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Berezin–Toeplitz Quantization for Eigenstates of the Bochner Laplacian on Symplectic Manifolds

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Abstract We study the Berezin–Toeplitz quantization using as quantum space the space of eigenstates of the renormalized Bochner Laplacian corresponding to eigenvalues localized near the origin on a symplectic manifold. We show that this quantization has the correct semiclassical behavior and construct the corresponding star-product.

To Gennadi Henkin, in memoriam.

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1 Introduction

Quantization is a recipe in physics for passing from a classical system to a quantum system by obeying certain natural rules. By a classical system we understand a classical phase space (a symplectic manifold) and classical observables (smooth functions). The quantum system consists of a quantum space (a Hilbert space of functions or sections of a bundle) and quantum observables (bounded linear operators on the quantum space). The quantum system should reduce to the classical one as the size of the objects gets large, that is, as the "Planck constant," which, heuristically, corresponds to the magnitude where the quantum phenomena become relevant, tends to zero. This is the so-called semiclassical limit.

The original concept of quantization goes back to Weyl, von Neumann, and Dirac. In the geometric quantization introduced by Kostant [16] and Souriau [27], the quantum space is the Hilbert space of square integrable holomorphic sections of a prequantum line bundle (see also [1–3,8,11]). Berezin–Toeplitz quantization is a particularly efficient version of the geometric quantization theory. Toeplitz operators and more generally Toeplitz structures were introduced in geometric quantization by Berezin [4] and Boutet de Monvel and Guillemin [7]. Using the analysis of Toeplitz structures [7], Bordemann et al. [6] and Schlichenmaier [25,26] showed that the Berezin–Toeplitz quantization on a compact Kähler manifold satisfies the correspondence principle asymptotically and introduced the Berezin–Toeplitz star product (cf. (1.17) and (1.18)) when $E = \mathbb{C}$ and $g^{TX}(\cdot,\cdot) = \omega(\cdot, J\cdot)$.

In order to generalize the Berezin–Toeplitz quantization to arbitrary symplectic manifolds, one has to find a substitute for the space of holomorphic sections of tensor powers of the prequantum line bundle. A natural candidate is the kernel of the Dirac operator, since it has similar features to the space of holomorphic sections in the Kähler case, especially the asymptotics of the kernels of the orthogonal projection on both spaces [10]. The Berezin–Toeplitz quantization with quantum space the kernel of the Dirac operator was carried over by Ma and Marinescu [22] see also [23].

Another appealing candidate is the space of eigenstates of the renormalized Bochner Laplacian [12,20,21] corresponding to eigenvalues localized near the origin, cf. (1.7), (1.8). In this paper, we construct the Berezin–Toeplitz quantization for these spaces and show that it has the correct semiclassical behavior. The difference between this case and the quantization by the kernel of the Dirac operator comes from the possible presence of eigenvalues localized near the origin but different from zero. In this situation the analysis becomes more difficult.

Let us note also that Charles [9] developed a Berezin–Toeplitz type quantization on compact symplectic prequantizable manifolds by using a semiclassical approach to the Boutet de Monvel and Guillemin theory [7], and Hsiao and Marinescu [13] constructed a Berezin–Toeplitz quantization for eigenstates of small eigenvalues in the case of arbitrary complex manifolds.

The readers are referred to the monograph [20] (also [18]) for a comprehensive study of the (generalized) Bergman kernel, Berezin–Toeplitz quantization and its applications.

Let us describe the setting and results in detail. Let (X, ω) be a compact symplectic manifold of real dimension $2n$. Let (L, h^L) be a Hermitian line bundle on X, and let ∇^L be a Hermitian connection on (L, h^L) with the curvature $R^L = (\nabla^L)^2$. Let (E, h^E) be a Hermitian vector bundle with Hermitian connection ∇^E. We will assume throughout the paper that L is a line bundle satisfying the prequantization condition

$$\frac{\sqrt{-1}}{2\pi} R^L = \omega. \tag{1.1}$$

We choose an almost complex structure J such that ω is J-invariant. The almost complex structure J induces a splitting $TX \otimes_{\mathbb{R}} \mathbb{C} = T^{(1,0)}X \oplus T^{(0,1)}X$, where $T^{(1,0)}X$ and $T^{(0,1)}X$ are the eigenbundles of J corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$, respectively.

Let g^{TX} be a J-invariant Riemannian metric on TX. Let $\mathrm{d}v_X$ be the Riemannian volume form of (X, g^{TX}). The L^2-Hermitian product on the space $\mathscr{C}^\infty(X, L^p \otimes E)$ of smooth sections of $L^p \otimes E$ on X, with $L^p := L^{\otimes p}$, is given by

$$\langle s_1, s_2 \rangle = \int_X \langle s_1(x), s_2(x) \rangle \mathrm{d}v_X(x), \tag{1.2}$$

where $\langle \cdot, \cdot \rangle$ in the integrand is the pointwise Hermitian product on $L^p \otimes E$ induced by h^L, h^E. Let ∇^{TX} be the Levi–Civita connection on (X, g^{TX}), and let $\nabla^{L^p \otimes E}$ be the connection on $L^p \otimes E$ induced by ∇^L and ∇^E. Let $\{e_k\}$ be a local orthonormal frame of (TX, g^{TX}). The Bochner Laplacian acting on $\mathscr{C}^\infty(X, L^p \otimes E)$ is given by

$$\Delta^{L^p \otimes E} = -\sum_k \left[\left(\nabla^{L^p \otimes E}_{e_k}\right)^2 - \nabla^{L^p \otimes E}_{\nabla^{TX}_{e_k} e_k} \right]. \tag{1.3}$$

Let $\Phi \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ be Hermitian (i.e., self-adjoint with respect to h^E). The renormalized Bochner Laplacian is defined by

$$\Delta_{p,\Phi} = \Delta^{L^p \otimes E} - \tau p + \Phi, \quad \text{with } \tau = \frac{\sqrt{-1}}{2} \sum_k R^L(e_k, J e_k). \tag{1.4}$$

Write $|\cdot|_{g^{TX}}$ for the Hermitian norm induced by g^{TX} on $T^{(1,0)}X$, and set

$$\mu_0 = \inf_{u \in T_x^{(1,0)}X,\ x \in X} R^L(u, \overline{u}) / |u|^2_{g^{TX}}. \tag{1.5}$$

By [12], [19, Corollary 1.2], [20, Theorem 8.3.1], there exists $C_L > 0$ independent of p such that

$$\mathrm{Spec}(\Delta_{p,\Phi}) \subset [-C_L, C_L] \cup [2\mu_0 p - C_L, +\infty), \tag{1.6}$$

where $\mathrm{Spec}(A)$ denotes the spectrum of the operator A. Since $\Delta_{p,\Phi}$ is an elliptic operator on a compact manifold, it has discrete spectrum and its eigensections are smooth. Let

$$\mathcal{H}_p := \bigoplus_{\lambda \in [-C_L, C_L]} \mathrm{Ker}(\Delta_{p,\Phi} - \lambda) \subset \mathscr{C}^\infty(X, L^p \otimes E) \tag{1.7}$$

be the direct sum of eigenspaces of $\Delta_{p,\Phi}$ corresponding to the eigenvalues lying in $[-C_L, C_L]$.

