in X$ such that $\varphi_t(y) \neq x$, we have the following estimate as $p \to +\infty$,*

$$\exp\left(2\pi\sqrt{-1}tpQ_p(f)\right)(x,y) = O(p^{-\infty}). \tag{1.5}$$

Furthermore, there exists $a_r(t,x) \in \mathbb{C}$ $(r \in \mathbb{N})$ *smooth in* $x \in X$ *and* $t \in \mathbb{R}$*, such that for any* $k \in \mathbb{N}^*$*, as* $p \to +\infty$ *we have*

$$\begin{aligned}\exp&\left(2\pi\sqrt{-1}tpQ_p(f)\right)(\varphi_t(x),x)\\ &= p^n e^{2\pi\sqrt{-1}tpf(x)}\left(\sum_{r=0}^{k-1} p^{-r}a_r(t,x) + O(p^{-k})\right)\tau_{t,p},\end{aligned} \tag{1.6}$$

with $a_0(t,x) \neq 0$ *for all* $t \in \mathbb{R}$ *and* $x \in X$.

This follows from the more precise Proposition 3.3, which gives, in particular, a formula for the first coefficient $a_0(t,x)$. As explained there, this result shows that the quantum dynamics approximates the classical dynamics at the semi-classical limit $p \to +\infty$ in a precise sense. In Theorem 3.4, we also use the results described in Sect. 2 to give an associated semi-classical trace formula. Note that the estimate (1.5) is not uniform in $(t \in \mathbb{R},\ x,\ y \in X)$, and the estimate (1.6) shows that there is in fact a jump when $\varphi_t(y)$ tends to x.

In Sect. 4, we consider a time-independent Hamiltonian $f \in \mathscr{C}^\infty(X,\mathbb{R})$, and we use the setting of Sect. 2 to study the operators $\hat{g}(pQ_p(f-c)) \in \mathrm{End}(\mathscr{H}_p)$ defined for all $p \in \mathbb{N}$ by the formula

$$\hat{g}(pQ_p(f-c)) := \int_{\mathbb{R}} g(t)e^{2\pi\sqrt{-1}tpc}\left(\varphi_{t,p}^*\mathcal{T}_{p,t}\right)\mathrm{d}t, \tag{1.7}$$

where $g : \mathbb{R} \to \mathbb{R}$ is smooth with compact support, where $c \in \mathbb{R}$ is a regular value of f and where $\varphi_{t,p}^*\mathcal{T}_{p,t} \in \mathrm{End}(\mathscr{H}_{p,0})$ is the quantum evolution operator associated with f. Via the interpretation (1.3) in terms of quantum evolution operators, the *Gutzwiller trace formula* predicts a semi-classical estimate for the trace $\mathrm{Tr}[\hat{g}(pQ_p(f-c))]$ as $p \to +\infty$, in terms of the periodic orbits of the Hamiltonian flow of f included in the level set $f^{-1}(c)$. This formula was first worked out by Gutzwiller in [19, (36)] for usual Schrödinger operators over $\mathbb{R}^n$ as the *Planck constant* $\hbar$ tends to 0, using path integral methods. As explained in his book [20], this formula plays a fundamental role in the theory of *quantum chaos*, which studies the quantization of chaotic classical systems.

Specifically, let $\mathrm{Supp}\, g \subset \mathbb{R}$ be the support of g, and suppose that for all $t \in \mathrm{Supp}\, g$, the fixed point set $X^{\varphi_t} \subset X$ of $\varphi_t : X \to X$ is non-degenerate over a neighborhood of $f^{-1}(c)$ in the sense of Definition 2.7 and intersects $f^{-1}(c)$ transversally such that $X^{\varphi_t} \cap f^{-1}(c)$ is non-empty only for a finite subset $T \subset \mathrm{Supp}\, g$. Let $\{Y_j\}_{1\leqslant j\leqslant m}$ be the set of connected components of

$$\coprod_{t\in T} X^{\varphi_t} \cap f^{-1}(c), \tag{1.8}$$

and for any $1 \leqslant j \leqslant m$, write $t_j \in T$ for the time such that $Y_j \subset X^{\varphi_{t_j}} \cap f^{-1}(c)$. In particular, these hypotheses are automatically satisfied if $\mathrm{Supp}\, g \subset \mathbb{R}$ is a small enough neighborhood of 0, so that $T = \{0\}$ and $X^{\varphi_0} = X$.

Let $\lambda_j \in \mathbb{R}$ be the *action* of f over Y_j as in Definition 3.5, and let $\mathrm{Vol}_\omega(f^{-1}(c)) > 0$ be the volume of $f^{-1}(c)$ with respect to the natural *Liouville measure* (4.20) induced by ω and f on $f^{-1}(c)$.

Theorem 1.2 *Under the above assumptions, there exists $b_{j,r} \in \mathbb{C}$ ($r \in \mathbb{N}$), depending only on geometric data around Y_j for all $1 \leqslant j \leqslant m$, such that for any $k \in \mathbb{N}^*$ and as $p \to +\infty$, we have*

$$\mathrm{Tr}\left[\hat{g}(pQ_p(f-c))\right] = \sum_{j=1}^{m} p^{(\dim Y_j - 1)/2} g(t_j) e^{-2\pi\sqrt{-1}p\lambda_j} \left(\sum_{r=0}^{k-1} p^{-r} b_{j,r} + O(p^{-k})\right). \tag{1.9}$$

Furthermore, there is an explicit geometric formula for the first coefficients $b_{j,0}$ for all $1 \leqslant j \leqslant m$, and as $p \to +\infty$ we have

$$\mathrm{Tr}\left[\hat{g}(pQ_p(f-c))\right] = p^{n-1} g(0)\,\mathrm{Vol}_\omega(f^{-1}(c)) + O(p^{n-2}). \tag{1.10}$$

Note that formula (1.9) does not follow from (1.3) and Proposition 1.1 by integrating over $t \in \mathbb{R}$, due to the jump in the estimates (3.25) and (1.6) when $\varphi_t(y)$ tends to x. This is illustrated by Proposition 4.1, where it is shown how $\hat{g}(pQ_p(f-c))$ localizes around $f^{-1}(c)$ after integrating in $t \in \mathbb{R}$ via stationary phase estimates, with a precise control on the constant around $f^{-1}(c)$.

The general formula for the first coefficients $b_{j,0}$ of the expansion (1.9) is given in Theorem 4.3, and reduces to the so-called *Weyl term* (1.10) of the trace formula in the case $0 \in \mathrm{Supp}\, g$. However, the relevance of this formula for quantum chaos mainly lies in the terms associated with *isolated periodic orbits*, and one would like to consider general situations where this formula exhibits natural geometric quantities associated with such orbits.

To describe such situations, let us first consider the general case of a Hermitian vector bundle with connection (E, h^E, ∇^E) over $\mathbb{R} \times X$. Writing E_t for its restriction to X over $t \in \mathbb{R}$, we take more generally the quantum spaces $\mathscr{H}_{p,t}$ to be the almost holomorphic sections of $E_t \otimes L^p$ with respect to J_t, for all $p \in \mathbb{N}$, together with the L^2-Hermitian product induced by h^{E_t} and h^{L^p}. Following Definition 2.4, we can again consider the parallel transport $\mathcal{T}_{p,t} : \mathscr{H}_{p,0} \to \mathscr{H}_{p,t}$ with respect to the associated L^2-connection, and if $\varphi_t : X \to X$ is a Hamiltonian flow lifting to a bundle map $\varphi_t^E : E_0 \to E_t$ over X for all $t \in \mathbb{R}$, we again have a unitary evolution operator $\varphi_{t,p}^* \mathcal{T}_{p,t} \in \mathrm{End}(\mathscr{H}_{p,0})$. Then the right-hand side of formula (1.7) still makes sense, and Theorem 4.3 gives the general version of Theorem 1.2 in this context.

Consider now the canonical line bundle $(K_X, h^{K_X}, \nabla^{K_X})$ over $\mathbb{R} \times X$ associated with $\{J_t \in \mathrm{End}(TX)\}_{t \in \mathbb{R}}$ defined in Sect. 2 by the formula (2.14), and assume that (X, ω) admits a *metaplectic structure*, so that this canonical line bundle admits a square root $K_X^{1/2}$ over $\mathbb{R} \times X$ with induced metric and connection, called the *metaplectic correction*. Then φ_t admits a natural lift for all $t \in \mathbb{R}$, and we call the associated unitary operator $\varphi_{t,p}^* \mathcal{T}_{p,t}$ as above the *metaplectic quantum evolution operator*.

Theorem 1.3 *Assume that (X, ω) admits a metaplectic structure, and consider the assumptions of Theorem 1.2. Let $1 \leqslant j \leqslant m$ be such that $\dim Y_j = 1$ and such that $[J\xi_f, \xi_f] = 0$ over Y_j. Then the first coefficients of the analogous expansion (1.9) as $p \to +\infty$ for the metaplectic quantum evolution operator satisfy the following formula,*

$$b_{j,0} = (-1)^{\frac{n-1}{2}} \frac{t(Y_j)}{|\det_{N_x}(\mathrm{Id}_N - d\varphi_{t_j}|_N)|^{1/2}}, \tag{1.11}$$

for a natural choice of square root and for any $x \in Y_j$, where N is the normal bundle of Y_j inside $Tf^{-1}(c)$ and where $t(Y_j) > 0$ is the primitive period of Y_j as a periodic orbit of the flow $t \mapsto \varphi_t$ inside $f^{-1}(c)$.

For (X, J_t, ω) Kähler for all $t \in \mathbb{R}$ and endowed with a metaplectic structure, one can show using [21, (5.2)] that the generator of the metaplectic quantum evolution operator considered above coincides with the metaplectic Kostant–Souriau operator considered by Charles in [12, Th. 1.5].

Theorem 1.3 follows from Theorem 4.4, which gives also a formula for general (E, h^E, ∇^E) as an integral along the associated periodic orbit. In the case of usual Schrödinger operators over a compact Riemannian manifold the Gutzwiller trace formula has been established by Guillemin and Uribe [18, Th. 2.8], Paul and Uribe [32, Th. 5.3] and Meinrenken [30, Th. 3], while in the case of Toeplitz operators over the smooth boundary of a compact strictly pseudoconvex domain, it has been established by Boutet de Monvel and Guillemin [9, Th, 9, Th. 10]. In the case of Berezin–Toeplitz operators over a compact prequantized Kähler manifold with metaplectic structure, instead of Kostant–Souriau operators over a general compact prequantized symplectic manifold as in Theorem 1.2, it has been established by Borthwick, Paul and Uribe [8, Th. 4.2] using the theory of Boutet de Monvel and Guillemin [9].

In all the works cited above, the corresponding formulas for the first coefficients involve an undetermined subprincipal symbol term with no obvious geometric interpretation. In contrast, the general formula for the first coefficient given in (4.22) is completely explicit in terms of local geometric data. Furthermore, the formula for isolated periodic orbits given in Theorem 1.3 is the same as the corresponding formulas in all the cases mentioned above, but without the undetermined subprincipal symbol term. This makes it much simpler to use in practical applications.

In fact, let the Hamiltonian $f \in \mathscr{C}^\infty(X, \mathbb{R})$ and the almost complex structure $J \in \mathrm{End}(TX)$ be such that

$$d\iota_{J\xi_f}\omega = \omega \ \text{ over } \ f^{-1}(I), \tag{1.12}$$

for some interval $I \subset \mathbb{R}$ of regular values of f containing $c \in \mathbb{R}$. As explained at the end of Sect. 4, this induces a *contact form* $\alpha \in \Omega^1(\Sigma, \mathbb{R})$ on $\Sigma := f^{-1}(c)$, and ξ_f generates the *Reeb flow* of (Σ, α). Then Proposition 4.5 shows how Theorem 1.2 can be used to detect the periods of the non-degenerate isolated periodic orbits of this flow, and Theorem 1.3 allows in principle to compute the associated action. This is of particular interest in contact topology, where the study of non-degenerate isolated periodic orbits of the Reeb flow is a major topic, usually tackled via methods of Floer homology. Note that to recover the action from the expansion (1.9) in practice, one needs a completely explicit formula for the first coefficient, and formula (1.11) is the best that one can hope for.

In Theorem 4.2, we also establish semi-classical estimates on the Schwartz kernel of the operator $\hat{g}(pQ_p(f-c))$ as $p \to +\infty$, analogous to the corresponding estimates in [8, Th. 1.1] for Berezin–Toeplitz operators over compact prequantized Kähler manifolds admitting a metaplectic structure. Once again, our formula for the first order term is completely explicit in terms of geometric data, without the undetermined subprincipal symbol term appearing in the corresponding formula in [8, Th. 2.7].

In the case of compact prequantized Kähler manifolds and when the Hamiltonian flow $\varphi_t : X \to X$ acts by biholomorphisms, so that one can define the quantization of φ_t simply by its induced action on holomorphic sections, the pointwise semi-classical estimates of Theorem 4.2 for $E = \mathbb{C}$ have been obtained by Paoletti in [31, Th. 1.2]. In this case, the general version of Theorem 1.2 is a direct consequence of the following *Kirillov formula*,

$$\operatorname{Tr}\left[\varphi^*_{t_j+t,p}\right] = \int_{X^{\varphi_{t_j}}} \operatorname{Td}_{\varphi_{t_j}^{-1},-t\xi_f}(TX)\,\operatorname{ch}_{\varphi_{t_j}^{-1},-t\xi_f}(L^p), \tag{1.13}$$

as described in [6, (2.38)] for any fixed $1 \leqslant j \leqslant m$ and $|t| > 0$ small enough, using the stationary phase lemma as $p \to +\infty$. Paoletti recovers this special case in [31, Th. 1.3] without using formula (1.13).

The theory of Berezin–Toeplitz operators over compact prequantized Kähler manifolds with $E = \mathbb{C}$ was first developed by Bordemann, Meinreken and Schlichenmaier [7] and Schlichenmaier [33]. Their approach is based on the work of Boutet de Monvel and Sjöstrand on the Szegö kernel in [10], and the theory of Toeplitz structures developed by Boutet de Monvel and Guillemin in [9]. The present paper is based instead on the approach of Ma and Marinescu using Bergman kernels, and we refer to the book [26] for a detailed presentation of this method.

The quantization of symplectic maps over compact prequantized Kähler manifolds has first been considered by Zelditch in [36], using a unitary version of the theory of Toeplitz structures of [9], and Zelditch and Zhou use it in [37, Th. 0.9] to establish the pointwise semi-classical estimates of Theorem 4.2 in the Kähler case for $E = \mathbb{C}$. Note that Theorem 1.2 is not a consequence of these pointwise semi-classical estimates, as they are not uniform in $x \in X$. The applications of parallel transport to the quantum dynamics associated with Hamiltonian flows have also been explored by Charles [13, Th. 5.3] in the case of compact prequantized Kähler manifolds with metaplectic structure, where he establishes an analogue of Proposition 1.1 in the language of Fourier integral operators.

2 Setting

Let (X, ω) be a compact symplectic manifold without boundary of dimension $2n$, and let (L, h^L) be a Hermitian line bundle over X, endowed with a Hermitian connection ∇^L satisfying the prequantization condition (1.1). Let J be an almost complex structure on TX compatible with ω, and let g^{TX} be the Riemmanian metric on X defined by

$$g^{TX}(\cdot,\cdot) := \omega(\cdot, J\cdot). \tag{2.1}$$

We write ∇^{TX} for the associated Levi–Civita connection on TX, and dv_X for the Riemannian volume form of (X, g^{TX}). It satisfies the *Liouville formula* $dv_X = \omega^n/n!$.

For any Hermitian vector bundle with connection (E, h^E, ∇^E) over X, we write $\langle\cdot,\cdot\rangle_E$ and $|\cdot|_E$ for the Hermitian product and norm induced by h^E, and write R^E for the curvature of ∇^E. We denote by $\mathbb{C}$ the trivial line bundle with trivial Hermitian metric and connection. For any $p \in \mathbb{N}$, we write L^p for the pth tensor power of L, and for any Hermitian vector bundle with connection (E, h^E, ∇^E), we set

$$E_p := L^p \otimes E, \tag{2.2}$$

equipped with the Hermitian metric h^{E_p} and connection ∇^{E_p} induced by h^L, h^E and ∇^L, ∇^E. The *L^2-Hermitian product* $\langle\cdot,\cdot\rangle_p$ on $\mathscr{C}^\infty(X, E_p)$ is given for any $s_1, s_2 \in \mathscr{C}^\infty(X, E_p)$ by the formula

$$\langle s_1, s_2\rangle_p := \int_X \langle s_1(x), s_2(x)\rangle_{E_p} dv_X(x). \tag{2.3}$$

Let $L^2(X, E_p)$ be the completion of $\mathscr{C}^\infty(X, E_p)$ with respect to $\langle\cdot,\cdot\rangle_p$.