In mathematical physics terms, the operator $\Delta_{p,\Phi}$ is a semiclassical Schrödinger operator, and the space $\mathcal{H}_p$ is the space of its bound states as $p \to \infty$. The space $\mathcal{H}_p$ proves to be an appropriate replacement for the space of holomorphic sections $H^0(X, L^p \otimes E)$ from the Kähler case. Indeed, if (X, ω) is Kähler, then $\mathcal{H}_p = H^0(X, L^p \otimes E)$ for p large enough. Moreover, for an arbitrary compact prequantized symplectic manifold (X, ω) as above, the dimension of the space $\mathcal{H}_p$ is given for p large enough as in the Kähler case by the Riemann–Roch–Hirzebruch formula: see [19, Corollary 1.2], [20, Theorem 8.3.1], [5, Theorem 4.9],

$$d_p := \dim \mathcal{H}_p = \langle \mathrm{Td}(X)\,\mathrm{ch}(L^p \otimes E), [X] \rangle \sim p^n (\mathrm{rank}\, E)\, \mathrm{vol}_\omega(X), \quad p \gg 1. \tag{1.8}$$

Another striking similarity is the fact that the kernel of the orthogonal projection on $\mathcal{H}_p$ has an asymptotic expansion analogous to the Bergman kernel expansion for Kähler manifolds, see [20,21]. We will use the asymptotic expansion of [20,21] together with the approach of [22] to Berezin–Toeplitz quantization in order to derive the properties of Toeplitz operators modeled on the projection on $\mathcal{H}_p$.

Let $P_{\mathcal{H}_p}$ be the orthogonal projection from $\mathscr{C}^\infty(X, L^p \otimes E)$ onto $\mathcal{H}_p$. The kernel $P_{\mathcal{H}_p}(x, x')$ of $P_{\mathcal{H}_p}$ with respect to $dv_X(x')$ is called a generalized Bergman kernel [21]. Note that $P_{\mathcal{H}_p}(x, x') \in (L^p \otimes E)_x \otimes (L^p \otimes E)^*_{x'}$. For a smooth section $f \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ of the bundle $\mathrm{End}(E)$, we define the Berezin–Toeplitz quantization of f by

$$T_{f,p} = P_{\mathcal{H}_p} f P_{\mathcal{H}_p} \in \mathrm{End}(L^2(X, L^p \otimes E)), \tag{1.9}$$

where we denote for simplicity by f the endomorphism of $L^2(X, L^p \otimes E)$ induced by f, namely, $s \mapsto fs$, with $(fs)(x) = f(x)(s(x))$, for $s \in L^2(X, L^p \otimes E)$ and $x \in X$.

Definition 1.1 A Toeplitz operator is a sequence $\{T_p\} = \{T_p\}_{p \in \mathbb{N}}$ of linear operators

$$T_p : L^2(X, L^p \otimes E) \longrightarrow L^2(X, L^p \otimes E) \tag{1.10}$$

with the following properties:

(i) For any $p \in \mathbb{N}$, we have

$$T_p = P_{\mathcal{H}_p} T_p P_{\mathcal{H}_p}; \tag{1.11}$$

(ii) There exist a sequence $g_l \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ such that for all $k \geqslant 0$ there exists $C_k > 0$ with

$$\left\| T_p - P_{\mathcal{H}_p} \left(\sum_{l=0}^{k} p^{-l} g_l \right) P_{\mathcal{H}_p} \right\| \leqslant C_k p^{-k-1}, \tag{1.12}$$

where $\|\cdot\|$ denotes the operator norm on the space of the bounded operators.

If each T_p is self-adjoint, then $\{T_p\}_{p\in\mathbb{N}}$ is called self-adjoint.

We express (1.12) symbolically by

$$T_p = \sum_{l=0}^{k} p^{-l} T_{g_l,p} + \mathcal{O}(p^{-k-1}). \tag{1.13}$$

If (1.12) holds for any $k \in \mathbb{N}$, then we write

$$T_p = \sum_{l=0}^{\infty} p^{-l} T_{g_l,p} + \mathcal{O}(p^{-\infty}). \tag{1.14}$$

The main result of this paper is as follows.

Theorem 1.2 *Let (X, J, ω) be a compact symplectic manifold, (L, h^L, ∇^L), (E, h^E, ∇^E) be Hermitian vector bundles as above, and g^{TX} be a J-compatible Riemannian metric on TX. Let $f, g \in \mathscr{C}^\infty(X, \mathrm{End}(E))$. Then the product of the Toeplitz operators $T_{f,p}$ and $T_{g,p}$ is a Toeplitz operator, more precisely, it admits the asymptotic expansion in the sense of (1.14):*

$$T_{f,p} T_{g,p} = \sum_{r=0}^{\infty} p^{-r} T_{C_r(f,g),p} + \mathcal{O}(p^{-\infty}), \tag{1.15}$$

where C_r are bidifferential operators, $C_r(f, g) \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ and $C_0(f, g) = fg$.

If $f, g \in \mathscr{C}^\infty(X)$, then we have

$$C_1(f, g) - C_1(g, f) = \sqrt{-1}\{f, g\}\,\mathrm{Id}_E, \tag{1.16}$$

where $\{\cdot,\cdot\}$ is the Poisson bracket on $(X, 2\pi\omega)$, and therefore the correspondence principle holds asymptotically,

$$[T_{f,g}, T_{g,p}] = \frac{\sqrt{-1}}{p} T_{\{f,g\},p} + O(p^{-2}), \quad p \to \infty. \tag{1.17}$$

Corollary 1.3 *Let $f, g \in \mathscr{C}^\infty(X, \mathrm{End}(E))$. Set*

$$f * g := \sum_{k=0}^{\infty} C_k(f, g)\hbar^k \in \mathscr{C}^\infty(X, \mathrm{End}(E))[[\hbar]], \tag{1.18}$$

where $C_r(f, g)$ are determined by (1.15). Then (1.18) defines an associative star-product on $\mathscr{C}^\infty(X, \mathrm{End}(E))$.

Theorem 1.4 *For any $f \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ the operator norm of $T_{f,\,p}$ satisfies*

$$\lim_{p\to\infty} \|T_{f,\,p}\| = \|f\|_\infty := \sup_{0\neq u\in E_x, x\in X} |f(x)(u)|_{h^E}/|u|_{h^E} . \tag{1.19}$$

In the special case when the Riemannian metric g^{TX} is associated with ω, we can even calculate $C_1(f, g)$, not only the difference $C_1(f, g) - C_1(g, f)$ from (1.16). To state the result, let

$$\begin{aligned} (\nabla^E)^{1,0} &: \mathscr{C}^\infty(X, \mathrm{End}(E)) \to \mathscr{C}^\infty(X, T^{*(1,0)}X \otimes \mathrm{End}(E)),\\ (\nabla^E)^{0,1} &: \mathscr{C}^\infty(X, \mathrm{End}(E)) \to \mathscr{C}^\infty(X, T^{*(0,1)}X \otimes \mathrm{End}(E)), \end{aligned} \tag{1.20}$$

be the $(1, 0)$-component and $(0, 1)$-component respectively of the connection ∇^E, and let $\langle\cdot,\cdot\rangle$ denote the pairing induced by g^{TX} on $T^*X \otimes \mathrm{End}(E)$ with values in $\mathrm{End}(E)$.