Definition 2.1 For any $p \in \mathbb{N}$, the *Bochner Laplacian* Δ^{E_p} of $(E_p, h^{E_p}, \nabla^{E_p})$ is the second-order differential operator acting on $\mathscr{C}^\infty(X, E_p)$ by the formula

$$\Delta^{E_p} := -\sum_{j=1}^{2n} \left[(\nabla^{E_p}_{e_j})^2 - \nabla^{E_p}_{\nabla^{TX}_{e_j} e_j} \right], \tag{2.4}$$

where $\{e_j\}_{j=1}^{2n}$ is any local orthonormal frame of (TX, g^{TX}).

This defines an unbounded self-adjoint elliptic operator on $L^2(X, E_p)$, and by standard elliptic theory, its spectrum $\mathrm{Spec}(\Delta^{E_p})$ is discrete and contained in $\mathbb{R}$.

Definition 2.2 For any $p \in \mathbb{N}$, the *renormalized Bochner Laplacian* Δ_p is the second-order differential operator acting on $\mathscr{C}^\infty(X, E_p)$ by the formula

$$\Delta_p := \Delta^{E_p} - 2\pi np - \sum_{j=1}^{n} R^E(w_j, \bar{w}_j), \tag{2.5}$$

where $\{w_j\}_{j=1}^n$ is an orthonormal basis of $T^{(1,0)}X$ for the Hermitian metric induced by g^{TX}.

As above, Δ_p is an unbounded self-adjoint elliptic operator on $L^2(X, E_p)$, and has discrete spectrum $\mathrm{Spec}(\Delta_p)$ contained in $\mathbb{R}$. Furthermore, we have the following refinement of [17, Th.2.a].

Theorem 2.3 [25, Cor. 1.2] *There exist constants* $\widetilde{C},\ C > 0$ *such that for all* $p \in \mathbb{N}$,

$$\mathrm{Spec}(\Delta_p) \subset \ [-\widetilde{C}, \widetilde{C}\,] \cup\,]4\pi np - C, +\infty[, \tag{2.6}$$

and the constants $\widetilde{C},\ C > 0$ *are uniform in the choice of* $J \in \mathrm{End}(TX)$ *varying smoothly in a compact set of parameters. Furthermore, the direct sum*

$$\mathscr{H}_p := \bigoplus_{\lambda \in [-\widetilde{C}, \widetilde{C}\,]} \mathrm{Ker}(\lambda - \Delta_p) \tag{2.7}$$

is naturally included in $\mathscr{C}^\infty(X, E_p)$, *and there is* $p_0 \in \mathbb{N}$ *such that for any* $p \geqslant p_0$, *we have*

$$\dim \mathscr{H}_p = \int_X \mathrm{Td}(T^{(1,0)}X)\,\mathrm{ch}(E)\exp(p\omega), \tag{2.8}$$

where $\mathrm{Td}(T^{(1,0)}X)$ *represents the Todd class of* $T^{(1,0)}X$ *and* $\mathrm{ch}(E)$ *represents the Chern character of* E. *The integer* $p_0 \in \mathbb{N}$ *is uniform in the choice of* $J \in \mathrm{End}(TX)$ *varying smoothly in a compact set of parameters.*

For any $p \in \mathbb{N}$, the Hilbert space $\mathscr{H}_p \subset L^2(X, E_p)$ defined by (2.7) is called the *space of almost holomorphic sections* of E_p. In the special case when J is integrable, so that (X, J, ω, g^{TX}) is a Kähler manifold and the Hermitian bundles (L, h^L) and (E, h^E) admit natural holomorphic structures such that ∇^L and ∇^E are their Chern connections, then the subspace $\mathscr{H}_p \subset \mathscr{C}^\infty(X, L^p)$ coincides with the space of holomorphic sections of E_p for all $p \geqslant p_0$. In fact, as explained for example in [26, § 1.4.3, § 1.5], by the *Bochner–Kodaira formula*, the formula (2.5) is twice the *Kodaira Laplacian* of E_p, and there is a *spectral gap*, so that $\widetilde{C} = 0$ in (2.6). It is then a basic fact of Hodge theory that the kernel of the Kodaira Laplacian in $\mathscr{C}^\infty(X, E_p)$ is precisely the space of holomorphic sections of E_p, for all $p \in \mathbb{N}$.

The goal of this section is to describe the results of [21] about the dependence of this quantization scheme on the choice of an almost complex structure $J \in \mathrm{End}(TX)$. To this end, we consider a smooth path

$$t \longmapsto J_t \in \mathrm{End}(TX), \quad \text{for all } \ t \in \mathbb{R}, \tag{2.9}$$

of almost complex structures over X compatible with ω. We will see $\{J_t \in \mathrm{End}(TX)\}_{t\in\mathbb{R}}$ as the endomorphism of the vertical tangent bundle TX over $\mathbb{R} \times X$ of the tautological fibration

$$\begin{aligned} \pi : \mathbb{R} \times X &\longrightarrow \mathbb{R} \\ (t, x) &\longmapsto t, \end{aligned} \tag{2.10}$$

restricting to $J_t \in \mathrm{End}(TX)$ over $t \in \mathbb{R}$. We then have an induced vertical Riemannian metric on TX over $\mathbb{R} \times X$, defined by its restriction on the fiber X over any $t \in \mathbb{R}$ via the formula

$$g_t^{TX}(\cdot,\cdot) := \omega(\cdot, J_t\,\cdot). \tag{2.11}$$

Following Bismut in [3, Def. 1.6] and in [4, (1.2)], we consider the induced *vertical Levi–Civita connection* ∇^{TX} on the subbundle TX of the tangent bundle of $\mathbb{R}\times X$, defined by the formula

$$\nabla^{TX} := \Pi^{TX}\nabla^{\mathbb{R}\oplus TX}\Pi^{TX}, \tag{2.12}$$

where $\nabla^{\mathbb{R}\oplus TX}$ is the Levi–Civita connection on the total space of $\mathbb{R}\times X$ for the Riemannian metric defined on $T(\mathbb{R}\times X) = \mathbb{R}\oplus TX$ as the orthogonal sum of the canonical metric of $(\mathbb{R})$ and the metric (g_t^{TX}) over $(t\in\mathbb{R})$, with $\Pi^{TX} : \mathbb{R}\oplus TX \to TX$ the canonical projection. Note that by the Liouville formula $dv_X = \omega^n/n!$, the Riemannian volume form dv_X of (X, g_t^{TX}) does not depend on $t\in\mathbb{R}$.

Let $TX_{\mathbb{C}} := TX\otimes_{\mathbb{R}}\mathbb{C}$ be the complexification of the vertical tangent bundle TX over $\mathbb{R}\times X$. The family of complex structures $\{J_t\in\mathrm{End}(TX)\}_{t\in\mathbb{R}}$ induces a splitting

$$TX_{\mathbb{C}} = T^{(1,0)}X\oplus T^{(0,1)}X \tag{2.13}$$

into the eigenspaces of J_t corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$ over $\{t\}\times X$ for all $t\in\mathbb{R}$. We endow $TX_{\mathbb{C}}$ with the Hermitian product h^{TX} given by $g_t^{TX}(\cdot,\bar{\cdot})$ over $\{t\}\times X$ for all $t\in\mathbb{R}$. The *canonical line bundle* associated with $\{J_t\in\mathrm{End}(TX)\}_{t\in\mathbb{R}}$ is the line bundle

$$K_X := \det(T^{(1,0)*}X) \tag{2.14}$$

over $\mathbb{R}\times X$ equipped with the Hermitian metric h^{K_X} and connection ∇^{K_X} induced by the vertical Hermitian metric h^{TX} and the vertical Levi–Civita connection (2.12) via the splitting (2.13).

For any Hermitian vector bundle with connection (E, h^E, ∇^E) over $\mathbb{R}\times X$, we write $(E_t, h^{E_t}, \nabla^{E_t})$ for the Hermitian vector bundle with connection induced on X by restriction to the fiber over $t\in\mathbb{R}$. For all $t\in\mathbb{R}$, we write

$$\tau_t^E : E_0 \longrightarrow E_t \tag{2.15}$$

for the bundle isomorphism over X induced by parallel transport in E with respect to ∇^E along horizontal directions of $\pi : \mathbb{R}\times X\to\mathbb{R}$. We still write (L, h^L, ∇^L) for the Hermitian line bundle with connection over $\mathbb{R}\times X$ defined by pullback of (L, h^L, ∇^L) over X via the second projection, and write $(E_p, h^{E_p}, \nabla^{E_p})$ for the tensor product $E_p = E\otimes L^p$ over $\mathbb{R}\times X$ for any $p\in\mathbb{N}$, with induced Hermitian metric and connection. For any $p\in\mathbb{N}$ and $t\in\mathbb{R}$, we write $\Delta_{p,t}$ for the renormalized Bochner Laplacian acting on $\mathscr{C}^\infty(X, E_{p,t})$ associated with the metric g_t^{TX} as in Definition 2.2, and write $\mathscr{H}_{p,t}\subset\mathscr{C}^\infty(X, E_{p,t})$ for the associated space of almost holomorphic sections defined in Theorem 2.3.

Let us assume that there exists $p_0\in\mathbb{N}$ such that the two intervals in (2.6) are disjoint and such that $\mathscr{H}_{p,t}$ satisfies the Riemann–Roch–Hirzebruch formula (2.8) for all $p\geqslant p_0$ and $t\in\mathbb{R}$. By Theorem 2.3, such a $p_0\in\mathbb{N}$ always exists if we ask J_t and

$(E_t, h^{E_t}, \nabla^{E_t})$ to be independent of $t \in \mathbb{R}$ outside a compact set of $\mathbb{R}$. On the other hand, this assumption will be automatically satisfied for all $t \in \mathbb{R}$ in the main case of interest considered in Sect. 3. In the sequel, we fix such a $p_0 \in \mathbb{N}$.

Following for instance [2, § 9.2], for all $t \in \mathbb{R}$ and $p \geqslant p_0$, we define the orthogonal projection operator $P_{p,t} : L^2(X, E_{p,t}) \to \mathscr{H}_{p,t}$ with respect to the associated L^2-Hermitian product (2.3) by the following contour integral in the complex plane,

$$P_{p,t} := \int_\Gamma \left(\lambda - \Delta_{p,t}\right)^{-1} d\lambda, \tag{2.16}$$

where $\Gamma \subset \mathbb{C}$ is a circle of center 0 and radius $a > 0$ satisfying $\widetilde{C} < a < 4\pi p - C$. This shows that the projection operators $P_{p,t}$ depend smoothly on $t \in \mathbb{R}$, and as the dimension of $\mathrm{Im}(P_{p,t}) = \mathscr{H}_{p,t}$ is constant in $t \in \mathbb{R}$ by assumption, this defines a finite dimensional bundle over $\mathbb{R}$, which can be seen as a subbundle of the infinite dimensional vector bundle with fiber $\mathscr{C}^\infty(X, E_{p,t})$ over $t \in \mathbb{R}$.

Definition 2.4 For any $p \geqslant p_0$, the *quantum bundle* $(\mathscr{H}_p, h^{\mathscr{H}_p}, \nabla^{\mathscr{H}_p})$ is the bundle of almost holomorphic sections over $\mathbb{R} \simeq \{J_t \in \mathrm{End}(TX)\}_{t\in\mathbb{R}}$ defined via (2.16) as above, endowed with the *L^2-Hermitian structure* $h^{\mathscr{H}_p}$ induced by the L^2-Hermitian product of $L^2(X, E_{p,t})$ for all $t \in \mathbb{R}$, and with the *L^2-Hermitian connection* $\nabla^{\mathscr{H}_p}$, defined on the canonical vector field ∂_t of $\mathbb{R}$ via its action on the total space $\mathscr{C}^\infty(\mathbb{R} \times X, E_p)$ by the formula

$$\nabla^{\mathscr{H}_p}_{\partial_t} := P_{p,t} \nabla^{E_p}_{\partial_t} P_{p,t}, \tag{2.17}$$

for all $t \in \mathbb{R}$. By convention, we set $\mathscr{H}_p = \{0\}$ for all $p < p_0$.

By an argument of [5, Th. 1.14], the L^2-connection $\nabla^{\mathscr{H}_p}$ preserves the L^2-Hermitian product $h^{\mathscr{H}_p}$. For any $p \in \mathbb{N}$ and $t \in \mathbb{R}$, let $\mathscr{L}(\mathscr{H}_{p,0}, \mathscr{H}_{p,t})$ be the space of linear operators from $\mathscr{H}_{p,0}$ to $\mathscr{H}_{p,t}$, and write $\|\cdot\|_{p,0,t}$ for the operator norm of $\mathscr{L}(\mathscr{H}_{p,0}, \mathscr{H}_{p,t})$ induced by $h^{\mathscr{H}_p}$. We consider the *parallel transport*

$$\mathcal{T}_{p,t} \in \mathscr{L}(\mathscr{H}_{p,0}, \mathscr{H}_{p,t}) \tag{2.18}$$

in the quantum bundle $\mathscr{H}_p$ over $\mathbb{R}$ with respect to $\nabla^{\mathscr{H}_p}$. Recall that $\tau^E_t : E_0 \to E_t$ has been defined by (2.15). The following Theorem shows that the parallel transport has the semi-classical behavior of a *Toeplitz operator* as $p \to +\infty$.

Theorem 2.5 [21, Th. 3.16] *There exists a sequence $\{\mu_{l,t} \in \mathscr{C}^\infty(X, E_{p,t} \otimes E^*_{p,0})\}_{l\in\mathbb{N}}$, smooth in $t \in \mathbb{R}$, such that for all $k \in \mathbb{N}^*$, there exists $C_k > 0$ such that*

$$\left\| \mathcal{T}_{p,t} - \sum_{l=0}^{k-1} p^{-l} P_{p,t} \mu_{l,t} P_{p,0} \right\|_{p,0,t} \leqslant C_k p^{-k}, \tag{2.19}$$

for all $p \in \mathbb{N}$ and $t \in \mathbb{R}$. Furthermore, there is a natural function $\mu_t \in \mathscr{C}^\infty(X, \mathbb{C})$ such that the first coefficient $\mu_{0,t}$ satisfies

$$\mu_{0,t} = \mu_t \tau^E_t. \tag{2.20}$$

To describe the function $\mu_t \in \mathscr{C}^\infty(X, \mathbb{C})$ of (2.20), let us describe the local setting involved in the proof of the above theorem in [21]. For any $t \in \mathbb{R}$, using the fact that the almost complex structures $J_0 \in \mathrm{End}(TX)$ and $J_t \in \mathrm{End}(TX)$ are both compatible with the same symplectic form ω, we get a splitting

$$TX_{\mathbb{C}} = T^{(1,0)}X_0 \oplus T^{(0,1)}X_t \tag{2.21}$$

into the holomorphic subspace $T^{(1,0)}X_0$ of $TX_{\mathbb{C}}$ associated to $J_0 \in \mathrm{End}(TX)$ and the anti-holomorphic subspace $T^{(0,1)}X_t$ of $TX_{\mathbb{C}}$ associated with $J_t \in \mathrm{End}(TX)$ as in (2.13). We write

$$\Pi_0^t \in \mathrm{End}(TX_{\mathbb{C}}) \tag{2.22}$$

for the projection operator onto $T^{(1,0)}X_0$ with kernel $T^{(0,1)}X_t$. In a dual way, we write $\overline{\Pi}_t^0 \in \mathrm{End}(TX_{\mathbb{C}})$ for the projection operator onto $T^{(0,1)}X_t$ with kernel $T^{(1,0)}X_0$. Considering its restriction to $T^{(0,1)}X_0$ and via the isomorphism $T^{(0,1)}X_t \simeq T^{(1,0)*}X_t$ induced by g_t^{TX} for all $t \in \mathbb{R}$, it induces an isomorphism

$$\det(\overline{\Pi}_t^0) : K_{X,0} \longrightarrow K_{X,t} \tag{2.23}$$

of the respective canonical line bundles over X. Recall the connection ∇^{K_X} on the canonical line bundle K_X over $\mathbb{R} \times X$ of (2.14), inducing $\tau_t^{K_X} : K_{X,0} \to K_{X,t}$ by (2.15). Then by [21, (5.4)], the function $\mu_t \in \mathscr{C}^\infty(X, \mathbb{C})$ of (2.20) satisfies

$$\bar{\mu}_t^2(x) = \det(\overline{\Pi}_t^0)^{-1}\tau_t^{K_X}, \tag{2.24}$$

for all $t \in \mathbb{R}$, via the canonical identification $K_{X,0} \otimes K_{X,0}^* \simeq \mathbb{C}$.

The main tool of the proof of Theorem 2.5 in [21] is the local study of the *Schwartz kernel* with respect to dv_X of the parallel transport operator. For any linear operator $\mathcal{K}_{p,t} \in \mathscr{L}(\mathscr{H}_{p,0}, \mathscr{H}_{p,t})$, write $\mathcal{K}_{p,t}(\cdot,\cdot) \in \mathscr{C}^\infty(X \times X, E_{p,t} \boxtimes E_{p,0}^*)$ for the Schwartz kernel with respect to dv_X of the bounded operator

$$\mathcal{K}_{p,t} := P_{p,t}\mathcal{K}_{p,t}P_{p,0} : L^2(X, E_{p,0}) \longrightarrow L^2(X, E_{p,t}), \tag{2.25}$$

defined for any $s \in \mathscr{C}^\infty(X, E_{p,0})$ and $x \in X$ by the formula

$$\mathcal{K}_{p,t}s\,(x) = \int_X \mathcal{K}_{p,t}(x, y)s(y)\, dv_X(y). \tag{2.26}$$

The existence of a smooth Schwartz kernel is an immediate consequence of the fact that the image of (2.25) is finite dimensional. In the case $t = 0$, so that $\mathcal{K}_{p,0} \in \mathrm{End}(\mathscr{H}_{p,0})$ and $\mathcal{K}_{p,0}(x, x) \in \mathrm{End}(E_{p,0})_x$ for all $x \in X$, we have the following basic trace formula,

$$\mathrm{Tr}[\mathcal{K}_{p,0}] = \int_X \mathrm{Tr}\left[\mathcal{K}_{p,0}(x, x)\right] dv_X(x). \tag{2.27}$$

Fix $\varepsilon > 0$ and consider a collection of diffeomorphisms

$$B^{T_{x_0}X}(0,\varepsilon) \xrightarrow{\sim} V_{x_0} \subset X, \tag{2.28}$$

varying smoothly with $x_0 \in X$, sending 0 to $x_0 \in X$ and with differential at 0 inducing the identity of $T_{x_0}X$. For any $x_0 \in X$ and $t \in \mathbb{R}$, we pullback (L, h^L, ∇^L) and $(E_t, h^{E_t}, \nabla^{E_t})$ over V_{x_0} in this chart, and identify them with their central fiber L_{x_0} and E_{t,x_0} by parallel transport along radial lines of $B^{T_{x_0}X}(0,\varepsilon)$. We then identify L_{x_0} with $\mathbb{C}$ by the choice of a unit vector. For any $\mathcal{K}_{p,t} \in \mathscr{L}(\mathscr{H}_{p,0}, \mathscr{H}_{p,t})$, we write $\mathcal{K}_{p,t,x_0}(\cdot,\cdot)$ for the image in this trivialization of its Schwartz kernel over $V_{x_0} \times V_{x_0}$. Then $\mathcal{K}_{p,t,x_0}(\cdot,\cdot)$ can be seen as the evaluation at $x_0 \in X$ of the pullback of $E_t \otimes E_0^*$ over the fibered product $B^{TX}(0,\varepsilon) \times_X B^{TX}(0,\varepsilon)$ over X, and for any $m \in \mathbb{N}$, let $|\cdot|_{\mathscr{C}^m(X)}$ be a local $\mathscr{C}^m$-norm on this bundle induced by derivation with respect to $x_0 \in X$.

For any $Z,\ Z' \in T_{x_0}X$ and $t \in \mathbb{R}$, we use the following local model

$$\mathscr{T}_{t,x_0}(Z,Z') := \exp\Big(-\pi\Big[\big\langle \Pi_0^t(Z-Z'),(Z-Z')\big\rangle + \sqrt{-1}\omega(Z,Z')\Big]\Big), \tag{2.29}$$

where $\langle\cdot,\cdot\rangle$ is the scalar product on $T_{x_0}X$ induced by the metric g_0^{TX} defined by (2.11). For any $F_{t,x_0}(Z,Z') \in E_t \otimes E_0^*$ polynomial in $Z,\ Z' \in T_{x_0}X$, we write

$$F\mathscr{T}_{t,x_0}(Z,Z') := F_{t,x_0}(Z,Z')\mathscr{T}_{t,x_0}(Z,Z') \in E_t \otimes E_0^*. \tag{2.30}$$

For any function $h \in \mathscr{C}^\infty(X,\mathbb{R})$, we will use repeatedly in the sequel the following form of Taylor expansion around any $x_0 \in X$ up to order $k-1 \in \mathbb{N}$, as $|Z| \to 0$ in the chart (2.28) above,

$$\begin{aligned} h(Z) &= h(x_0) + \sum_{r=1}^{k-1}\sum_{|\alpha|=r} \frac{\partial^r h}{\partial Z^\alpha}\frac{Z^\alpha}{\alpha!} + O(|Z|^k) \\ &= h(x_0) + \sum_{r=1}^{k-1} p^{-r/2}\sum_{|\alpha|=r}\frac{\partial^r h}{\partial Z^\alpha}\frac{(\sqrt{p}Z)^\alpha}{\alpha!} + p^{-\frac{k}{2}}O(|\sqrt{p}Z|^k). \end{aligned} \tag{2.31}$$

Write $d^X(\cdot,\cdot)$ for the Riemannian distance of (X, g_0^{TX}), and for any $m \in \mathbb{N}$, let $|\cdot|_{\mathscr{C}^m}$ be the local $\mathscr{C}^m$-norm induced by derivation with respect to ∇^{E_p} over $\mathbb{R} \times X$. Then Theorem 2.5 is based on the following result.

Theorem 2.6 [21, Th. 3.17] *Consider a collection of charts of the form (2.28), varying smoothly with $x_0 \in X$, sending 0 to x_0 and with differential at 0 inducing the identity of $T_{x_0}X$. Then for any $m,\ k \in \mathbb{N}$, $\theta \in]0,1[$ and any compact subset $K \subset \mathbb{R}$, there is $C_k > 0$ such that for all $p \in \mathbb{N}$ and $t \in K$, we have*

$$\big|\mathcal{T}_{p,t}(x,y)\big|_{\mathscr{C}^m} \leqslant C_k p^{-k} \quad \textit{as soon as} \quad d^X(x,y) > \varepsilon p^{-\frac{\theta}{2}}. \tag{2.32}$$

Furthermore, there is a family $\{G_{r,t,x_0}(Z,Z') \in E_{t,x_0} \otimes E^*_{0,x_0}\}_{r\in\mathbb{N}}$ *of polynomials in* $Z, Z' \in T_{x_0}X$ *of the same parity as* r*, depending smoothly on* $x_0 \in X$*, such that for any* $m, m', l, k \in \mathbb{N}$*,* $\delta \in]0,1[$ *and any compact subset* $K \in \mathbb{R}$*, there is* $C > 0$ *and* $\theta \in]0,1[$ *such that for any* $x_0 \in X$*,* $p \in \mathbb{N}$ *and* $Z, Z' \in T_{x_0}X$ *with* $|Z|, |Z'| < \varepsilon p^{-\theta/2}$*, we have*

$$\sup_{|\alpha|+|\alpha'|=m} \Big| \frac{\partial^l}{\partial t^l} \frac{\partial^\alpha}{\partial Z^\alpha} \frac{\partial^{\alpha'}}{\partial Z'^{\alpha'}} \Big(p^{-n} \mathcal{T}_{p,t,x_0}(Z,Z') - \sum_{r=0}^{k1} p^{-r/2} G_r \mathscr{T}_{t,x_0}(\sqrt{p}Z, \sqrt{p}Z') \Big) \Big|_{\mathscr{C}^{m'}(X)} \leqslant C p^{-\frac{k-m}{2}+\delta}, \tag{2.33}$$

where $G_{0,t,x_0}(Z,Z')$ *is constant in* $Z, Z' \in T_{x_0}X$*, given by*

$$G_{0,t,x_0}(Z,Z') = \bar{\mu}_t^{-1}(x_0)\tau^E_{t,x_0}. \tag{2.34}$$

Let us now consider a diffeomorphism $\varphi : X \to X$ preserving the symplectic form ω, together with a lift $\varphi^L : L \to L$ to the total space of L preserving metric and connection, and assume that for some $t_0 \in \mathbb{R}$, we have

$$J_{t_0} = d\varphi\, J_0\, d\varphi^{-1}. \tag{2.35}$$

Assume further that there is a lift $\varphi^E : E_0 \to E_{t_0}$ of φ preserving metric and connection, and write $\varphi_p : E_{p,0} \to E_{p,t_0}$ for the induced lift on E_p, for all $p \in \mathbb{N}$. For any $s \in \mathscr{C}^\infty(X, E_{p,t_0})$, we define the *pullback* $\varphi_p^* s \in \mathscr{C}^\infty(X, E_{p,0})$ by the formula

$$(\varphi_p^* s)(x) := \varphi_p^{-1} s(\varphi(x)). \tag{2.36}$$

This induces by restriction a unitary isomorphism

$$\varphi_p^* : \mathscr{H}_{p,t_0} \xrightarrow{\sim} \mathscr{H}_{p,0}. \tag{2.37}$$

Then Theorem 2.8 gives a semi-classical estimate for the trace of the endomorphism $\varphi_p^* \mathcal{T}_{p,t_0} \in \mathrm{End}(\mathscr{H}_{p,0})$ as $p \to +\infty$, using the trace formula (2.27). To this end, we need the following assumption.

Definition 2.7 The fixed point set $X^\varphi \subset X$ of a diffeomorphism $\varphi : X \to X$ is said to be *non-degenerate* over an open set $U \subset X$ if $X^\varphi \cap U$ is a proper submanifold of $\overline{U}$ satisfying

$$T_x X^\varphi = \mathrm{Ker}(\mathrm{Id}_{T_xX} - d\varphi_x) \ \text{ for all } \ x \in X^\varphi \cap U. \tag{2.38}$$

As the lift φ^L preserves h^L and ∇^L, its value $\beta \in \mathbb{C}$ via the canonical identification $L \otimes L^* \simeq \mathbb{C}$ is locally constant over X^φ, and satisfies $|\beta| = 1$.

Let $Y \subset X$ be a submanifold, and let N be a subbundle of TX over Y transverse to TY. Write g^N for the Euclidean metric on N induced by the metric g_0^{TX} defined by

(2.11), and write $|\mathrm{d}v|_{TX}$ and $|\mathrm{d}v|_N$ for the Riemannian densities of (TX, g_0^{TX}) and (N, g^N). We denote by $|\mathrm{d}v|_{TX/N}$ the density over Y defined by the formula

$$|\mathrm{d}v|_{TX} = |\mathrm{d}v|_{TX/N}|\mathrm{d}v|_N. \tag{2.39}$$

We write $P^N : TX \to N$ for the orthogonal projection with respect to g_0^{TX} of vector bundles over Y.

Theorem 2.8 [21, Th. 4.3] *Assume that the fixed point set X^φ of $\varphi : X \to X$ is non-degenerate over X, and write $\{X_j\}_{1\leqslant j\leqslant m}$ for the set of its connected components. Then there exist densities ν_r over X^φ for any $r \in \mathbb{N}$ such that for any $k \in \mathbb{N}^*$ and as $p \to +\infty$,*

$$\operatorname{Tr}[\varphi_p^* \mathcal{T}_{p,t_0}] = \sum_{j=1}^{m} p^{\dim X_j/2} e^{-2\pi\sqrt{-1}p\lambda_j} \left(\sum_{r=0}^{k-1} p^{-r} \int_{X_j} \nu_r + O(p^{-k}) \right), \tag{2.40}$$

where $e^{2\pi\sqrt{-1}\lambda_j}$ is the constant value of φ^L over X_j, for some $\lambda_j \in \mathbb{R}$. Furthermore, for any $x \in X^\varphi$ we have

$$\begin{aligned}\frac{\nu_0}{|dv|_{TX/N}}(x) &= \operatorname{Tr}_{E_x}[\varphi^{E,-1}\tau_{t_0}^E] \left(\det(\overline{\Pi}_{t_0}^0)^{-1}\tau_{t_0}^{K_X}\right)_x^{-\frac{1}{2}} \int_{N_x} \mathscr{T}_{t_0,x}(d\varphi.Z, Z)dZ \\ &= \operatorname{Tr}_{E_x}[\varphi^{E,-1}\tau_{t_0}^E] \left(\det(\overline{\Pi}_{t_0}^0)^{-1}\tau_{t_0}^{K_X}\right)_x^{-\frac{1}{2}} \\ &\qquad \det{}_{N_x}^{-\frac{1}{2}} \left[P^N(\Pi_0^{t_0} - d\varphi^{-1}\overline{\Pi}_{t_0}^0)(\mathrm{Id}_{TX} - d\varphi)P^N \right],\end{aligned} \tag{2.41}$$

for some natural choices of square roots, where N is any subbundle of TX over X^φ transverse to TX^φ.

The first coefficient (2.41) acquires a geometric interpretation in the special case when the bundles TX^φ and N are both preserved by φ and J_0. In order to describe it, let $\varphi^{K_X} : K_{X,0} \longrightarrow K_{X,t_0}$ be the natural action induced by φ, and recall that $\tau_{t_0}^{K_X} : K_{X,0} \longrightarrow K_{X,t_0}$ has been defined in (2.15). Then $\varphi^{K_X,-1}\tau_{t_0}^{K_X} \in \mathscr{C}^\infty(X, \mathbb{C})$ via the canonical identification $K_{X,0} \otimes K_{X,0}^* \simeq \mathbb{C}$, and one can compute the following.

Proposition 2.9 [21, Lemma 5.1] *Assume that the fixed point set X^φ of $\varphi : X \to X$ is non-degenerate over X, and that there exists a subbundle N of TX over X^φ transverse to TX^φ such that TX^φ and N are both preserved by φ and J_0. Then we have the following formula, for all $x \in X^\varphi$,*

$$\begin{aligned}&\left(\det(\overline{\Pi}_{t_0}^0)^{-1}\tau_{t_0}^{K_X}\right)_x^{-\frac{1}{2}} \int_{N_x} \mathscr{T}_{t_0,x}(d\varphi.Z, Z)dZ \\ &\quad = (-1)^{\frac{\dim N_x}{4}} (\varphi^{K_X,-1}\tau_{t_0}^{K_X})_x^{-\frac{1}{2}} \,|\det{}_{N_x}(\mathrm{Id}_N - d\varphi|_N)|^{-\frac{1}{2}},\end{aligned} \tag{2.42}$$

for some natural choices of square roots.

The previous result acquires an even cleaner formulation in the case when (E, h^E, ∇^E) satisfies

$$E^2 = K_X, \tag{2.43}$$

as a line bundle over $\mathbb{R} \times X$ with induced metric and connection. Such a line bundle exists if and only if the first Chern class $c_1(TX) \in H^2(X, \mathbb{Z})$ of TX is even, and the choice of a complex line bundle E satisfying (2.43) is called a *metaplectic structure* on X. We write $E =: K_X^{1/2}$, and call it the *metaplectic correction*. We then get the following straightforward corollary of Proposition 2.9.

Corollary 2.10 *Consider the assumptions of Proposition* 2.9*, and assume further that* X *admits a metaplectic structure. Then if* $E = K_X^{1/2}$ *is the associated metaplectic correction over* $\mathbb{R} \times X$*, the first coefficient* ν_0 *of* (2.41) *satisfies the formula*

$$\nu_0 = (-1)^{\frac{\dim N}{4}} \, |\det{}_N(\mathrm{Id}_N - d\varphi|_N)|^{-\frac{1}{2}} \, |dv|_{TX/N}, \tag{2.44}$$

for some natural choices of square roots.