Following an argument of [14], we get the last result of this paper.

Theorem 1.5 *If $g^{TX}(\cdot,\cdot) = \omega(\cdot, J\cdot)$, then for any $f, g \in \mathscr{C}^\infty(X, \mathrm{End}(E))$, the coefficient $C_1(f, g) \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ defined in (1.15) is given by*

$$C_1(f, g) = -\frac{1}{2\pi}\big\langle(\nabla^E)^{1,0} f, (\nabla^E)^{0,1} g\big\rangle. \tag{1.21}$$

Note that this formula is clearly compatible with the formula (1.16) for the Poisson bracket in the case $f, g \in \mathscr{C}^\infty(X)$. Note also that (1.21) is a direct generalization of the formula [24, (0.20)] for Kähler manifolds. The formula for the second coefficient C_1(f,g), as well as other formulas for the coefficients of the expansions of Toeplitz operators, is remarkably universal, that is, it has the same form in different geometric contexts (Kähler case [24], spin-c case [14], spectral spaces [13], see also [28] for an interpretation in graph-theoretic terms, and see [14] for further references).

We completed this paper a while ago. Recently Kordyukov informed us about his preprint [15] in which the Berezin–Toeplitz quantization by eigenstates of the Bochner Laplacian is reconsidered.

The organization of the paper is as follows. In Sect. 2, we recall the asymptotic expansion of the generalized Bergman kernel obtained in [17]. In Sect. 3, we obtain the asymptotic expansion of the kernel of a Toeplitz operator. In Sect. 4, we show that the asymptotic expansion is also a sufficient condition for a family to be Toeplitz. In Sect. 5, we conclude that the set of Toeplitz operators forms an algebra. In Sect. 6, we prove Theorem 1.5.

2 The Asymptotic Expansion of the Generalized Bergman Kernel

Let a^X be the injectivity radius of (X, g^{TX}). Let $d(x, y)$ denote the Riemannian distance from x to y on (X, g^{TX}). By [20, Proposition 8.3.5] (cf. also [17, (1.11)]), we have the following far off-diagonal behavior of the generalized Bergman kernel.

Proposition 2.1 *For any $b > 0$ and any $k, l \in \mathbb{N}$ and $0 < \theta < 1$, there exists $C_{b,k,l,\theta} > 0$ such that*

$$\left| P_{\mathcal{H}_p}(x, x') \right|_{\mathscr{C}^k(X \times X)} \leqslant C_{b,k,l,\theta}\, p^{-l}, \quad \text{for } d(x, x') > bp^{-\frac{\theta}{2}}, \tag{2.1}$$

here the $\mathscr{C}^k$-norm is induced by $\nabla^L, \nabla^E, h^L, h^E$ and g^{TX}.

Let $\varepsilon \in (0, a^X/4)$ be fixed. We denote by $B^X(x, \varepsilon)$ and $B^{T_x X}(0, \varepsilon)$ the open balls in X and $T_x X$ with center x and radius ε, respectively. We identify $B^{T_x X}(0, \varepsilon)$ with $B^X(x, \varepsilon)$ by using the exponential map of (X, g^{TX}).

Let $x_0 \in X$. For $Z \in B^{T_{x_0} X}(0, \varepsilon)$, we identify (L_Z, h^L_Z), (E_Z, h^E_Z) and $(L^p \otimes E)_Z$ to $(L_{x_0}, h^L_{x_0})$, $(E_{x_0}, h^E_{x_0})$ and $(L^p \otimes E)_{x_0}$ by parallel transport with respect to the connections ∇^L, ∇^E and $\nabla^{L^p \otimes E}$ along the curve $\gamma_Z : [0, 1] \ni u \to \exp^X_{x_0}(uZ)$. This is the basic trivialization we use in this paper.

Using this trivialization we identify $f \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ to a family $\{f_{x_0}\}_{x_0 \in X}$ where f_{x_0} is the function f in normal coordinates near x_0, i.e., $f_{x_0} : B^{T_{x_0} X}(0, \varepsilon) \to \mathrm{End}(E_{x_0})$, $f_{x_0}(Z) = f \circ \exp^X_{x_0}(Z)$. In general, for functions expressed in normal coordinates centered at $x_0 \in X$, we will add a subscript x_0 to indicate the base point x_0.

Similarly, $P_{\mathcal{H}_p}(x, x')$ induces in terms of the basic trivialization a smooth section

$$(Z, Z') \longmapsto P_{\mathcal{H}_p, x_0}(Z, Z')$$

of $\pi^* \mathrm{End}(E)$ over $\{(Z, Z') \in TX \times_X TX : |Z|, |Z'| < \varepsilon\}$, which depends smoothly on x_0. Here $\pi : TX \times_X TX \to X$ is the natural projection from the fibered product $TX \times_X TX$ on X and we identify a section $S \in \mathscr{C}^\infty\big(TX \times_X TX, \pi^* \mathrm{End}(E)\big)$ with the family $(S_x)_{x \in X}$, where $S_x = S|_{\pi^{-1}(x)}$.

Let dv_{TX} be the Riemannian volume form on $(T_{x_0} X, g^{T_{x_0} X})$. Let $\kappa_{x_0}(Z)$ be the smooth positive function defined by the equation

$$dv_X(Z) = \kappa_{x_0}(Z) dv_{TX}(Z), \quad \kappa_{x_0}(0) = 1, \tag{2.2}$$

where the subscript x_0 of $\kappa_{x_0}(Z)$ indicates the base point $x_0 \in X$.

Writing $\langle \cdot, \cdot \rangle$ for the $\mathbb{C}$-bilinear product induced by g^{TX} on $T^{(1,0)} X$, we identify the 2-form R^L with the Hermitian matrix $\dot{R}^L \in \mathrm{End}(T^{(1,0)} X)$ such that for any $W, Y \in T^{(1,0)} X$,

$$R^L(W, \overline{Y}) = \big\langle \dot{R}^L W, \overline{Y} \big\rangle. \tag{2.3}$$

Choose $\{w_j\}_{j=1}^n$ an orthonormal basis of $T^{(1,0)}_{x_0}X$ such that

$$\dot{R}^L(x_0) = \mathrm{diag}(a_1, \ldots, a_n) \in \mathrm{End}(T^{(1,0)}_{x_0}X). \tag{2.4}$$

Then $a_j > 0$, for all $1 \le j \le n$. We fix an orthonormal basis of $T_{x_0}X$ given by $e_{2j-1} = \frac{1}{\sqrt{2}}(w_j + \overline{w}_j)$ and $e_{2j} = \frac{\sqrt{-1}}{\sqrt{2}}(w_j - \overline{w}_j)$. Then $z_j = Z_{2j-1} + \sqrt{-1}Z_{2j}$ is a complex coordinate of $Z \in \mathbb{R}^{2n} \simeq (T_{x_0}X, J)$.