In the sequel, we will write $|\cdot|_p$ for the norm induced on $E_p \otimes E_p^*$ by h^{E_p}, for all $p \in \mathbb{N}$.

3 Quantum evolution operators

Let (X, ω) be a compact symplectic manifold without boundary endowed with (L, h^L, ∇^L) satisfying the prequantization condition (1.1), and consider a smooth function $f \in \mathscr{C}^\infty(X, \mathbb{R})$. The *Hamiltonian vector field* $\xi_f \in \mathscr{C}^\infty(X, TX)$ associated with f is defined by the formula

$$\iota_{\xi_f} \omega = df. \tag{3.1}$$

The *Hamiltonian flow* of f is the flow of diffeomorphisms $\varphi_t : X \to X$ defined for all $t \in \mathbb{R}$ by

$$\begin{cases} \frac{\partial}{\partial t} \varphi_t = \xi_f, \\ \varphi_0 = \mathrm{Id}_X. \end{cases} \tag{3.2}$$

By definition (3.1) of a Hamiltonian vector field and by Cartan formula, the Hamiltonian flow $\varphi_t : X \to X$ preserves ω for all $t \in \mathbb{R}$. Let $\widetilde{\xi}_f \in \mathscr{C}^\infty(L, TL)$ be the horizontal lift of ξ_f to the total space of L with respect to ∇^L, and let $\mathbf{t} \in \mathscr{C}^\infty(L, TL)$ be the canonical vector field on the total space of L defined by

$$\mathbf{t} = \frac{\partial}{\partial t}\Big|_{t=0} e^{2\pi\sqrt{-1}t}, \tag{3.3}$$

for the action of $e^{2\pi\sqrt{-1}t}$ by complex multiplication in the fibers. Then the flow (3.2) lifts to a flow $\varphi_t^L : L \to L$ on the total space of L over X, defined for all $t \in \mathbb{R}$ by the formula

$$\begin{cases} \frac{\partial}{\partial t}\varphi_t^L = \widetilde{\xi}_f + f\mathbf{t}, \\ \varphi_0^L = \mathrm{Id}_L. \end{cases} \tag{3.4}$$

Note that both φ_t and φ_t^L define 1-parameter groups, as both vector fields defining them do not depend on $t \in \mathbb{R}$. From the definition (3.1) of the Hamiltonian vector field of f, we see that φ_t^L is the unique lift of φ_t to L preserving the connection ∇^L, for all $t \in \mathbb{R}$. More specifically, for any $t \in \mathbb{R}$, recall the pullback of $s \in \mathscr{C}^\infty(X, L)$ by φ_t defined by formula (2.36). Then for any vector field $v \in \mathscr{C}^\infty(X, TX)$, we get from (3.1) and (3.4) that

$$\varphi_t^* \nabla_v^L s = \nabla_{d\varphi_t.v}^L \varphi_t^* s. \tag{3.5}$$

On the other hand, we also deduce from (3.4) the following *Kostant formula*, for all $t \in \mathbb{R}$,

$$\frac{\partial}{\partial t}\varphi_t^* s = \left(\nabla_{\xi_f}^L - 2\pi\sqrt{-1}f\right)\varphi_t^* s. \tag{3.6}$$

For any $p \in \mathbb{N}$ and $t \in \mathbb{R}$, let us write $\varphi_{t,p}$ for the flow induced by φ_t^L on the total space of L^p, and $\varphi_{t,p}^*$ for the associated pullback as in (2.36). Then for any $s \in \mathscr{C}^\infty(X, L^p)$, the Kostant formula (3.6) becomes

$$\frac{\partial}{\partial t}\varphi_{t,p}^* s = \left(\nabla_{\xi_f}^{L^p} - 2\pi\sqrt{-1}pf\right)\varphi_{t,p}^* s. \tag{3.7}$$

This formula characterizes $f \in \mathscr{C}^\infty(X, \mathbb{R})$ as the *Kostant moment map* for the action of $\mathbb{R}$ on $(L^p, h^{L^p}, \nabla^{L^p})$ induced by φ_t for all $t \in \mathbb{R}$.

Note that (3.1) implies that $f(\varphi_t(x)) = f(x)$ for all $t \in \mathbb{R}$ and $x \in X$, and (3.2) implies that $d\varphi_t.\xi_f = \xi_f$ for all $t \in \mathbb{R}$. For all $x \in X$, we write

$$\tau_{t,p} : L_x^p \longrightarrow L_{\varphi_t(x)}^p \tag{3.8}$$

for the parallel transport along the path $s \mapsto \varphi_s(x)$ for $s \in [0, t]$. We can then reformulate the Kostant formula (3.7) as

$$\varphi_{t,p}^{-1}\tau_{t,p} = e^{-2\pi\sqrt{-1}tpf} \in \mathscr{C}^\infty(X, \mathbb{C}), \tag{3.9}$$

via the canonical identification $L \otimes L^* \simeq \mathbb{C}$.

Let us now consider an almost complex structure $J_0 \in \mathrm{End}(TX)$ over X compatible with ω. Then for any $t \in \mathbb{R}$, the formula

$$J_t := d\varphi_t\, J_0\, d\varphi_t^{-1} \in \mathrm{End}(TX) \tag{3.10}$$

defines a path $\{J_t \in \mathrm{End}(TX)\}_{t\in\mathbb{R}}$ of almost complex structures over X compatible with ω. For any $t \in \mathbb{R}$, we write $\mathscr{H}_{p,t}$ for the space of almost holomorphic sections with respect to J_t defined in Theorem 2.3. Then for any $t_0 \in \mathbb{R}$ and $p \in \mathbb{N}$, the pullback induces by restriction a bijective linear map

$$\varphi_{t_0,p}^* : \mathscr{H}_{p,t+t_0} \longrightarrow \mathscr{H}_{p,t}, \tag{3.11}$$

for all $t \in \mathbb{R}$. For any fixed $p \in \mathbb{N}$, this implies, in particular, that the dimension $\dim \mathscr{H}_{p,t}$ does not depend on $t \in \mathbb{R}$, so that the quantum bundle $(\mathscr{H}_p, h^{\mathscr{H}_p}, \nabla^{\mathscr{H}_p})$ of Definition 2.4 is well defined over $\mathbb{R}$ for all $p \in \mathbb{N}$. Recall the tautological fibration $\pi : \mathbb{R} \times X \to X$ considered in (2.10) together with all the data induced by $\{J_t \in \mathrm{End}(TX)\}_{t\in\mathbb{R}}$, and consider the flow over $\mathbb{R} \times X$ defined for all $t_0 \in \mathbb{R}$ by

$$\begin{aligned} \Phi_{t_0} : \mathbb{R} \times X &\longrightarrow \mathbb{R} \times X \\ (t, x) &\longmapsto (t + t_0, \varphi_{t_0}(x)). \end{aligned} \tag{3.12}$$

For any $p \in \mathbb{N}$ and $t_0 \in \mathbb{R}$, the lift $\varphi_{t_0,p}$ of φ_{t_0} to $(L^p, h^{L^p}, \nabla^{L^p})$ over X induces tautologically a lift $\Phi^*_{t_0,p}$ of Φ_{t_0} to the pullback of $(L^p, h^{L^p}, \nabla^{L^p})$ over $\mathbb{R} \times X$ via the second projection. For any section $s \in \mathscr{C}^\infty(\mathbb{R} \times X, L^p)$ over $\mathbb{R} \times X$ and any $t \in \mathbb{R}$, write $s_t \in \mathscr{C}^\infty(X, L^p)$ for the section over X defined by $s_t(x) := s(t, x)$ for all $x \in X$. Then for any $t_0 \in \mathbb{R}$ and $p \in \mathbb{N}$, the pullback of $s \in \mathscr{C}^\infty(\mathbb{R} \times X, L^p)$ by Φ_{t_0} is given for any $t \in \mathbb{R}$ by the formula

$$(\Phi^*_{t_0,p} s)_t = \varphi^*_{t_0,p} s_{t+t_0}. \tag{3.13}$$

By (3.11), the pullback $\Phi^*_{t_0,p}$ preserves the smooth sections $\mathscr{C}^\infty(\mathbb{R}, \mathscr{H}_p)$ of the quantum bundle, seen as a subspace of $\mathscr{C}^\infty(\mathbb{R} \times X, L^p)$ as in Definition 2.4.

We still write ξ_f for the pullback of the Hamiltonian vector field $\xi_f \in \mathscr{C}^\infty(X, TX)$ to a vertical vector field over $\pi : \mathbb{R} \times X \to \mathbb{R}$ via the second projection, and write ∂_t for the horizontal vector field over $\pi : \mathbb{R} \times X \to \mathbb{R}$ induced by the canonical vector field of $\mathbb{R}$. By definition of the pullback of (L, h^L, ∇^L) to $\mathbb{R} \times X$, for any $s \in \mathscr{C}^\infty(\mathbb{R} \times X, L^p)$ and $t \in \mathbb{R}$, we have

$$\left(\nabla^{L^p}_{\partial_t} s\right)_t = \frac{\partial}{\partial t} s_t. \tag{3.14}$$

Recall Definition 2.4 for the connection $\nabla^{\mathscr{H}_p}$, and note that for all $t,\ t_0 \in \mathbb{R}$, we have

$$\Phi^*_{t_0,p} P_{p,t} = P_{p,t+t_0} \Phi^*_{t_0,p}. \tag{3.15}$$

Using (3.11) to (3.15), for any smooth section $s \in \mathscr{C}^\infty(\mathbb{R}, \mathscr{H}_p) \subset \mathscr{C}^\infty(\mathbb{R} \times X, L^p)$ and seeing the orthogonal projection $P_p : \mathscr{C}^\infty(\mathbb{R} \times X, L^p) \to \mathscr{C}^\infty(\mathbb{R}, \mathscr{H}_p)$ as a global endomorphism, we get the following quantized version of the Kostant formula (3.7), for all $t_0 \in \mathbb{R}$,

$$\begin{aligned} \frac{\partial}{\partial t}\Big|_{t=t_0} \Phi^*_{t,p} s &= \left(\nabla^{L^p}_{\xi_f} - 2\pi\sqrt{-1} p f\right) \Phi^*_{t_0,p} s + \nabla^{L^p}_{\partial_t} \Phi^*_{t_0,p} s \\ &= P_p \left(\nabla^{L^p}_{\xi_f + \partial_t} - 2\pi\sqrt{-1} p f\right) \Phi^*_{t_0,p} P_p s \\ &= \left(P_p \nabla^{L^p}_{\partial_t} P_p + P_p (\nabla^{L^p}_{\xi_f} - 2\pi\sqrt{-1} p f) P_p\right) \Phi^*_{t_0,p} s \\ &= \left(\nabla^{\mathscr{H}_p}_{\partial_t} - 2\pi\sqrt{-1} p P_p \left(f + \frac{\sqrt{-1}}{2\pi p} \nabla^{L^p}_{\xi_f}\right) P_p\right) \Phi^*_{t_0,p} s. \end{aligned} \tag{3.16}$$

Seeing the path $\{J_t \in \mathrm{End}(TX)\}_{t\in\mathbb{R}}$ as lying in the space $\mathscr{J}_\omega$ of almost complex structures compatible with ω and comparing with the usual Kostant formula (3.7), we can interpret the *quantized Kostant formula* (3.16) by stating that the Kostant–Souriau operator $Q_p(f) \in \mathrm{End}(\mathscr{H}_p)$ given by formula (1.4) induces a moment map on the quantum bundle $\mathscr{H}_p$ over $\mathscr{J}_\omega$ for the natural action of the group of Hamiltonian diffeomorphisms $\mathrm{Ham}(X,\omega)$ on $\mathscr{J}_\omega$ defined by (3.10).

The relevance of the Kostant–Souriau operator in Kähler geometry goes back to the work of Cahen, Gutt and Rawnsley [11] relating it to Berezin's quantization of Kähler manifolds [1], and the moment map picture described above has been introduced by Donaldson in [14]. As explained for example in [16, §1], we can consider the line bundle $\det(\mathscr{H}_p)$ over any compact submanifold of $\mathscr{J}_\omega$ for $p \in \mathbb{N}$ big enough, and the curvature of the connection $\nabla^{\det\mathscr{H}_p}$ induced by the L^2-connection (2.17) on $\det(\mathscr{H}_p)$ defines a natural symplectic form via the prequantization formula (1.1). In the spin^c Dirac operator case, which implies the Kähler case, it follows from the asymptotics of the curvature of $\nabla^{\mathscr{H}_p}$ as $p \to +\infty$ established by Ma and Zhang in [29, Th. 2.1]. Then the quantized Kostant formula (3.16) shows that the Hamiltonian flow associated with the function $\det(Q_p(f))$ on $\mathscr{J}_\omega$ is precisely the action of the Hamiltonian flow of f on $\mathscr{J}_\omega$ defined by (3.10).

On the other hand, as described for example in [35, § 9.7], the quantum dynamics is given by the 1-parameter family of unitary operators generated by the quantum Hamiltonian operator acting on a *fixed* space of quantum states. For any $p \in \mathbb{N}$, we thus consider the quantum Hamiltonian operator $Q_p(f)$ restricted to the space $\mathscr{H}_{p,0}$ of almost holomorphic sections with respect to our initial almost complex structure J_0. This induces a one-parameter family $\exp\left(2\pi\sqrt{-1}tpQ_p(f)\right) \in \mathrm{End}(\mathscr{H}_{p,0})$ of unitary operators defined for all $t \in \mathbb{R}$ by

$$\begin{cases} \frac{\partial}{\partial t}\exp\left(2\pi\sqrt{-1}tpQ_p(f)\right) = 2\pi\sqrt{-1}pQ_p(f)\exp\left(2\pi\sqrt{-1}tpQ_p(f)\right), \\ \exp\left(2\pi\sqrt{-1}tpQ_p(f)\right)\big|_{t=0} = \mathrm{Id}_{\mathscr{H}_{p,0}}. \end{cases} \tag{3.17}$$

Writing $\mathcal{T}_{p,t} : \mathscr{H}_{p,0} \to \mathscr{H}_{p,t}$ for the parallel transport with respect to $\nabla^{\mathscr{H}_p}$ over $\mathbb{R}$ as in Sect. 2, the following Lemma establishes formula (1.3).

Lemma 3.1 *For any $p \in \mathbb{N}$ and all $t \in \mathbb{R}$, we have the following equality*

$$\exp\left(-2\pi\sqrt{-1}tpQ_p(f)\right) = \varphi_{t,p}^*\mathcal{T}_{p,t} \in \mathrm{End}(\mathscr{H}_{p,0}). \tag{3.18}$$

Proof By Definition 2.4 of the L^2-connection $\nabla^{\mathscr{H}_p}$, using (3.14) and the fact that $\Phi_{t_0,p}^*$ commutes with P_p when acting on $\mathscr{C}^\infty(\mathbb{R}\times X, L^p)$ for any $t_0 \in \mathbb{R}$ and $p \in \mathbb{N}$, we have

$$\Phi_{t_0,p}^*\nabla_{\partial_t}^{\mathscr{H}_p} = \nabla_{\partial_t}^{\mathscr{H}_p}\Phi_{t_0,p}^*. \tag{3.19}$$

Furthermore, as $\varphi_{t_0,p}^*$ commutes with $\left(\nabla_{\xi_f}^{L^p} - 2\pi\sqrt{-1}pf\right)$ when acting on $\mathscr{C}^\infty(X, L^p)$, by (1.4) and the pullback formula (3.13), we get

$$\Phi_{t_0,p}^*Q_p(f) = Q_p(f)\Phi_{t_0,p}^*. \tag{3.20}$$

This implies the following analogue of (3.9) for the quantized Kostant formula (3.16), for all $t \in \mathbb{R}$,

$$\frac{\partial}{\partial t}\varphi_{t,p}^* \mathcal{T}_{p,t} = -2\pi\sqrt{-1}\, p Q_p(f) \varphi_{t,p}^* \mathcal{T}_{p,t}, \tag{3.21}$$

which follows from the quantized Kostant formula (3.16) in the same way as (3.9) follows from the usual Kostant formula (3.7). This proves the lemma. □

Before giving the applications of the results described in Sect. 2 to the quantum evolution operator defined above, let us illustrates its behavior via the following definition.