By [17, Theorem 2.1] and [22, Theorem 1.18], we obtain the following version of the off diagonal expansion of the generalized Bergman kernel.

Theorem 2.2 *For any $x_0 \in X$ and $r \in \mathbb{N}$, there exist polynomials $J_{r,x_0}(Z, Z') \in \mathrm{End}(E_{x_0})$ in Z, Z' with the same parity as r and with $\deg J_{r,x_0} \leqslant 3r$ such that by setting*

$$\mathscr{F}_{r,x_0}(Z, Z') = J_{r,x_0}(Z, Z')\mathscr{P}(Z, Z'), \quad J_{0,x_0}(Z, Z') = \mathrm{Id}_{E_{x_0}}, \tag{2.5}$$

with

$$\mathscr{P}(Z, Z') = \prod_{j=1}^{n} \frac{a_j}{2\pi} \exp\left[-\frac{1}{4}\sum_{j=1}^{n} a_j\big(|z_j|^2 + |z'_j|^2 - 2z_j\overline{z}'_j\big)\right], \tag{2.6}$$

the following statement holds: for any $b > 0$ and $k_0, m, m' \in \mathbb{N}$, there exists $C_{b,k_0,m,m'} > 0$ such that for $|\alpha| + |\alpha'| \leqslant m'$ and any $|Z|, |Z'| < bp^{-\frac{1}{2}+\theta}$ with

$$\theta = \frac{1}{2\big(2n + 8 + 2k_0 + 3m' + 2m\big)}, \tag{2.7}$$

we have

$$\begin{aligned}&\left|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}}\left(p^{-n}P_{\mathcal{H}_p,x_0}(Z, Z') - \sum_{r=0}^{k}\mathscr{F}_{r,x_0}\big(\sqrt{p}Z, \sqrt{p}Z'\big)\kappa_{x_0}^{-1/2}(Z)\kappa_{x_0}^{-1/2}(Z')p^{-\frac{r}{2}}\right)\right|_{\mathscr{C}^m(X)}\\ &\leqslant C_{b,k_0,m,m'}\,p^{-\frac{k_0}{2}-1},\end{aligned} \tag{2.8}$$

where $k = k_0 + m' + 2$ and $\mathscr{C}^m(X)$ is the $\mathscr{C}^m$ norm for the parameter $x_0 \in X$.

In particular when $m' = 0$, the following statement holds: for any $b > 0$ and $k, m \in \mathbb{N}$, there exists $C > 0$ such that for any $|Z|, |Z'| < bp^{-\frac{1}{2}+\theta_2}$ with

$$\theta_2 = \frac{1}{4\big(n + k + m + 2\big)}, \tag{2.9}$$

we have

$$\begin{aligned}&\Big|p^{-n}P_{\mathcal{H}_p,x_0}(Z,Z')-\sum_{r=0}^{k}\mathscr{F}_{r,x_0}(\sqrt{p}Z,\sqrt{p}Z')\kappa_{x_0}^{-1/2}(Z)\kappa_{x_0}^{-1/2}(Z')p^{-\frac{r}{2}}\Big|_{\mathscr{C}^m(X)}\\&\leqslant Cp^{-k/2}.\end{aligned}\tag{2.10}$$

Note that the more expansion terms in (2.10), the smaller of the expansion domain for the variables Z and Z'. This serves as the main ingredient for the generalized Bergman kernel case.

By [22, Lemma 2.2], for any polynomials $F, G \in \mathbb{C}[Z, Z']$ there exists $\mathscr{K}[F, G] \in \mathbb{C}[Z, Z']$ such that

$$\big((F\mathscr{P})\circ(G\mathscr{P})\big)(Z,Z')=\mathscr{K}[F,G](Z,Z')\mathscr{P}(Z,Z').\tag{2.11}$$

3 Asymptotic Expansion of Toeplitz Operators

For $f \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ we define the Toeplitz operator $T_{f,p}$ on $L^2(X, L^p \otimes E)$ by (1.9). The Schwartz kernel of $T_{f,p}$ is given by

$$T_{f,p}(x,x')=\int_X P_{\mathcal{H}_p}(x,x'')f(x'')P_{\mathcal{H}_p}(x'',x')\mathrm{d}v_X(x'').\tag{3.1}$$

Note that if $f \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ is self-adjoint, i.e., $f(x) = f(x)^*$ for all $x \in X$, then the operator $T_{f,p}$ is self-adjoint.

We examine now the asymptotic expansion of the kernel of Toeplitz operators $T_{f,p}$. Outside the diagonal of $X \times X$, we have the following analogue of [22, Lemma 4.2].

Lemma 3.1 *Let $\theta \in (0, 1)$ and $f \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ fixed. For any $b > 0$ and $k, l \in \mathbb{N}$, there exists $C_{b,k,l} > 0$ such that*

$$\Big|T_{f,p}(x,x')\Big|_{\mathscr{C}^k(X\times X)}\leqslant C_{b,k,l}\,p^{-l},\tag{3.2}$$

for all $p \geqslant 1$ and all $(x, x') \in X \times X$ with $d(x, x') > bp^{-\theta}$, where the $\mathscr{C}^k$-norm is induced by $\nabla^L, \nabla^E, h^L, h^E$ and g^{TX}.

Proof From Proposition 2.1 and (2.10), we know that for any $k \in \mathbb{N}$ there exist $C_k > 0$ and $M_k > 0$ such that for all $(x, x') \in X \times X$,

$$\Big|P_{\mathcal{H}_p}(x,x')\Big|_{\mathscr{C}^k(X\times X)}\leqslant C_k\,p^{M_k}.\tag{3.3}$$

We split the integral in (3.1) in a sum of two integrals as follows:

$$T_{f,p}(x,x')=\left(\int_{B^X(x,\frac{b}{2}p^{-\theta})}+\int_{X\setminus B^X(x,\frac{b}{2}p^{-\theta})}\right)P_{\mathcal{H}_p}(x,x'')f(x'')P_{\mathcal{H}_p}(x'',x')dv_X(x''). \tag{3.4}$$

Assume that $d(x,x')>bp^{-\theta}$. Then

$$\begin{aligned} d(x'',x') &> \frac{b}{2}p^{-\theta} \quad \text{for } x''\in B^X\left(x,\frac{b}{2}p^{-\theta}\right),\\ d(x,x'') &\geqslant \frac{b}{2}p^{-\theta} \quad \text{for } x''\in X\Big\backslash B^X\left(x,\frac{b}{2}p^{-\theta}\right). \end{aligned} \tag{3.5}$$

Now from (2.1) and (3.3)–(3.5), we get (3.2). The proof of Lemma 3.1 is complete. □

We concentrate next on a neighborhood of the diagonal of $X\times X$ in order to obtain the asymptotic expansion of the kernel $T_{f,p}(x,x')$.