Definition 3.2 For any $x_0 \in X$ and any unit vector $\zeta \in L_{x_0}$, the associated *coherent state* is the sequence $\{s_{x_0,p} \in \mathscr{H}_{p,0}\}_{p\in\mathbb{N}}$ defined for all $x \in X$ by

$$s_{x_0,p}(x) = P_{p,0}(x, x_0)\zeta^p, \tag{3.22}$$

where $P_{p,0}(\cdot,\cdot) \in \mathscr{C}^\infty(X \times X, L^p \boxtimes (L^p)^*)$ is the Schwartz kernel with respect to dv_X of the orthogonal projection operator $P_{p,0} : \mathscr{C}^\infty(X, L^p) \to \mathscr{H}_{p,0}$.

Coherent states represent the quantization of a classical particle located at $x_0 \in X$ in phase space. As one can readily check from the definition, it is characterized by the fact that its orthogonal in $\mathscr{H}_{p,0}$ consists of sections vanishing at $x_0 \in X$. As shown in [27, Th. 0.1, § 1.1], the sequence $\{s_{x_0,p}\}_{p\in\mathbb{N}}$ decreases rapidly as $p \to +\infty$ outside any open set containing x_0, while $|s_{x_0,p}(x_0)|_{L^p}$ is of order p^n. Now using the following tautological identity of operators acting on $\mathscr{C}^\infty(X, L^p)$,

$$\exp\left(2\pi\sqrt{-1}tpQ_p(f)\right) = \exp\left(2\pi\sqrt{-1}tpQ_p(f)\right) P_{p,0}, \tag{3.23}$$

and by the classical formula for the Schwartz kernel of the composition of two operators, for any $t \in \mathbb{R}$ and $x \in X$, we get from Definition 3.2,

$$\begin{aligned}
&\exp\left(2\pi\sqrt{-1}tpQ_p(f)\right) s_{x_0,p}(x) \\
&\quad = \int_X \exp\left(2\pi\sqrt{-1}tpQ_p(f)\right)(x, w) P_{p,0}(w, x_0)\zeta^p\, dv_X(w) \\
&\quad = \exp\left(2\pi\sqrt{-1}tpQ_p(f)\right)(x, x_0)\zeta^p.
\end{aligned} \tag{3.24}$$

This shows that the last line of (3.24), seen as a section of L^p with respect to the variable $x \in X$, can be interpreted as the quantum evolution at time $t \in \mathbb{R}$ of the quantization of a classical particle at $x_0 \in X$. In particular, we expect this section to decrease rapidly as $p \to +\infty$ outside any open set containing the classical evolution $\varphi_t(x_0) \in X$, while its value at $\varphi_t(x_0)$ should be of order p^n. The following result shows that this is indeed the case.

Proposition 3.3 *For any $\varepsilon > 0$, $k, m \in \mathbb{N}$, $\theta \in]0,1[$ and any compact subset $K \subset \mathbb{R}$, there exists $C_k > 0$ such that*

$$\left|\exp\left(2\pi\sqrt{-1}tpQ_p(f)\right)(x,y)\right|_{\mathscr{C}^m} \leqslant C_k p^{-k} \text{ as soon as } d^X(x,\varphi_t(y)) > \varepsilon p^{-\frac{\theta}{2}}, \tag{3.25}$$

for all $t \in K$. Furthermore, there exist $a_r(t,x) \in \mathbb{C}$ for any $r \in \mathbb{N}$, depending smoothly on $x \in X$ and $t \in \mathbb{R}$, such that for any $k \in \mathbb{N}^*$,

$$\exp\left(2\pi\sqrt{-1}tpQ_p(f)\right)(\varphi_t(x),x) = p^n e^{2\pi\sqrt{-1}tpf(x)}\left(\sum_{r=0}^{k-1} p^{-r}a_r(t,x) + O(p^{-k})\right)\tau_{t,p}, \tag{3.26}$$

with first coefficient a_0 satisfying the formula

$$a_0(t,x)^2 = \left(\det(\overline{\Pi}^0_{-t})^{-1}\tau^{K_X}_{-t}\right)^{-1}_x. \tag{3.27}$$

In particular, it satisfies $a_0(t,x) \neq 0$ for all $t \in \mathbb{R}$ and $x \in X$.

Proof Recall that for any $x,\ y \in X$ and $t \in \mathbb{R}$, we get from Lemma 3.1 and formula (2.36) that

$$\exp\left(-2\pi\sqrt{-1}tpQ_p(f)\right)(x,y) = \varphi_{t,p}^{-1}\mathcal{T}_{p,t}(\varphi_t(x),y). \tag{3.28}$$

Then (3.25) is a consequence of the Kostant formula (3.6), together with the rapid decrease in $\mathcal{T}_{p,t}(\cdot,\cdot)$ outside of the diagonal given by (2.32).

Using the exponentiation (3.9) of Kostant formula, rewrite (3.28) as

$$\begin{aligned}\exp\left(2\pi\sqrt{-1}tpQ_p(f)\right)(\varphi_t(x),x) &= \varphi_{t,p}\mathcal{T}_{p,-t}(\varphi_{-t}(\varphi_t(x)),x)\\ &= e^{2\sqrt{-1}\pi tpf(x)}\tau_{t,p}\mathcal{T}_{p,-t}(x,x).\end{aligned} \tag{3.29}$$

We can thus apply Theorem 2.6 with $x_0 = x$ for $Z = Z' = 0$, and noting that $J_{2q+1}(0,0) = 0$ for all $q \in \mathbb{N}$ for parity reasons, we then get the expansion (3.26), with first coefficient satisfying $a_0(t,x) = \bar{\mu}^{-1}_{-t}(x)$, for all $x \in X$ and $t \in \mathbb{R}$. This implies formula (3.27) via the formula (2.24) for $\mu \in \mathscr{C}^\infty(X,\mathbb{C})$. □

Recall the non-degeneracy assumption of Definition 2.7. We also have the following semi-classical trace formula for the quantum evolution operator, where we use the notations of Theorem 2.8.

Theorem 3.4 *Let $t \in \mathbb{R}$ be such that the fixed point set X^{φ_t} of $\varphi_t : X \to X$ is non-degenerate, and write $\{X_j\}_{1\leqslant j\leqslant m}$ for the set of its connected components. Then there exist densities ν_r over X^{φ_t} for any $r \in \mathbb{N}$ such that for any $k \in \mathbb{N}^*$ and as $p \to +\infty$,*

$$\begin{aligned}&\operatorname{Tr}\left[\exp\left(-2\pi\sqrt{-1}tpQ_p(f)\right)\right]\\ &= \sum_{j=1}^{q} p^{\dim X_j/2} e^{-2\pi\sqrt{-1}p\lambda_j}\left(\sum_{r=0}^{k-1} p^{-r}\int_{X_j}\nu_r + O(p^{-k})\right),\end{aligned} \tag{3.30}$$

where $e^{2\pi\sqrt{-1}\lambda_j}$ is the constant value of φ^L over X_j, for some $\lambda_j \in \mathbb{R}$. Furthermore, we have

$$\nu_0 = \left(\det(\overline{\Pi}_t^0)^{-1}\tau_t^{K_X}\right)^{-\frac{1}{2}} \det{}_N^{-\frac{1}{2}}\left[P^N(\Pi_0^t - d\varphi_t^{-1}\overline{\Pi_t^0})(\mathrm{Id}_{TX} - d\varphi_t)P^N\right]|dv|_{TX/N}, \tag{3.31}$$

for some natural choices of square roots.

Proof Using Lemma 3.1, this is a straightforward consequence of Theorem 2.8. □

The coefficient $\lambda_j \in \mathbb{R}$ appearing in the expansion (3.30) has a natural geometric interpretation, which fits in a more general context. In fact, note that the evolution equations (3.2) and (3.4) generalize to *time-dependent Hamiltonians* $F \in \mathscr{C}^\infty(\mathbb{R} \times X, \mathbb{R})$, so that $f \in \mathscr{C}^\infty(X, \mathbb{R})$ is replaced by $f_t \in \mathscr{C}^\infty(X, \mathbb{R})$ depending on $t \in \mathbb{R}$, with $f_t(x) := F(t, x)$ for all $x \in X$. In that case, we also get a Hamiltonian flow $\varphi_t : X \to X$ together with a lift φ_t^L to the total space of L preserving metric and connection. The main difference here is that the term replacing $\left(\nabla_{\xi_f}^L - 2\pi\sqrt{-1}f\right)$ in the Kostant formula (3.6) will depend on time, and the corresponding exponentiation as in (3.9) at $x \in X$ reads

$$(\varphi_{t,p}^{-1}\tau_{t,p})_x = e^{-2\pi\sqrt{-1}p\int_0^t f_s(\varphi_s(x))ds}. \tag{3.32}$$

Let now $x \in X$ and $t \in \mathbb{R}$ be such that $\varphi_t(x) = x$, and via the canonical identification $L_x \otimes L_x^* \simeq \mathbb{C}$, let us write

$$\varphi_{t,x}^L =: e^{2\sqrt{-1}\pi\lambda_t(x)}. \tag{3.33}$$

Note that the path $s \mapsto \varphi_s(x)$ defines a loop γ inside X. Assuming that this loop bounds an immersed disk $D \subset X$, by the prequantizaton condition (1.1) and via (3.32) above, we get

$$\lambda_t(x) = \int_D \omega + \int_0^t f_s(\varphi_s(x))ds. \tag{3.34}$$

This is a familiar quantity in symplectic topology, called the *action* of F around the loop γ. Recall that as φ^L preserves the connection ∇^L, the quantity $\lambda_t(x) \in \mathbb{R}$ is constant when $x \in X$ varies continuously over a submanifold of fixed points of φ_t. This discussion motivates the following definition.

Definition 3.5 Let $F \in \mathscr{C}^\infty(\mathbb{R}\times X, \mathbb{R})$ be a time-dependent Hamiltonian, and let $t \in \mathbb{R}$ be such that its Hamiltonian flow $\varphi_t : X \to X$ at time $t \in \mathbb{R}$ has non-degenerate fixed point set $X^\varphi \subset X$ in the sense of Definition 2.7. Then for any connected component Y of X^φ, the associated real number $\lambda_0 \in \mathbb{R}$ defined over Y by

$$\varphi_t^L =: e^{2\sqrt{-1}\pi\lambda_0} \tag{3.35}$$

is called the *action* of F over Y.

In the general case of a time-dependent Hamiltonian $F \in \mathscr{C}^\infty(\mathbb{R} \times X, \mathbb{R})$, we get a quantum Hamiltonian as before from the corresponding quantized Kostant formula

(3.16), and we can define the associated evolution operator by equation (3.17). Then the analogue of Lemma 3.1 holds in that case, and the analogues of Proposition 3.3 and Theorem 3.4 hold in the same way.

Finally, the situation described above also generalizes to the case when $(L^p, h^{L^p}, \nabla^{L^p})$ over $\mathbb{R} \times X$ is replaced by $E_p = L^p \otimes E$ with induced metric and connection, where (E, h^E, ∇^E) is a Hermitian vector bundle with connection over $\mathbb{R} \times X$. In that case, we make the further assumption that the Hamiltonian flow $\varphi_t : X \to X$ lifts to a bundle map $\varphi_t^E : E_0 \to E_t$ over X preserving metric and connection for all $t \in \mathbb{R}$, and we still write $\varphi_{t,p}$ for the corresponding action on E_p for any $p \in \mathbb{N}$. Then the analogue of Lemma 3.1 for the quantum evolution operator defined by equation (3.17) holds as before, and using the general setting of Sect. 2, we also get the corresponding analogues of Proposition 3.3 and Theorem 3.4. The case of $E = K_X^{1/2}$, with the lift induced by $\varphi_t^{K_X} : K_{X,0} \to K_{X,t}$ for all $t \in \mathbb{R}$, will be of particular interest in the next section.

4 Gutzwiller trace formula

Consider the setting of the previous section, with Hamiltonian $f \in \mathscr{C}^\infty(X, \mathbb{R})$ not depending on time, and a Hermitian vector bundle with connection (E, h^E, ∇^E) over $\mathbb{R} \times X$, so that L^p is replaced by $E_p = L^p \otimes E$ for all $p \in \mathbb{N}$. Let $g : \mathbb{R} \to \mathbb{R}$ be a smooth function with compact support, and for all $t \in \mathbb{R}$, set

$$\hat{g}(t) := \int_{\mathbb{R}} g(t) e^{-2\pi\sqrt{-1}t}\, dt. \tag{4.1}$$

We define the family of operators $\{\hat{g}(pQ_p(f)) \in \mathrm{End}(\mathscr{H}_p)\}_{p\in\mathbb{N}}$ by the formula

$$\hat{g}(pQ_p(f)) := \int_{\mathbb{R}} g(t) \left(\varphi_{t,p}^* \mathcal{T}_{p,t}\right) dt. \tag{4.2}$$

In the particular case of $E = \mathbb{C}$, so that Lemma 3.1 holds, we get

$$\hat{g}(pQ_p(f)) = \int_{\mathbb{R}} g(t) \exp(-2\pi\sqrt{-1}tpQ_p(f))dt, \tag{4.3}$$

recovering the usual definition via functional calculus from (4.1). Note that formula (1.7) for $\hat{g}(pQ_p(f-c))$ with $c \in \mathbb{R}$ follows from (3.2), (3.4) and (2.36), as replacing $f \in \mathscr{C}^\infty(X, \mathbb{R})$ by $f - c$ does not change the Hamiltonian flow $\varphi_t : X \to X$ but multiplies its lift to L by $e^{-2\pi\sqrt{-1}tc}$. In the context of semi-classical analysis, the *Gutzwiller trace formula* predicts a semi-classical estimate for the trace $\mathrm{Tr}[\hat{g}(pQ_p(f - c))]$ as $p \to +\infty$, where $c \in \mathbb{R}$ is a regular value of f, showing that it localizes around the periodic orbits of the Hamiltonian flow of f inside the level set $f^{-1}(c)$.

The following preliminary result shows that the Schwartz kernel of $\hat{g}(pQ_p(f-c))$ decreases rapidly outside $f^{-1}(c)$ as $p \to +\infty$.

Proposition 4.1 *Let $c \in \mathbb{R}$ be a regular value of $f \in \mathscr{C}^\infty(X, \mathbb{R})$. For any $k \in \mathbb{N}$, there exists $C_k > 0$ such that for any $y \in X$, $x \in X \backslash f^{-1}(c)$ and $p \in \mathbb{N}$, we have*

$$\left|\hat{g}(pQ_p(f-c))(x,y)\right|_p \leqslant \frac{C_k}{|f(x)-c|^k} p^{n-\frac{k}{2}}. \tag{4.4}$$

Proof To simplify notations, we assume $E = \mathbb{C}$, the case of general E being completely analogous. Recall the definition (3.8) of $\tau_{t,p}$, and $p \in \mathbb{N}$. For any $x \in X$ and $t_0 \in \mathbb{R}$, consider a chart around $x_0 := \varphi_{t_0}(x) \in X$ as in (2.28), such that the radial line generated by ξ_{f,x_0} in $B^{T_{x_0}X}(0,\varepsilon)$ is sent to the path $s \mapsto \varphi_s(x_0)$ in V_{x_0}. Then L^p is identified with $L^p_{x_0}$ along this path by the parallel transport $\tau_{t,p}$, for all $p \in \mathbb{N}$ and $|t| < \varepsilon$. Thus for any $Z \in B^{T_{x_0}X}(0,\varepsilon)$ sent to $y \in V_{x_0}$ in the chart (2.28) and for any $t \in \mathbb{R}$ small enough, we have

$$\tau_{t,p}^{-1}\mathcal{T}_{p,t_0+t}(\varphi_{t_0+t}(x), y) = \mathcal{T}_{p,t_0+t,x_0}(t\xi_{f,x_0}, Z). \tag{4.5}$$

Using $\tau_{t_0+t,p} = \tau_{t,p}\tau_{t_0,p}$ for all $t_0 \in T$ and $|t| < \varepsilon$, we can apply Theorem 2.6 in such charts for all $t_0 \in \mathbb{R}$, so that for any $k \in \mathbb{N}$ we get $C_k > 0$ such that for any $x, y \in X$, $t \in \operatorname{Supp} g$ and all $p \in \mathbb{N}$, we have

$$\left|\frac{\partial^k}{\partial t^k}\tau_{t,p}^{-1}\mathcal{T}_{p,t}(\varphi_t(x), y)\right|_p \leqslant C_k p^{n+\frac{k}{2}}. \tag{4.6}$$

Recall from (3.1) that the Hamiltonian flow of f is the same as the Hamiltonian flow of $f-c$, for all $c \in \mathbb{R}$. Then exponentiating Kostant formula as in (3.9) and as $\operatorname{Supp} g$ is compact, we can integrate by parts to get from the definition (4.2) of $\hat{g}(pQ_p(f))$ that for any $x, y \in X$ and $k \in \mathbb{N}$,

$$\begin{aligned}
&\hat{g}(pQ_p(f-c))(x,y)\\
&= \int_{\mathbb{R}} g(t)\tau_{t,p}^{-1}\mathcal{T}_{p,t}(\varphi_t(x), y) e^{-2\pi\sqrt{-1}tp(f(x)-c)}\mathrm{d}t\\
&= \frac{1}{(-2\pi\sqrt{-1}p(f(x)-c))^k}\int_{\mathbb{R}} g(t)\tau_{t,p}^{-1}\mathcal{T}_{p,t}(\varphi_t(x), y)\frac{\partial^k}{\partial t^k} e^{-2\pi\sqrt{-1}tp(f(x)-c)}\mathrm{d}t\\
&= (2\pi\sqrt{-1})^{-k}\frac{p^{-k}}{(f(x)-c)^k}\int_{\mathbb{R}}\frac{\partial^k}{\partial t^k}\left(g(t)\tau_{t,p}^{-1}\mathcal{T}_{p,t}(\varphi_t(x), y)\right) e^{-2\pi\sqrt{-1}tp(f(x)-c)}\mathrm{d}t.
\end{aligned} \tag{4.7}$$

This proves the result by (4.6). □

Let us now estimate the Schwartz kernel of $\hat{g}(pQ_p(f-c))$ as $p \to +\infty$. Proposition 4.1 shows that it localizes around the level set $f^{-1}(c)$, in contrast with Proposition 3.3. In the following theorems and their proofs, we will use freely the notations of Sects. 1 and 2.