Let $\{\Xi_p\}_{p\in\mathbb{N}}$ be a sequence of linear operators $\Xi_p : L^2(X,L^p\otimes E)\to L^2(X,L^p\otimes E)$ with smooth kernel $\Xi_p(x,y)$ with respect to $dv_X(y)$. Recall that $\pi : TX\times_X TX\to X$ is the natural projection. Under our trivialization, $\Xi_p(x,y)$ induces a smooth section $\Xi_{p,x_0}(Z,Z')$ of $\pi^*\big(\mathrm{End}(E)\big)$ over $TX\times_X TX$ with $Z,Z'\in T_{x_0}X$, $|Z|,|Z'|<a^X$. Recall also that $\mathscr{P}_{x_0}=\mathscr{P}$ was defined by (2.6).

Consider the following condition for $\{\Xi_p\}_{p\in\mathbb{N}}$.

Condition A There exists a family $\{Q_{r,x_0}\}_{r\in\mathbb{N},x_0\in X}$ such that

(a) $Q_{r,x_0}\in\mathrm{End}(E_{x_0})[Z,Z']$;
(b) $\{Q_{r,x_0}\}_{r\in\mathbb{N},x_0\in X}$ is smooth with respect to the parameter $x_0\in X$ and there exist $b_1,b_0\in\mathbb{N}$ such that $\deg Q_r\leqslant b_1r+b_0$;
(c) For any $k,m\in\mathbb{N}$, there exists $\theta_{k,m}\in(0,1/2)$ such that for any $b>0$, there exists $C_{b,k,m}>0$ such that for every $x_0\in X$, $Z,Z'\in T_{x_0}X$ with $|Z|,|Z'|<bp^{-\frac{1}{2}+\theta_{k,m}}$ and $p\in\mathbb{N}^*$, the following estimate holds:

$$\left|p^{-n}\Xi_{p,x_0}(Z,Z')\kappa_{x_0}^{1/2}(Z)\kappa_{x_0}^{1/2}(Z')-\sum_{r=0}^{k}(Q_{r,x_0}\mathscr{P}_{x_0})(\sqrt{p}Z,\sqrt{p}Z')p^{-r/2}\right|_{\mathscr{C}^m(X)} \leqslant C_{b,k,m}p^{-k/2}. \tag{3.6}$$

(d) For any $\theta\in(0,1)$, $b>0$, $k,m\in\mathbb{N}$, there exists $C>0$ such that for any $p\in\mathbb{N}^*$, $d(x,x')>bp^{-\theta/2}$, we have

$$\left|\Xi_p(x,x')\right|_{\mathscr{C}^m(X\times X)}\leqslant Cp^{-k}.$$

Notation A For any $k, m \in \mathbb{N}$ we write the Eq. (3.6) for $|Z|, |Z'| < bp^{-\frac{1}{2}+\theta_{k,m}}$ as

$$p^{-n}\,\Xi_{p,x_0}(Z,Z') \cong \sum_{r=0}^{k} (Q_{r,x_0}\mathscr{P}_{x_0})(\sqrt{p}Z,\sqrt{p}Z')p^{-r/2} + \mathscr{O}_m(p^{-k/2}). \tag{3.7}$$

Remark 3.2 By Theorem 2.2, (2.9) and (2.10), we have

$$p^{-n}\,P_{\mathcal{H}_p,x_0}(Z,Z') \cong \sum_{r=0}^{k} (J_{r,x_0}\mathscr{P}_{x_0})(\sqrt{p}Z,\sqrt{p}Z')p^{-r/2} + \mathscr{O}_m(p^{-k/2}), \tag{3.8}$$

in the sense of Notation A with

$$\theta_{k,m} = \frac{1}{4(n+k+m+2)} \quad \text{for } k, m \in \mathbb{N}, \tag{3.9}$$

where $J_{r,x_0}(Z,Z') \in \mathrm{End}(E_{x_0})$ are the polynomials in Z, Z' defined in (2.5). Note that $J_{r,x_0}(Z,Z')$ has the same parity as r and $\deg J_{r,x_0} \leqslant 3r$, $J_{0,x_0} = \mathrm{Id}_{E_{x_0}}$.

The following result is about the near diagonal asymptotic expansion of the kernel $T_{f,p}(x,x')$. It is a version of [22, Lemma 4.6] in our situation.

Lemma 3.3 *Let $f \in \mathscr{C}^\infty(X, \mathrm{End}(E))$. There exists a family $\{Q_{r,x_0}(f)\}_{r\in\mathbb{N},x_0\in X}$ such that*

(a) $Q_{r,x_0}(f) \in \mathrm{End}(E_{x_0})[Z,Z']$ *are polynomials with the same parity as r;*
(b) $\{Q_{r,x_0}(f)\}_{r\in\mathbb{N},x_0\in X}$ *is smooth with respect to $x_0 \in X$, and* $\deg Q_{r,x_0} \leqslant 3r$*;*
(c) *For any $k_0, m \in \mathbb{N}$, we have*

$$p^{-n}T_{f,p,x_0}(Z,Z') \cong \sum_{r=0}^{k_0} \big(Q_{r,x_0}(f)\mathscr{P}_{x_0}\big)(\sqrt{p}Z,\sqrt{p}Z')p^{-r/2} + \mathscr{O}_m(p^{-k_0/2}), \tag{3.10}$$

in the sense of Notation A with

$$\theta_{k_0,m} = \frac{1}{4(n+k+m+2)} \text{ for some } k \geqslant k_0, \tag{3.11}$$

and $Q_{r,x_0}(f)$ are expressed by

$$Q_{r,x_0}(f) = \sum_{r_1+r_2+|\alpha|=r} \mathscr{K}\left[J_{r_1,x_0}, \frac{\partial^\alpha f_{x_0}}{\partial Z^\alpha}(0)\frac{Z^\alpha}{\alpha!}J_{r_2,x_0}\right]. \tag{3.12}$$

Especially,

$$Q_{0,x_0}(f) = f(x_0)\,\mathrm{Id}_{E_{x_0}}. \tag{3.13}$$

Proof For $k_0, m \in \mathbb{N}$ fixed, let $k \geqslant k_0$ to be determined. Set

$$\theta_2 = \frac{1}{4(n+k+m+2)}, \quad \theta_1 = 1 - 2\theta_2. \tag{3.14}$$

By (2.10), we have for any $|Z|, |Z'| < 2bp^{-\frac{1}{2}+\theta_2} = 2bp^{-\theta_1/2}$,

$$\begin{aligned} &\left| p^{-n} P_{\mathcal{H}_p,x_0}(Z,Z') - \sum_{r=0}^{k} \mathscr{F}_{r,x_0}(\sqrt{p}Z, \sqrt{p}Z')\kappa_{x_0}^{-1/2}(Z)\kappa_{x_0}^{-1/2}(Z')p^{-r/2} \right|_{\mathscr{C}^m(X)} \\ &\leqslant C_{k,l}\, p^{-k/2}. \end{aligned} \tag{3.15}$$