Theorem 4.2 *Let $c \in \mathbb{R}$ be a regular value of $f \in \mathscr{C}^\infty(X, \mathbb{R})$. If $x, y \in X$ do not satisfy $\varphi_t(x) = y$ for some $t \in \mathbb{R}$ or do not satisfy $f(x) = f(y) = c$, then for any $k \in \mathbb{N}$, there exists $C_k > 0$ such that for all $p \in \mathbb{N}$,*

$$\left|\hat{g}(pQ_p(f-c))(x,y)\right|_p < C_k p^{-k}. \tag{4.8}$$

Let $x,\ y \in f^{-1}(c)$, and write $T := \{t \in \mathbb{R} \mid \varphi_t(x) = y\}$. Then there exist $b_{t_0,r} \in \mathbb{C}$ for all $r \in \mathbb{N}$ and $t_0 \in T$ such that for any $k \in \mathbb{N}^*$ and as $p \to +\infty$,

$$\hat{g}(pQ_p(f-c))(x,y) = p^{n-\frac{1}{2}} \sum_{t_0 \in T} g(t_0) \left(\sum_{r=0}^{k-1} p^{-r} b_{t_0,r} + O(p^{-k}) \right) \left(\varphi_{t_0}^{E,-1} \tau_{t_0}^{E} \right) \tau_{t_0,p}^{-1}. \tag{4.9}$$

Furthermore, for any $t_0 \in T$, we have

$$b_{t_0,0}^2 = \left(\det(\overline{\Pi}_{t_0}^0)^{-1} \tau_{t_0}^{K_X} \right)_x^{-1} \langle \Pi_0^{t_0} \xi_{f,x}, \xi_{f,x} \rangle_{g_0^{TX}}^{-1}. \tag{4.10}$$

In particular, it satisfies $b_{t_0,0} \neq 0$, for all $t_0 \in T$.

Proof Note first that the estimate (4.8) is a straightforward consequence of either Theorem 2.6 or Proposition 4.1, respectively.

To establish the expansion (4.9), fix $x,\ y \in X$ such that $f(x) = f(y) = c \in \mathbb{R}$ is a regular value of f. Then $f(x) - c = 0$, and exponentiating the Kostant formula as in (3.9), we can write the Schwartz kernel of $\hat{g}(pQ_p(f-c))$ as

$$\hat{g}(pQ_p(f-c))(x,y) = \int_{\mathbb{R}} g(t) \tau_{t,p}^{-1} \varphi_t^{E,-1} \mathcal{T}_{p,t}(\varphi_t(x), y) \mathrm{d}t. \tag{4.11}$$

As $c \in \mathbb{R}$ is a regular value of f, the Hamiltonian vector field ξ_f does not vanish over $f^{-1}(c)$. By (2.32), this implies that for any $\theta \in]0,1[$ and $\varepsilon > 0$, we get the following estimate as $p \to +\infty$,

$$\hat{g}(pQ_p(f-c))(x,y) = \sum_{t_0 \in T} \int_{t_0 - \varepsilon p^{-\frac{\theta}{2}}}^{t_0 + \varepsilon p^{-\frac{\theta}{2}}} g(t) \tau_{t,p}^{-1} \varphi_t^{E,-1} \mathcal{T}_{p,t}(\varphi_t(x), y) \mathrm{d}t + O(p^{-\infty}), \tag{4.12}$$

where all terms but a finite number vanish by compacity of $\operatorname{Supp} g$.

Consider a chart around y as in (2.28), sending the radial line generated by $\xi_{f,y}$ in $B^{T_yX}(0,\varepsilon)$ to the path $s \mapsto \varphi_s(y)$ in V_y. Then L^p is identified with L_y^p along this path by the parallel transport $\tau_{t,p}$, for all $p \in \mathbb{N}$ and $|t| < \varepsilon$. Using $\tau_{t_0+t,p} = \tau_{t,p}\tau_{t_0,p}$ for all $t_0 \in T$ and $|t| < \varepsilon$, we can then apply Theorem 2.6 to get a family $\{G_{r,t,y}(Z,Z') \in E_{t,y} \otimes E_{0,y}^*\}_{r \in \mathbb{N}}$ of polynomials in $Z, Z' \in T_yX$ of the same parity as r and smooth in $t \in \mathbb{R}$, such that for any $\delta \in]0,1[$ and $k \in \mathbb{N}^*$, there is $\theta \in]0,1[$ such that for all $p \in \mathbb{N}$,

$$\begin{aligned} \hat{g}(pQ_p(f-c))(x,y) = \sum_{t_0 \in T} p^n \tau_{p,t_0}^{-1} \int_{-\varepsilon p^{-\frac{\theta}{2}}}^{\varepsilon p^{-\frac{\theta}{2}}} g(t_0+t) \varphi_{t_0+t}^{E,-1} \\ \sum_{r=0}^{k-1} p^{-\frac{r}{2}} G_r \mathscr{T}_{t_0+t,y}(\sqrt{p}t\xi_{f,y}, 0) \mathrm{d}t + p^{n-\frac{\theta}{2}} O(p^{-\frac{k}{2}+\delta}). \end{aligned} \tag{4.13}$$

Consider the right-hand side of (4.13), and let us apply the Taylor expansion in $t \in \mathbb{R}$ described in (2.31) on all terms depending on $t \in \mathbb{R}$, respectively, inside and outside the exponential of the local model (2.29) for $\mathscr{T}_{t_0+t,y}$. We then get $F_{r,t_0}(t) \in E_{t_0} \otimes E_0^*$, polynomial in $t \in \mathbb{R}$ and of the same parity as r for any $r \in \mathbb{N}$, such that for all $t_0 \in T$ and as $|t| \to 0$,

$$\begin{aligned}\sum_{r=0}^{k-1} p^{-\frac{r}{2}} g(t_0+t)\varphi_{t_0+t}^{E,-1} G_r \mathscr{T}_{t_0+t,y}(\sqrt{p}t\xi_{f,y}, 0) \\ = g(t_0)\varphi_{t_0}^{E,-1} \sum_{r=0}^{k-1} p^{-r/2} F_{r,t_0}(\sqrt{p}t) \exp\left(-p\pi \langle \Pi_0^{t_0}\xi_{f,y}, \xi_{f,y}\rangle t^2\right) \\ + p^{-\frac{k}{2}} O(|\sqrt{p}t|^{M_k}),\end{aligned} \tag{4.14}$$

for some $M_k \in \mathbb{N}^*$ depending on the degrees of the polynomials $G_{r,t_0+t,y}$ for all $|t| < \varepsilon$, $t_0 \in T \cap \operatorname{Supp} g$ and $1 \leqslant r \leqslant k$. Furthermore, from the formula (2.34) for the first coefficient $G_{0,t,y}$, we know that $F_{0,t_0}(t) = \bar{\mu}_{t_0}^{-1}(y)\tau_{t_0,y}^E$ for all $t \in \mathbb{R}$. Note that by the definition (2.22) of $\Pi_0^{t_0}$, the real part of $\langle \Pi_0^{t_0}\xi_f, \xi_f\rangle$ is strictly positive, so that the right-hand side of (4.14) decreases exponentially in $t \in \mathbb{R}$. Writing

$$\delta_k = \delta + \frac{M_k(1-\theta)}{2}, \tag{4.15}$$

and after a change of variable $t \mapsto t/\sqrt{p}$, we then get

$$\begin{aligned}\hat{g}(pQ_p(f-c))(x,y) = \sum_{t_0\in T} g(t_0)\bar{\mu}_{t_0}^{-1}(y)\varphi_{t_0}^{E,-1}\tau_{t_0}^E\tau_{p,t_0}^{-1} p^{n-\frac{1}{2}} \\ \sum_{r=0}^{k-1} p^{-\frac{r}{2}} \int_{\mathbb{R}} F_{r,t_0}(t) \exp\left(-\pi \langle \Pi_0^{t_0}\xi_{f,y}, \xi_{f,y}\rangle t^2\right) \mathrm{d}t \\ + p^{n-\frac{1}{2}} O(p^{-\frac{k}{2}+\delta_k}).\end{aligned} \tag{4.16}$$

As $F_{2q+1,t_0}(t)$ is odd as a function of $t \in \mathbb{R}$ for all $q \in \mathbb{N}$, we get

$$\int_{\mathbb{R}} F_{2q+1,t_0}(t) \exp\left(-\pi \langle \Pi_0^{t_0}\xi_{f,y}, \xi_{f,y}\rangle t^2\right) \mathrm{d}t = 0. \tag{4.17}$$

As $\delta_k \to \delta$ when $\theta \to 1$ by (4.15) and as δ can be chosen arbitrary small, this gives the expansion (4.9), and we get the formula (4.10) for the first coefficient via the classical formula for Gaussian integrals, using the formula (2.24) and the fact that $F_{0,t_0}(t) = \bar{\mu}_{t_0}^{-1}(y)\tau_{t_0}^E$ for all $t \in \mathbb{R}$. □

Consider now the hypotheses and notations of Theorem 1.2. Recall that $c \in \mathbb{R}$ is a regular value of f, so that $\Sigma := f^{-1}(c)$ is a smooth manifold. Then there exists $\varepsilon > 0$ and diffeomorphisms

$$\Psi_{t_0} : f^{-1}(]c-\varepsilon, c+\varepsilon[) \xrightarrow{\sim}]c-\varepsilon, c+\varepsilon[\times \Sigma \tag{4.18}$$

for all $t_0 \in T$, such that $f(u,x) = u$ for all $(u,x) \in]c-\varepsilon, c+\varepsilon[\times \Sigma$ under this identification, and such that

$$\Psi_{t_0}\left(X^{\varphi_{t_0}} \cap f^{-1}(]c-\varepsilon, c+\varepsilon[)\right) = \bigcup_{\substack{1\leqslant j\leqslant m \\ t_j = t_0}}]c-\varepsilon, c+\varepsilon[\times Y_j, \tag{4.19}$$

where the connected components Y_j of $X^{\varphi_{t_j}} \cap f^{-1}(c)$ are seen as submanifolds of Σ, for all $1 \leqslant j \leqslant m$. We endow Σ with the Riemannian metric $g^{T\Sigma}$ induced by $g_0^{TX} := \omega(J_0\cdot, \cdot)$ via the inclusion $\Sigma = f^{-1}(c) \subset X$. For any $1 \leqslant j \leqslant m$, let $|dv|_{Y_j}$ be the Riemannian density over Y_j induced by $g^{T\Sigma}$ and let N be the normal bundle of Y_j inside Σ. We write $P^N : T\Sigma \to N$ for the orthogonal projection with respect to $g^{T\Sigma}$ over Y_j for all $1 \leqslant j \leqslant m$.

Recall that the *Liouville measure* on the level set $f^{-1}(c)$ is induced by the volume form

$$\frac{\iota_v\, \omega^n}{(n-1)!} \in \Omega^{2n-1}(f^{-1}(c), \mathbb{R}), \tag{4.20}$$

for any $v \in \mathscr{C}^\infty(f^{-1}(c), TX)$ satisfying $\omega(\xi_f, v) = 1$, and does not depend on such a choice. We write $\mathrm{Vol}_\omega(f^{-1}(c)) > 0$ for the volume of $f^{-1}(c)$ with respect to (4.20). Recalling Definition 3.5, the following theorem is a version of the *Gutzwiller trace formula* in geometric quantization of compact prequantized symplectic manifolds, and is the main result of this section.

Theorem 4.3 *Under the above assumptions, there exist $b_{j,r} \in \mathbb{C}$ for all $r \in \mathbb{N}$ and $1 \leqslant j \leqslant m$, such that for any $k \in \mathbb{N}^*$, we have as $p \to +\infty$,*

$$\mathrm{Tr}\left[\hat{g}(pQ_p(f-c))\right] = \sum_{j=1}^m p^{(\dim Y_j - 1)/2} g(t_j) e^{-2\pi\sqrt{-1}p\lambda_j} \left(\sum_{r=0}^{k-1} p^{-r} b_{j,r} + O(p^{-k})\right), \tag{4.21}$$

where $\lambda_j \in \mathbb{C}$ is the action of f over Y_j. Furthermore, for all $1 \leqslant j \leqslant m$ we have

$$b_{j,0} = \int_{Y_j} \mathrm{Tr}_E[\varphi_{t_j}^{E,-1}\tau_{t_j}^E] \left(\det(\overline{\Pi}_{t_j}^0)^{-1}\tau_{t_j}^{K_X}\right)^{-\frac{1}{2}} \det{}_N^{-\frac{1}{2}}\left[P^N(\Pi_0^{t_j} - d\varphi_{t_j}^{-1}\overline{\Pi}_{t_j}^0)(\mathrm{Id}_{TX} - d\varphi_{t_j})P^N\right] \frac{|dv|_{Y_j}}{|\xi_f|_{g_0^{TX}}}, \tag{4.22}$$

for some natural choices of square roots. In particular, we have as $p \to +\infty$,

$$\mathrm{Tr}\left[\hat{g}(pQ_p(f-c))\right] = p^{n-1} g(0)\, \mathrm{rk}(E)\, \mathrm{Vol}_\omega(f^{-1}(c)) + O(p^{n-2}). \tag{4.23}$$

Proof First note that replacing $f \in \mathscr{C}^\infty(X,\mathbb{R})$ by $f-c$, we are reduced to the case $c=0$. Consider thus $f \in \mathscr{C}^\infty(X,\mathbb{R})$ satisfying the hypotheses of Theorem 1.2 with $c=0$. Using the trace formula (2.27) and the definition of $\hat{g}(pQ_p(f))$ in (4.2) for all $p \in \mathbb{N}$, we know that

$$\begin{aligned}\operatorname{Tr}\left[\hat{g}(pQ_p(f))\right] &= \int_X \operatorname{Tr}\left[\hat{g}(pQ_p(f))(x,x)\right] dv_X(x)\\ &= \int_X \int_{\mathbb{R}} g(t) \operatorname{Tr}\left[\varphi_{t,p}^{-1}\mathcal{T}_{p,t}(\varphi_t(x),x)\right] dt\, dv_X(x).\end{aligned} \tag{4.24}$$

For any $\varepsilon > 0$, write

$$U(\varepsilon) := f^{-1}(]-\varepsilon,\varepsilon[). \tag{4.25}$$

Then by Proposition 4.1, for any $\theta \in]0,1[$ and $k \in \mathbb{N}$, we get $C_k > 0$ such that for all $x \in X \backslash U(\varepsilon p^{-\frac{\theta}{2}})$,

$$\left|\hat{g}(pQ_p(f))(x,x)\right|_p \leqslant \frac{C_k}{\varepsilon^k} p^{n-\frac{k(1-\theta)}{2}}, \tag{4.26}$$

so that in particular, we have as $p \to +\infty$,

$$\operatorname{Tr}\left[\hat{g}(pQ_p(f))\right] = \int_{U(\varepsilon p^{-\theta/2})} \operatorname{Tr}\left[\hat{g}(pQ_p(f))(x,x)\right] dv_X(x) + O(p^{-\infty}). \tag{4.27}$$

Recall that $T := \{t \in \operatorname{Supp} g \mid \exists\, x \in f^{-1}(0),\ \varphi_t(x) = x\}$ is finite, and let $\varepsilon > 0$ be such that all $u \in]-\varepsilon,\varepsilon[$ are regular values of f, so that the Hamiltonian vector field ξ_f does not vanish on the closure of $U(\varepsilon)$. Then in the same way as in (4.12), we get from the rapid decrease (2.32) of $\mathcal{T}_{p,t}(\cdot,\cdot)$ outside the diagonal that as $p \to +\infty$,

$$\begin{aligned}&\operatorname{Tr}\left[\hat{g}(pQ_p(f))\right]\\ &\quad= \sum_{t_0 \in T} \int_{U(\varepsilon p^{-\theta/2})} \int_{t_0-\varepsilon p^{-\theta/2}}^{t_0+\varepsilon p^{-\theta/2}} g(t) \operatorname{Tr}\left[\varphi_{t,p}^{-1}\mathcal{T}_{p,t}(\varphi_t(x),x)\right] dt\, dv_X(x) + O(p^{-\infty}).\end{aligned} \tag{4.28}$$

Take $\varepsilon > 0$ small enough so that the identification Ψ_{t_0} of (4.18) holds for any $t_0 \in T$. Recall from (3.1) that the Hamiltonian vector field ξ_f is tangent to the level sets of f. Then for any $t_0 \in T$, we get diffeomorphisms $\varphi_{u,t} : \Sigma \to \Sigma$, depending smoothly on $u \in]-\varepsilon,\varepsilon[$ and $t \in]t_0-\varepsilon, t_0+\varepsilon[$, such that

$$\varphi_t(u,x) = (u, \varphi_{u,t}(x)) \ \text{ and } \ \varphi_{u,t_0}(x) = x \ \text{ for all } \ x \in Y_j, \tag{4.29}$$

in the coordinates $(u,x) \in \Psi_{t_0}(U(\varepsilon))$ of (4.18) and for all $1 \leqslant j \leqslant m$ such that $t_j = t_0$. For any $\varepsilon > 0$ and $1 \leqslant j \leqslant m$, consider the normal geodesic neighborhood $V_j(\varepsilon) \subset \Sigma$ of Y_j inside $(\Sigma, g^{T\Sigma})$. Then by the non-degeneracy assumption of Definition 2.7 and as all $u \in]-\varepsilon,\varepsilon[$ are regular values of f, the map $\Phi : (u,t,x) \mapsto (u,t,\varphi_{u,t}(x))$ is also non-degenerate around the fixed point set $]-\varepsilon,\varepsilon[\times \{t_j\} \times Y_j$ inside $]-\varepsilon,\varepsilon[\times \mathbb{R} \times \Sigma$, so that working in local charts, we see that there exists $\varepsilon' > 0$ such that for all $\theta \in]0,1[$ and $p \in \mathbb{N}$,

$$d^{\Sigma}(x, \varphi_{u,t}(x)) > \varepsilon' p^{-\theta/2} \quad \text{as soon as} \quad (t,x), \notin \bigcup_{1 \leqslant j \leqslant m}]t_j - \varepsilon p^{-\theta/2}, t_j + \varepsilon p^{-\theta/2}[\times V_j(\varepsilon p^{-\theta/2}), \tag{4.30}$$

for all $u \in]-\varepsilon, \varepsilon[$. On the other hand, as $f(x,u) = u$ in the coordinates $(u,x) \in \Psi_{t_j}(U(\varepsilon))$ of (4.18), by the definition (3.1) of the Hamiltonian vector field ξ_f and the definition (2.11) of g_0^{TX}, we get a function $\varrho \in \mathscr{C}^\infty(U(\varepsilon), \mathbb{R})$ such that over $U(\varepsilon)$, we have

$$dv_X = \varrho\, du\, dv_\Sigma \quad \text{and} \quad \varrho(0,x) = |\xi_{f,x}|^{-1}_{g_0^{TX}} \quad \text{for all} \quad x \in \Sigma. \tag{4.31}$$

We can then rewrite (4.28) as

$$\operatorname{Tr}\left[\hat{g}(pQ_p(f))\right] = \sum_{j=1}^{m} \int_{V_j(\varepsilon p^{-\theta/2})} \int_{-\varepsilon p^{-\frac{\theta}{2}}}^{\varepsilon p^{-\frac{\theta}{2}}} \int_{-\varepsilon p^{-\frac{\theta}{2}}}^{\varepsilon p^{-\frac{\theta}{2}}} \operatorname{Tr}\left[I_{j,p}(t,u,x)\right] dt\, du\, dv_\Sigma(x) + O(p^{-\infty}), \tag{4.32}$$

where in the coordinates $(u,x) \in \Psi_{t_j}(U(\varepsilon))$ and for any $t \in]t_j - \varepsilon, t_j + \varepsilon[$, we set

$$I_{j,p}(t,u,x) = g(t_j + t)\varphi^{-1}_{t_j+t,p} \mathcal{T}_{p,t_j+t}((u, \varphi_{u,t_j+t}(x)), (u,x))\varrho(u,x). \tag{4.33}$$

Recall that N denotes the normal bundle of Y_j inside Σ equipped with the metric g^N induced by $g^{T\Sigma}$, and consider the natural identification

$$V_j(\varepsilon) \xrightarrow{\sim} B^N(\varepsilon) := \{w \in N \mid |w|_N < \varepsilon\}. \tag{4.34}$$

As $(\varphi_{t_j}(x) = x)$ implies $(\varphi_{t_j}(\varphi_t(x)) = \varphi_t(x))$ for all $t \in \mathbb{R}$ and $x \in X$ by the 1-parameter group property of φ_t, we know that its flow is transverse to the fibers of the ball bundle $B^N(\varepsilon)$ via the identification (4.34) for $\varepsilon > 0$ small enough. We can then pick $x_0 \in Y_j$ and $u \in]-\varepsilon, \varepsilon[$, and consider the natural embedding defined from the fiber $B^N_{x_0}(\varepsilon)$ of the ball bundle (4.34) over x_0 and a neighborhood $I \subset \mathbb{R}$ of 0 by

$$\begin{aligned} I \times B^N_{x_0}(\varepsilon) &\xrightarrow{\sim} W_u \subset \Sigma \\ (t,w) &\longmapsto \varphi_{u,t_j+t}(w). \end{aligned} \tag{4.35}$$

We identify in turn W_u with a subset of $T_{x_0}\Sigma$ via the inclusion

$$\begin{aligned} I \times B^N_{x_0}(\varepsilon) &\longrightarrow T_{x_0}\Sigma \\ (t,w) &\longmapsto w + t\xi_{f,u}, \quad \text{where } \xi_{f,u} := \frac{\partial}{\partial t}\varphi_{u,t}(x_0) \in T_{x_0}\Sigma. \end{aligned} \tag{4.36}$$

For any $u \in]-\varepsilon, \varepsilon[$, we identify L over W_u with the pullback of L over $B^N_{x_0}(\varepsilon)$ via parallel transport with respect to ∇^L along flow lines of $t \mapsto \varphi_{u,t}$, then trivialize L over $B^N_{x_0}(\varepsilon)$ via parallel transport with respect to ∇^L along radial lines. Then L^p is trivialized by the parallel transport $\tau_{t,p}$ along the flow lines of φ_t, and by the exponentiation of Kostant formula (3.9), we have in this trivialization,

$$\varphi_{t,p}^{-1} = e^{2\sqrt{-1}\pi ptu} \quad \text{for all } |t| < \varepsilon \text{ and } p \in \mathbb{N}. \tag{4.37}$$

Now by the definition (3.1) of ξ_f and as $\{u\} \times \Sigma$ corresponds to the level set $f^{-1}(u)$ via the identification Ψ_{t_j} of (4.18), we know that for any vector field $v \in \mathscr{C}^\infty(W_u, T\Sigma)$, we have

$$R^L\left(v, \xi_f\right) = \frac{2\pi}{\sqrt{-1}}\omega\left(v, \xi_f\right) = 0. \tag{4.38}$$

This shows that the trivialization of L described above coincides with a trivialization along radial lines of W_u in (4.36), for all $u \in]-\varepsilon, \varepsilon[$, so that we are under the hypotheses of Theorem 2.6. As in Definition 3.5 and using (4.19) under the identification Ψ_{t_j} of (4.18), define the action $\lambda_j \in \mathbb{R}$ by

$$\varphi_{t_j,p} =: e^{2\pi\sqrt{-1}p\lambda_j} \quad \text{over }]-\varepsilon, \varepsilon[\times Y_j. \tag{4.39}$$

By a standard computation, which can be found for example in [26, (1.2.31)] and which holds in any trivialization of L along radial lines, the connection ∇^L at $w \in B^N_{x_0}(\varepsilon)$ inside $W_u \subset T_{x_0}\Sigma$ as in (4.36) has the form

$$\nabla^L = d + \frac{1}{2}R^L(w, .) + O(|w|^2). \tag{4.40}$$

Using this formula together with the fact that φ^L preserves ∇^L, we get in our coordinates a smooth bounded function λ_{x_0} of $u \in]-\varepsilon, \varepsilon[$ and $w \in B^N_{x_0}(0, \varepsilon)$ such that

$$\varphi_{t_j,p}^{-1} = e^{-2\pi\sqrt{-1}p\lambda_j} \exp(p|w|^3\lambda_{x_0}(u, w)) \quad \text{over }]-\varepsilon, \varepsilon[\times V_j(\varepsilon). \tag{4.41}$$

We can then apply Theorem 2.6 in a chart of the form (2.28) around (u, x_0) and containing $W_u \subset T_{x_0}\Sigma \subset T_{(u,x_0)}X$ via the identifications (4.36) and (4.18), to get from (4.33) as $p \to +\infty$,

$$\begin{aligned} I_{j,p}(t, u, w) = {} & p^n e^{-2\pi\sqrt{-1}p\lambda_j} g(t_j + t) \exp(p|w|^3\lambda_{x_0}(u, w)) e^{2\sqrt{-1}\pi ptu} \rho(u, w) \\ & \varphi_{t_j+t}^{E,-1} \sum_{r=0}^{k-1} p^{-\frac{r}{2}} G_r \mathscr{T}_{t_j+t,(u,x_0)}(\sqrt{p}\varphi_{u,t_j+t}(w), \sqrt{p}w) + p^n O(p^{-\frac{k}{2}+\delta}), \end{aligned} \tag{4.42}$$

for all $t,\ u \in \mathbb{R}$ with $|t - t_j|,\ |u| < \varepsilon p^{-\theta/2}$ and $w \in B^N_{x_0}(\varepsilon p^{-\theta/2})$. Following (4.14), we apply the Taylor expansion described in (2.31) to the right-hand side of (4.42) in $t,\ u$ and w, inside and outside the exponential of the local model (2.29) for $\mathscr{T}_{t_j+t,(u,x_0)}$.

We then get a family $\{F_{r,x_0}(t,u,w) \in E_{t_j,x_0} \otimes E^*_{0,x_0}\}_{r\in\mathbb{N}}$ of polynomials in $w \in N_{x_0}$ and $t,\ u \in \mathbb{R}$, of the same parity as r, depending smoothly in $x_0 \in Y_j$ and with $F_{0,x_0}(t,u,w) = \bar{\mu}_{t_j}^{-1}(x_0)\tau^E_{t_j,x_0}$ for all t, u and w, such that for any $k \in \mathbb{N}$, there is $M_k \in \mathbb{N}^*$ such that as $p \to +\infty$,

$$I_{j,p}(t,u,w) = p^n e^{-2\pi\sqrt{-1}p\lambda_j} g(t_j)|\xi_{f,x_0}|^{-1}_{g_0^{TX}} \varphi^{E,-1}_{t_j,x_0} \sum_{r=0}^{k-1} p^{-\frac{r}{2}} F_{r,x_0}(\sqrt{p}t,\sqrt{p}u,\sqrt{p}w)$$
$$e^{2\sqrt{-1}\pi ptu}\mathscr{T}_{t_j,x_0}(\sqrt{p}t\xi_{f,x_0} + \sqrt{p}d\varphi_{t_j}.w,\sqrt{p}w) + p^{n-\frac{k}{2}+\delta}O(|\sqrt{p}t|^{M_k}+1). \tag{4.43}$$

Using the non-degeneracy assumption of Definition 2.7 together with the explicit formula (2.29) for the local model, we see that (4.43) decreases exponentially in $t \in \mathbb{R}$ and $w \in N_{x_0}$, but not in $u \in \mathbb{R}$. Now to get an exponential decrease in $u \in \mathbb{R}$, we will make the change of variables $t \mapsto t/\sqrt{p}$ and $u \mapsto u/\sqrt{p}$, integrate first with respect to $t \in \mathbb{R}$ and then with respect to $u \in \mathbb{R}$. With this in mind, from the local model (2.29), we get for any $t,\ u \in \mathbb{R}$ and $w \in N_{x_0}$,

$$e^{2\sqrt{-1}\pi tu}\mathscr{T}_{t_j,x_0}(t\xi_{f,x_0} + d\varphi_{t_j}.w, w) = \mathscr{T}_{t_j,x_0}(d\varphi_{t_j}.w, w)$$
$$\exp\Big[-\pi t^2\langle\Pi_0^{t_j}\xi_{f,x_0},\xi_{f,x_0}\rangle - \pi t\langle(\Pi_0^{t_j} + (\Pi_0^{t_j})^*)\xi_{f,x_0}, d\varphi_{t_j}.w - w\rangle + 2\sqrt{-1}\pi tu\Big]. \tag{4.44}$$

Using the classical formula for the Fourier transform of Gaussian integrals, we compute

$$\begin{aligned}
&\int_{\mathbb{R}}\Big(\int_{\mathbb{R}}\exp\Big[-\pi t^2\langle\Pi_0^{t_j}\xi_{f,x_0},\xi_{f,x_0}\rangle \\
&\qquad -\pi t\langle(\Pi_0^{t_j}+(\Pi_0^{t_j})^*)\xi, d\varphi_{t_j}.w - w\rangle + 2\sqrt{-1}\pi tu\Big]\mathrm{d}t\Big)\mathrm{d}u \\
&= \langle\Pi_0^{t_j}\xi_{f,x_0},\xi_{f,x_0}\rangle^{-\frac{1}{2}} \\
&\int_{\mathbb{R}}\exp\left[-\frac{\pi}{4}\frac{\Big(2u+\sqrt{-1}\langle(\Pi_0^{t_j}+(\Pi_0^{t_j})^*)\xi_{f,x_0}, d\varphi_{t_j}.w - w\rangle\Big)^2}{\langle\Pi_0^{t_j}\xi_{f,x_0},\xi_{f,x_0}\rangle}\right]\mathrm{d}u \\
&= \langle\Pi_0^{t_j}\xi_{f,x_0},\xi_{f,x_0}\rangle^{-\frac{1}{2}}\int_{\mathbb{R}}\exp\left[-\pi\frac{u^2}{\langle\Pi_0^{t_j}\xi_{f,x_0},\xi_{f,x_0}\rangle}\right]\mathrm{d}u = 1,
\end{aligned} \tag{4.45}$$

for a suitable choice of square root in the middle terms. In particular, this computation shows that we get an exponential decrease in $u \in \mathbb{R}$ after integration with respect to $t \in \mathbb{R}$. Using successive integration by parts, this computation readily generalizes to the case when the exponential is multiplied by a polynomial $F_{r,x_0}(t,u,w)$ of the same parity as $r \in \mathbb{N}$, to get as a result a polynomial $H_{r,x_0}(w) \in E_{t_j,x_0}\otimes E^*_{0,x_0}$ in $w \in N_{x_0}$ of the same parity as $r \in \mathbb{N}$, with $H_{0,x_0}(w) = \bar{\mu}_{t_j}^{-1}(x_0)\tau^E_{t_j,x_0}$ for all $w \in N_{x_0}$. Thus from (4.43), for all $k \in \mathbb{N}$ we get $\delta_k \in]0,1[$ as in (4.15) such that for all $w \in B^N_{x_0}(\varepsilon p^{-\theta/2})$ as $p \to +\infty$,

$$\begin{aligned}
&\int_{-\varepsilon p^{-\frac{\theta}{2}}}^{\varepsilon p^{-\frac{\theta}{2}}} \int_{-\varepsilon p^{-\frac{\theta}{2}}}^{\varepsilon p^{-\frac{\theta}{2}}} \operatorname{Tr}\left[I_{j,p}(t,u,w)\right] \mathrm{d}t\, \mathrm{d}u \\
&= p^{-1} \int_{-\varepsilon p^{\frac{1-\theta}{2}}}^{\varepsilon p^{\frac{1-\theta}{2}}} \int_{-\varepsilon p^{\frac{1-\theta}{2}}}^{\varepsilon p^{\frac{1-\theta}{2}}} \operatorname{Tr}\left[I_{j,p}(p^{-1/2}t, p^{-1/2}u, w)\right] \mathrm{d}t\, \mathrm{d}u \\
&= p^{-1} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \operatorname{Tr}\left[I_{j,p}(p^{-1/2}t, p^{-1/2}u, w)\right] \mathrm{d}t \right) \mathrm{d}u + O(p^{-\infty}) \\
&= p^{n-1} e^{-2\pi\sqrt{-1}p\lambda_j} g(t_j) \sum_{r=0}^{k-1} p^{-\frac{r}{2}} \operatorname{Tr}\left[\varphi_{t_j,x_0}^{E,-1} H_{r,x_0}(\sqrt{p}w)\right] \\
&\quad \mathscr{T}_{t_j,x_0}(\sqrt{p}d\varphi_{t_j}.w, \sqrt{p}w) + p^{n-1} O(p^{-\frac{k}{2}+\delta_k}).
\end{aligned} \tag{4.46}$$