For $|Z|, |Z'| < \frac{b}{2}p^{-\frac{1}{2}+\theta_2} = \frac{b}{2}p^{-\theta_1/2}$, we get from (3.1) that

$$T_{f,p,x_0}(Z,Z') = \int_X P_{\mathcal{H}_p,x_0}(Z,y) f(y) P_{\mathcal{H}_p,x_0}(y,Z')\mathrm{d}v_X(y). \tag{3.16}$$

We split the integral into integrals over $B^X(x, bp^{-\theta_1/2})$ and $X \setminus B^X(x, bp^{-\theta_1/2})$. We have

$$d(y, \exp_{x_0} Z) \geqslant d(y, x_0) - |Z| > \frac{b}{2}p^{-\theta_1/2} \;\text{ on } X \backslash B^X(x, bp^{-\theta_1/2}), \tag{3.17}$$

since on this set $d(y, x_0) > bp^{-\theta_1/2}$ holds. By Proposition 2.1 for θ_1 in (3.14), (3.3) and (3.17) we have for $|Z|, |Z'| < \frac{b}{2}p^{-\theta_1/2}$,

$$\begin{aligned} T_{f,p,x_0}(Z,Z') = &\int_{|Z''|<bp^{-\frac{\theta_1}{2}}} P_{\mathcal{H}_p,x_0}(Z,Z'') f_{x_0}(Z'') P_{\mathcal{H}_p,x_0}(Z'',Z')\kappa_{x_0}(Z'')\mathrm{d}v_{TX}(Z'') \\ &+ \mathscr{O}_m(p^{-\infty}). \end{aligned} \tag{3.18}$$

Then

$$\begin{aligned} &p^{-n} T_{f,p,x_0}(Z,Z')\kappa_{x_0}^{1/2}(Z)\kappa_{x_0}^{1/2}(Z') \\ &= p^{-n}\int_{|Z''|<bp^{-\frac{\theta_1}{2}}} P_{\mathcal{H}_p,x_0}(Z,Z'')\kappa_{x_0}^{\frac{1}{2}}(Z)\kappa_{x_0}^{\frac{1}{2}}(Z'') f_{x_0}(Z'') \\ &\quad \times P_{\mathcal{H}_p,x_0}(Z'',Z')\kappa_{x_0}^{\frac{1}{2}}(Z'')\kappa_{x_0}^{\frac{1}{2}}(Z')\mathrm{d}v_{TX}(Z'') + \mathscr{O}_m(p^{-\infty}). \end{aligned} \tag{3.19}$$

We consider the Taylor expansion of f_{x_0}:

$$\begin{aligned} f_{x_0}(Z) &= \sum_{|\alpha|\leqslant k} \frac{\partial^\alpha f_{x_0}}{\partial Z^\alpha}(0)\frac{Z^\alpha}{\alpha!} + O(|Z|^{k+1}) \\ &= \sum_{|\alpha|\leqslant k} p^{-|\alpha|/2}\frac{\partial^\alpha f_{x_0}}{\partial Z^\alpha}(0)\frac{(\sqrt{p}Z)^\alpha}{\alpha!} + p^{-\frac{k+1}{2}} O(|\sqrt{p}Z|^{k+1}). \end{aligned} \tag{3.20}$$

Combining the asymptotic expansion (3.15) and (3.20), to obtain the asymptotic expansion of (3.19), we need to consider $I_{r_1,|\alpha|,r_2}(T_{x_0}X)(Z,Z')$ defined by

$$\begin{aligned} & p^{-n+\frac{r_1+|\alpha|+r_2}{2}} I_{r_1,|\alpha|,r_2}(T_{x_0}X)(Z,Z') \\ &:= \int_{T_{x_0}X} (J_{r_1,x_0}\mathscr{P}_{x_0})(\sqrt{p}Z,\sqrt{p}Z'')\frac{\partial^\alpha f_{x_0}}{\partial Z^\alpha}(0)\frac{(\sqrt{p}Z'')^\alpha}{\alpha!} \\ &\quad\times (J_{r_2,x_0}\mathscr{P}_{x_0})(\sqrt{p}Z'',\sqrt{p}Z')\mathrm{d}v_{TX}(Z''). \end{aligned} \tag{3.21}$$

Clearly, we can define $I_{r_1,|\alpha|,r_2}\big(B^{T_{x_0}X}(0,a)\big)(Z,Z')$ and $I_{r_1,|\alpha|,r_2}\big(T_{x_0}X\backslash B^{T_{x_0}X}(0,a)\big)(Z,Z')$ for $a>0$ in the same manner. Then by (3.19),

$$\begin{aligned} & p^{-n}T_{f,p,x_0}(Z,Z')\kappa^{1/2}(Z)\kappa^{-1/2}(Z') \\ &= \sum_{r_1,|\alpha|,r_2\leqslant k} I_{r_1,|\alpha|,r_2}\big(B^{T_{x_0}X}(0,bp^{-\theta_1/2})\big)(Z,Z') \\ &\quad + I_1(Z,Z') + I_2(Z,Z') + I_3(Z,Z') + \mathscr{O}_m(p^{-\infty}), \end{aligned} \tag{3.22}$$

with

$$\begin{aligned} I_1(Z,Z') = \int_{|Z''|<bp^{-\theta_1/2}} &\Big[p^{-n}P_{\mathcal{H}_p}(Z,Z'')\kappa^{1/2}(Z)\kappa^{1/2}(Z'') \\ &- \sum_{r\leqslant k}(J_{r,x_0}\mathscr{P}_{x_0})(\sqrt{p}Z,\sqrt{p}Z'')p^{-r/2}\Big] \\ &\times f_{x_0}(Z'')P_{\mathcal{H}_p}(Z'',Z')\kappa^{1/2}(Z'')\kappa^{1/2}(Z')\mathrm{d}v_{TX}(Z''), \end{aligned} \tag{3.23}$$

and

$$\begin{aligned} I_2(Z,Z') = \int_{|Z''|<bp^{-\theta_1/2}} &\sum_{r_1\leqslant k}(J_{r_1,x_0}\mathscr{P}_{x_0})(\sqrt{p}Z,\sqrt{p}Z'')p^{-r_1/2} \\ &\times\Big[f_{x_0}(Z'') - \sum_{|\alpha|\leqslant k}\frac{\partial^\alpha f_{x_0}}{\partial Z^\alpha}(0)\frac{(\sqrt{p}Z'')^\alpha}{\alpha!}p^{-|\alpha|/2}\Big] \\ &\times P_{\mathcal{H}_p}(Z'',Z')\kappa^{1/2}(Z'')\kappa^{1/2}(Z')\mathrm{d}v_{TX}(Z''), \end{aligned}$$

$$
\begin{aligned}
I_3(Z,Z') = p^n \int_{|Z''|<bp^{-\theta_1/2}} & \sum_{r_1\leqslant k} (J_{r_1,x_0}\mathscr{P}_{x_0})(\sqrt{p}Z,\sqrt{p}Z'')p^{-r_1/2} \\
& \times \sum_{|\alpha|\leqslant k} \frac{\partial^\alpha f_{x_0}}{\partial Z^\alpha}(0)\frac{(\sqrt{p}Z'')^\alpha}{\alpha!}p^{-|\alpha|/2} \\
& \times \Big[p^{-n}P_{\mathcal{H}_p}(Z'',Z')\kappa^{1/2}(Z'')\kappa^{1/2}(Z') \\
& - \sum_{r_2\leqslant k}(J_{r_2}\mathscr{P}_{x_0})(\sqrt{p}Z'',\sqrt{p}Z')p^{-r_2/2}\Big]dv_{TX}(Z''). \qquad (3.24)
\end{aligned}
$$