Working locally, we can suppose that Y_j is orientable. Let $\mathrm{d}v_{Y_j}$ be the Riemannian volume form on Y_j induced by $g^{T\Sigma}$, let $\mathrm{d}w$ be the Euclidean volume form on the fibers of (N, g^N) and let $\rho \in \mathscr{C}^\infty(B^N(\varepsilon), \mathbb{R})$ be such that via the identification (4.34) of $V_j(\varepsilon)$ with the ball bundle $B^N(\varepsilon)$ over Y_j, we have

$$\mathrm{d}v_\Sigma = \rho\, \mathrm{d}w\, \mathrm{d}v_{Y_j} \quad \text{and} \quad \rho(0,x) = 1 \ \text{ for all } \ x \in Y_j. \tag{4.47}$$

Through the change of variable $w \mapsto w/\sqrt{p}$, using the exponential decrease in (4.46) in $w \in N_{x_0}$ and taking the Taylor expansion in $w \in N_{x_0}$ of $\rho(w, x_0)$ as described in (2.31) for all $x_0 \in Y_j$, we then get polynomials $K_{r,x_0} \in \mathbb{C}[w]$ of the same parity as $r \in \mathbb{N}$, with $K_{0,x_0}(w) = \bar{\mu}_{t_j}^{-1}(x_0) \operatorname{Tr}\left[\varphi_{t_j,x_0}^{E,-1} \tau_{t_j,x_0}^E\right]$, such that using (4.46), we can rewrite (4.32) as

$$\begin{aligned}
&\operatorname{Tr}\left[\hat{g}(pQ_p(f))\right] = p^{(\dim Y_j - 1)/2}\Big[\int_{x\in Y_j} \Big(\int_{B_x^N(\varepsilon p^{(1-\theta)/2})} e^{-2\pi\sqrt{-1}p\lambda_j} g(t_j) \\
&\qquad \rho(p^{-1/2}w, x) \sum_{r=0}^{k-1} p^{-\frac{r}{2}} K_{r,x}(w) \mathscr{T}_{t_j,x}(d\varphi_{t_j}.w, w)\mathrm{d}w \Big) \mathrm{d}v_{Y_j}(x) + O(p^{-\frac{k}{2}+\delta_k'}) \Big] \\
&= p^{(\dim Y_j - 1)/2}\Big[e^{-2\pi\sqrt{-1}p\lambda_j} g(t_j) \sum_{q=0}^{\lfloor \frac{k-1}{2} \rfloor} p^{-q} \\
&\qquad \int_{x\in Y_j} \Big(\int_{N_x} K_{2q,x}(w) \mathscr{T}_{t_j,x}(d\varphi.w, w)\mathrm{d}w \Big) \frac{\mathrm{d}v_{Y_j}}{|\xi_f|_{g_0^{TX}}}(x) + O(p^{-\frac{k}{2}+\delta_k'}) \Big],
\end{aligned} \tag{4.48}$$

for some $\delta_k' \in]0,1[$ satisfying $\delta_k' \to \delta$ as $\theta \to 1$, where all the terms with r odd vanished in the same way as in (4.17) by the odd parity of $K_{2q+1,x}$ for all $q \in \mathbb{N}$. This shows the expansion (4.21), and formula (4.22) follows from (4.48) and the second equality of (2.41). Finally, formula (4.23) follows from the fact that function $\mu_t \in \mathscr{C}^\infty(X, \mathbb{C})$ of (2.20) is constant equal to 1 for $t = 0$ and the fact that the vector field $v \in \mathscr{C}^\infty(f^{-1}(c), TX)$ defining the volume form (4.20) over $f^{-1}(c)$ can be chosen to be $v = J_0\xi_f/|\xi_f|^2_{g_0^{TX}}$. □

Under the assumptions of Theorem 1.2, consider now the case when $Y_j \subset \Sigma$ satisfy $\dim Y_j = 1$ for some $1 \leqslant j \leqslant m$, so that $Y_j = \{\varphi_s(x)\}_{0 \leqslant s < t(Y_j)}$, for some $x \in f^{-1}(c)$ satisfying $\varphi_t(x) = x$. Here $t(Y_j) > 0$ is the smallest time $t > 0$ for which $\varphi_t(x) = x$, called the *primitive period* of Y_j. We then get the following special case of Theorem 4.3, recovering the explicit geometric term associated with isolated periodic orbits in the Gutzwiller trace formula of Theorem 1.3.

Theorem 4.4 *Consider the hypotheses of Theorem 1.2, and let* $1 \leqslant j \leqslant m$ *be such that* $\dim Y_j = 1$ *and such that* $[J_0\xi_f, \xi_f] = 0$ *over* Y_j. *Then the term* $b_{j,0} \in \mathbb{C}$ *of* (4.22) *is given by*

$$b_{j,0} = (-1)^{\frac{n-1}{2}} \int_{Y_j} \frac{(\varphi_{t_j}^{K_X,-1}\tau_{t_j}^{K_X})^{-\frac{1}{2}} \operatorname{Tr}[\varphi_{t_j}^{E,-1}\tau_{t_j}^{E}]}{|\det{}_N(\mathrm{Id}_N - d\varphi_{t_j}|_N)|^{1/2}} \frac{|dv|_{Y_j}}{|\xi_f|_{g_0^{TX}}}, \tag{4.49}$$

for some natural choices of square roots. If X *admits a metaplectic structure* (2.43) *and taking* $E = K_X^{1/2}$ *to be the associated metaplectic correction, then formula* (4.49) *becomes*

$$b_{j,0} = (-1)^{\frac{n-1}{2}} \frac{t(Y_j)}{|\det_{N_x}(\mathrm{Id}_N - d\varphi_{t_j}|_N)|^{1/2}}, \tag{4.50}$$

not depending on $x \in Y_j$.

Proof The assumption $[J_0\xi_f, \xi_f] = 0$ over Y_j means that $d\varphi_t . J\xi_f = J\xi_f$ over Y_j for all $t \in \mathbb{R}$. As $d\varphi_t.\xi_f = \xi_f$ by definition and as $d\varphi_t$ preserves the symplectic form ω, by definition (2.11) of g_0^{TX}, this implies that $d\varphi_t$ preserves the normal bundle $N \subset T\Sigma$ of Y_j inside Σ, for all $t \in \mathbb{R}$. We are then under the assumptions of Proposition 2.9, and formula (4.49) is a consequence of Theorem 4.3.

Note now that from the 1-parameter group property of φ_t, for any $t \in \mathbb{R}$ and $x \in Y_j$, we have

$$d\varphi_{t_j,\varphi_t(x)} = d\varphi_{t,x} d\varphi_{t_j,x} d\varphi_{t,x}^{-1}. \tag{4.51}$$

This shows that the quantity $\det_{N_x}(\mathrm{Id}_N - d\varphi_{t_j}|_N)$ is actually independent of $x \in Y_j$. Considering the case $E = K_X^{1/2}$ and recalling that ξ_f is the tangent vector field of the curve Y_j, we get

$$\int_{Y_j} (\varphi_{t_j}^{K_X,-1}\tau_{t_j}^{K_X})^{-\frac{1}{2}} \operatorname{Tr}_E[\varphi_{t_j}^{E,-1}\tau_{t_j}^{E}] \frac{|dv|_{Y_j}}{|\xi_f|_{g_0^{TX}}} = \int_{Y_j} \frac{|dv|_{Y_j}}{|\xi_f|_{g_0^{TX}}} = t(Y_j). \tag{4.52}$$

This gives formula (4.50). □

Let us now show how these results can be applied to contact topology. To that end, consider $f \in \mathscr{C}^\infty(X, \mathbb{R})$ and an almost complex structure $J \in \operatorname{End}(TX)$ satisfying

$$d\iota_{J\xi_f}\omega = \omega \;\text{ over }\; f^{-1}(I), \tag{4.53}$$

over some interval $I \subset \mathbb{R}$ of regular values of f. Then considering the Riemannian metric $g^{TX} = \omega(\cdot, J\cdot)$ as in (2.11), the form

$$-\frac{\iota_{J\xi_f}\omega}{|\xi_f|^2_{g^{TX}}} \in \Omega^1(X, \mathbb{R}) \tag{4.54}$$

restricts to a *contact form* $\alpha \in \Omega^1(\Sigma, \mathbb{R})$ over $\Sigma := f^{-1}(c)$ for any $c \in I$, meaning that $\alpha \wedge d\alpha^{n-1}$ is a volume form over Σ. The restriction of the Hamiltonian vector field $\xi_f \in \mathscr{C}^\infty(X, TX)$ over $f^{-1}(c)$ induces the *Reeb vector field* of (Σ, α), which is the unique vector field $\xi \in \mathscr{C}^\infty(\Sigma, T\Sigma)$ satisfying

$$\begin{cases} \iota_\xi \alpha = 1, \\ \iota_\xi d\alpha = 0. \end{cases} \tag{4.55}$$

The corresponding *Reeb flow* is the flow of diffeomorphisms $\varphi_t : \Sigma \to \Sigma$ generated by ξ, and its periodic orbits are called the *Reeb orbits* of (Σ, α). An isolated Reeb orbit of period $t_0 \in \mathbb{R}$ is said to be *non-degenerate* if it satisfies Definition 2.7 as a fixed point set of φ_{t_0} inside Σ.

Conversely, given a compact manifold Σ endowed with a contact form $\alpha \in \Omega^1(\Sigma, \mathbb{R})$, we define a symplectic form $\omega^{S\Sigma} \in \Omega^2(S\Sigma, \mathbb{R})$ over $S\Sigma := \mathbb{R} \times \Sigma$ by the formula

$$\omega^{S\Sigma} := -d(e^u \pi^* \alpha), \qquad \text{with} \qquad \begin{aligned} \pi : S\Sigma := \mathbb{R} \times \Sigma &\longrightarrow \Sigma. \\ (u, x) &\longmapsto x \end{aligned} \tag{4.56}$$

The symplectic manifold $(S\Sigma, \omega^{S\Sigma})$ is called the *symplectization* of (Σ, α). Consider the function $f \in \mathscr{C}^\infty(S\Sigma, \mathbb{R})$ defined by

$$f(u, x) := e^u \ \text{ for all } \ (u, x) \in \mathbb{R} \times \Sigma. \tag{4.57}$$

Then its Hamiltonian vector field $\xi_f \in \mathscr{C}^\infty(S\Sigma, TS\Sigma)$ restricts over any level set of f to the Reeb vector field $\xi \in \mathscr{C}^\infty(\Sigma, T\Sigma)$ of (Σ, α). On the other hand, consider an almost complex structure $J \in \mathrm{End}(TS\Sigma)$ compatible with $\omega^{S\Sigma}$ satisfying

$$J\xi_f = \frac{\partial}{\partial u}, \tag{4.58}$$

in the coordinates $(u, x) \in \mathbb{R} \times \Sigma = S\Sigma$. Such an almost complex structure always exists. Finally, consider the situation when an open set of the form $I \times \Sigma \subset S\Sigma$ can be symplectically embedded into a compact prequantized symplectic manifold (X, ω) without boundary. We can then extend the Hamiltonian and the almost complex structure defined by (4.57) and (4.58) to the whole of (X, ω) using partitions of unity. The typical case when such an embedding exists is when Σ is the boundary of a star-shaped domain in $\mathbb{R}^{2n}$, with contact form $\alpha \in \Omega^1(\Sigma, \mathbb{R})$ given by the restriction of the standard *Liouville form* $\lambda \in \Omega^1(\mathbb{R}^{2n}, \mathbb{R})$. Then using a Darboux chart, an open set of the form $I \times \Sigma \subset S\Sigma$ can always be symplectically embedded in (X, ω). More generally, by the results of [15, Th. 1.3] and [24, Cor. 1.11], any *fillable* contact manifold satisfies this property. The following result then shows how to use the above picture to extract pieces of information on isolated non-degenerate Reeb orbits of (Σ, α) from the geometric quantization of (X, ω).

Proposition 4.5 *Let Σ be a compact manifold without boundary endowed with a contact form $\alpha \in \Omega^1(\Sigma, \mathbb{R})$, and assume that an open set of the form $I \times \Sigma$ in its symplectization $(S\Sigma, \omega^{S\Sigma})$ can be symplectically embedded in a compact pre-quantized symplectic manifold (X, ω) without boundary. Consider the Hamiltonian $f \in \mathscr{C}^\infty(X, \mathbb{R})$ and the almost complex structure $J \in \mathrm{End}(TX)$ defined via (4.57) and (4.58).*

Then for any $c \in I$, Theorem 1.2 holds as soon as the fixed point set of the Reeb flow $\varphi_t : \Sigma \to \Sigma$ is non-degenerate for all $t \in \mathrm{Supp}\, g$, and Theorem 1.3 holds for the non-degenerate isolated Reeb orbits of (Σ, α) in case (X, ω) is endowed with a metaplectic structure.

Proof Recall that the Hamiltonian vector field $\xi_f \in \mathscr{C}^\infty(S\Sigma, TS\Sigma)$ of $f \in \mathscr{C}^\infty(S\Sigma, \mathbb{R})$ defined by (4.57) restricts to the Reeb vector field $\xi \in \mathscr{C}^\infty(\Sigma, T\Sigma)$ of (Σ, α) over $\{u\} \times \Sigma$, for all $u \in \mathbb{R}$. Then the Hamiltonian flow $\varphi_t : X \to X$ of f over $I \times \Sigma$ is of the form $\varphi_t(u, x) = (u, \varphi_t(x))$ for all $(u, x) \in I \times \Sigma$. It thus satisfies the hypotheses of Theorem 1.2 as long as the fixed point set of the Reeb flow $\varphi_t : \Sigma \to \Sigma$ is non-degenerate for all $t \in \mathrm{Supp}\, g$, the discreteness of $T \subset \mathrm{Supp}\, g$ following from the fact that $d\varphi_t$ preserves $\mathrm{Ker}\, \alpha \subset T\Sigma$ for all $t \in \mathbb{R}$, so that by (4.55) there is no $v \in \mathrm{Ker}\, \alpha_x$ such that $\xi_x + d\varphi_t.v = v$ for any $x \in \Sigma$ satisfying $\varphi_t(x) = x$. On the other hand, the formula (4.58) shows

$$[J\xi_f, \xi_f] = \left[\frac{\partial}{\partial u}, \xi\right] = 0 \text{ over } \mathbb{R} \times \Sigma, \tag{4.59}$$

so that the hypotheses of Theorem 1.3 are satisfied as well. This shows the result. □

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Compliance with ethical standards

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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