We claim that for k large,

$$
\left|I_j(Z,Z')\right|_{\mathscr{C}^m(X)} \leqslant Cp^{-k_0/2} \quad \text{for } j=1,2,3. \qquad (3.25)
$$

In fact, by (3.3), there exists $C_0>0$ and $M_0>0$ such that for all $(x,x')\in X\times X$,

$$
\left|P_{\mathcal{H}_p}(x,x')\right|_{\mathscr{C}^0(X\times X)} \leqslant C_0 p^{M_0}. \qquad (3.26)
$$

Combining (3.15), (3.23) and (3.26) yields

$$
|I_1(Z,Z')|_{\mathscr{C}^m(X)} \leqslant Cp^{-\frac{k}{2}+M_0}. \qquad (3.27)
$$

By (3.20), (3.26) and the fact that $\deg J_r \leqslant 3r$,

$$
\begin{aligned}
\left|I_2(Z,Z')\right|_{\mathscr{C}^m(X)} &\leqslant C(1+\sqrt{p}|Z|)^{3k}\cdot p^{-\frac{k+1}{2}}\cdot p^{M_0} \\
&\leqslant Cp^{-\frac{k+1}{2}+3k\theta_2+M_0}.
\end{aligned} \qquad (3.28)
$$

By (3.15) and the fact that $\deg J_r \leqslant 3r$, we have for $|Z|,|Z'|<\frac{b}{2}p^{-\theta_1/2}$,

$$
\left|I_3(Z,Z')\right|_{\mathscr{C}^m(X)} \leqslant C(1+\sqrt{p}|Z|)^{3k}p^{-\frac{k}{2}}. \qquad (3.29)
$$

From (3.27)–(3.29), choose $k>k_0$ big enough such that

$$
k+1-6k\theta_2-2M_0 = k\Big(1-\frac{3}{2(n+k+m+2)}\Big)-2M_0+1>k_0. \qquad (3.30)
$$

Then the claim (3.25) holds. By (3.22) and (3.25),

$$
\begin{aligned}
&\Big|p^{-n}T_{f,p,x_0}(Z,Z')\kappa^{1/2}(Z)\kappa^{-1/2}(Z') \\
&\quad - \sum_{r_1,\alpha,r_2\leqslant k} I_{r_1,|\alpha|,r_2}(B^{T_{x_0}X}(0,bp^{-\theta_1/2}))(Z,Z')\Big|_{\mathscr{C}^m(X)} \leqslant Cp^{-k_0/2}.
\end{aligned} \qquad (3.31)
$$

Note by (1.5) and (2.6),

$$\left|\mathscr{P}(\sqrt{p}Z,\sqrt{p}Z')\right| = \prod_j \frac{a_j}{2\pi} e^{-\frac{p}{4}\sum_j a_j|z_j-z'_j|^2} \leqslant Ce^{-\frac{p}{4}\mu_0|Z-Z'|^2}. \tag{3.32}$$

By (3.32) and the fact that $\deg J_r \leqslant 3r$, we obtain

$$\begin{aligned}
&\left|I_{r_1,|\alpha|,r_2}\big(T_{x_0}X\backslash B^{T_{x_0}X}(0,bp^{-\theta_1/2})\big)(Z,Z')\right|_{\mathscr{C}^m(X)}\\
&\quad\leqslant Cp^n\int_{|Z''|>bp^{-\theta_1/2}}(1+\sqrt{p}|Z|+\sqrt{p}|Z''|)^{3r_1}(1+\sqrt{p}|Z'|+\sqrt{p}|Z''|)^{3r_2}\\
&\qquad\times(\sqrt{p}|Z''|)^{|\alpha|}\ \exp\Big(-\frac{\mu_0}{2}\sqrt{p}|Z-Z''|-\frac{\mu_0}{2}\sqrt{p}|Z''-Z'|\Big)dv_{TX}(Z'').
\end{aligned} \tag{3.33}$$

Note that for any $|Z|,|Z'|<\frac{b}{2}p^{-\theta_1/2}$ and $|Z''|>bp^{-\theta_1/2}$, we have

$$|Z|<|Z''|,\quad |Z'|<|Z''|,\quad |Z-Z''|\geqslant\frac{1}{2}|Z''|,\quad |Z'-Z''|\geqslant\frac{1}{2}|Z''|. \tag{3.34}$$

Substituting (3.34) into (3.33) yields for any $|Z|,|Z'|<\frac{b}{2}p^{-\theta_1/2}$,

$$\begin{aligned}
&\left|I_{r_1,|\alpha|,r_2}\big(T_{x_0}X\backslash B^{T_{x_0}X}(0,bp^{-\theta_1/2})\big)(Z,Z')\right|_{\mathscr{C}^m(X)}\\
&\leqslant Cp^n\int_{|Z''|>bp^{-\theta_1/2}}\big(1+\sqrt{p}|Z''|\big)^{3(r_1+r_2)}(\sqrt{p}|Z''|)^{|\alpha|}e^{-\frac{\mu_0}{2}\sqrt{p}|Z''|}dv_{TX}(Z'')\\
&\leqslant Cp^n\exp\left(-\frac{b}{4}\mu_0p^{\theta_2}\right)\int_{|Z''|>bp^{-\theta_1/2}}\big(1+\sqrt{p}|Z''|\big)^{3(r_1+r_2)+|\alpha|}e^{-\frac{\mu_0}{4}\sqrt{p}|Z''|}dv_{TX}(Z'')\\
&\leqslant C\exp\left(-\frac{b}{4}\mu_0p^{\theta_2}\right)\int_{|Z''|>bp^{\theta_2}}\big(1+|Z''|\big)^{3(r_1+r_2)+|\alpha|}e^{-\frac{\mu_0}{4}|Z''|}dv_{TX}(Z'')\\
&\leqslant C\exp\left(-\frac{b}{4}\mu_0p^{\theta_2}\right).
\end{aligned} \tag{3.35}$$

Combining (3.31) and (3.35), we obtain

$$\begin{aligned}
&\left|p^{-n}T_{f,p,x_0}(Z,Z')\kappa^{1/2}(Z)\kappa^{1/2}(Z')-\sum_{r_1,|\alpha|,r_2\leqslant k}I_{r_1,|\alpha|,r_2}(T_{x_0}X)(Z,Z')\right|_{\mathscr{C}^m(X)}\\
&\quad\leqslant Cp^{-\frac{k_0}{2}}.
\end{aligned} \tag{3.36}$$

Clearly,

$$\begin{aligned}\sum_{r_1,|\alpha|,r_2\leqslant k} & I_{r_1,|\alpha|,r_2}(T_{x_0}X)(Z,Z') \\ &= \left(\sum_{r_1+|\alpha|+r_2\leqslant k_0} + \sum_{r_1+|\alpha|+r_2=k_0+1}^{3k}\right) I_{r_1,|\alpha|,r_2}(T_{x_0}X)(Z,Z').\end{aligned} \tag{3.37}$$

By (2.11) and (3.21),

$$\begin{aligned}& I_{r_1,|\alpha|,r_2}(T_{x_0}X)(Z,Z') \\ &\quad = p^{-(r_1+|\alpha|+r_2)/2}\left(\mathscr{K}\left[J_{r_1,x_0}, \frac{\partial^\alpha f_{x_0}}{\partial Z^\alpha}(0)\frac{Z^\alpha}{\alpha!}J_{r_2,x_0}\right]\mathscr{P}\right)(\sqrt{p}Z,\sqrt{p}Z').\end{aligned} \tag{3.38}$$

In view of (3.36)–(3.38), to finish the proof of Lemma 3.3, it suffices to prove that the $\mathscr{C}^m$ norm with respect to the parameter $x_0 \in X$ of the term

$$\sum_{r_1+|\alpha|+r_2=k_0+1}^{3k} I_{r_1,|\alpha|,r_2}(T_{x_0}X)(Z,Z'), \quad \text{for } |Z|, |Z'| < \frac{b}{2}p^{-\theta_1/2}, \tag{3.39}$$

is controlled by $Cp^{-k_0/2}$ for large k.

Estimating $I_{r_1,|\alpha|,r_2}(T_{x_0}X)(Z,Z')$ for $|Z|, |Z'| < \frac{b}{2}p^{-\theta_1/2}$, using (3.32), (3.38) and the fact that $\deg J_r \leqslant 3r$, we obtain

$$\begin{aligned}&\left|I_{r_1,|\alpha|,r_2}(T_{x_0}X)(Z,Z')\right|_{\mathscr{C}^m(X)} \\ &\quad\leqslant Cp^{-s/2}\big(1+\sqrt{p}|Z|+\sqrt{p}|Z'|\big)^{3s}\exp\Big(-\frac{p}{4}\mu_0|Z-Z'|^2\Big) \\ &\quad\leqslant Cp^{-\frac{s}{2}}p^{\frac{1-\theta_1}{2}\cdot 3s} = Cp^{-s(1-6\theta_2)/2},\end{aligned} \tag{3.40}$$

with $s = r_1+|\alpha|+r_2$. If $s > k_0$, then

$$s(1-6\theta_2) \geqslant (k_0+1)\left(1-\frac{6}{4(n+k+m+2)}\right). \tag{3.41}$$

Choose k big enough such that

$$(k_0+1)\left(1-\frac{6}{4(n+k+m+2)}\right) > k_0. \tag{3.42}$$

Then

$$\left|I_{r_1,|\alpha|,r_2}(T_{x_0}X)(Z,Z')\right|_{\mathscr{C}^m(X)} \leqslant Cp^{-k_0/2} \quad \text{for } r_1+|\alpha|+r_2 = s > k_0. \tag{3.43}$$

To sum up, we have proved the following statement: for fixed k_0, choose $k > k_0$ such that (3.30) and (3.42) hold. Set

$$\theta_2 = \frac{1}{4(n+k+m+2)}, \quad \theta_1 = 1 - 2\theta_2, \tag{3.44}$$

then for any $|Z|, |Z'| < \frac{b}{2} p^{-\theta_1/2}$, we have

$$\begin{aligned} &\left| p^{-n} T_{f,p,x_0}(Z,Z')\kappa_{x_0}^{1/2}(Z)\kappa_{x_0}^{1/2}(Z') - \sum_{r=0}^{k_0} \big(Q_{r,x_0}(f)\mathscr{P}_{x_0}\big)(\sqrt{p}Z, \sqrt{p}Z')p^{-r/2} \right|_{\mathscr{C}^m(X)} \\ &\leqslant Cp^{-k_0/2}, \end{aligned} \tag{3.45}$$

where $Q_{r,x_0}(f)$ is given by (3.12). This completes the proof of Lemma 3.3. □

Remark 3.4 Let Ξ_p be a sequence of operators satisfying Condition A and assume that $\Xi_p = P_{\mathcal{H}_p}\Xi_p P_{\mathcal{H}_p}$ for all $p \in \mathbb{N}$. Applying the proof of Lemma 3.3, by splitting integrals and studying different integration regions, we deduce by Theorem 2.2 and (3.6):

For any $k, m, m' \in \mathbb{N}$, there exists $\theta_{k,m,m'} \in (0, 1/2)$ such that for any $b > 0$, there exists $C > 0$ such that for every $x_0 \in X$, $Z, Z' \in T_{x_0}X$ with $|Z|, |Z'| < bp^{-\frac{1}{2}+\theta_{k,m,m'}}$ and $p \in \mathbb{N}^*$, $|\alpha| + |\alpha'| \leqslant m'$, we have

$$\begin{aligned} &\left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \Big(p^{-n} \Xi_{p,x_0}(Z,Z')\kappa_{x_0}^{1/2}(Z)\kappa_{x_0}^{1/2}(Z') \right. \\ &\left. \quad - \sum_{r=0}^{k} (Q_{r,x_0}\mathscr{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z')p^{-r/2} \Big) \right|_{\mathscr{C}^m(X)} \leqslant Cp^{-(k-m')/2}. \end{aligned} \tag{3.46}$$

In fact, by $\Xi_p = P_{\mathcal{H}_p}\Xi_p P_{\mathcal{H}_p}$, for $|Z|, |Z'| < bp^{-\frac{1}{2}+\theta_{k,m,m'}}$, we have the analogue of (3.16):

$$\Xi_{p,x_0}(Z,Z') = \int_X P_{\mathcal{H}_{p,x_0}}(Z,y)\Xi_p(y,Z')P_{\mathcal{H}_{p,x_0}}(y,Z')dv_X(y). \tag{3.47}$$

Then the estimate (3.46) follows from Theorem 2.2, (3.6), (3.15) and (3.47) in the same manner as (3.45) follows from (3.15), (3.16) and (3.20).

4 A Criterion for Toeplitz Operators

In this section, we prove a useful criterion which ensures that a given family of bounded linear operators is a Toeplitz operator.

Theorem 4.1 *Let $\{T_p : L^2(X, L^p \otimes E) \to L^2(X, L^p \otimes E)\}$ be a family of bounded linear operators which satisfies the following three conditions:*

(i) *For any* $p \in \mathbb{N}$, $P_{\mathcal{H}_p} T_p P_{\mathcal{H}_p} = T_p$.
(ii) *For any* $b > 0$, $l \in \mathbb{N}$ *and* $0 < \theta < 1$, *there exists* $C_{b,l,\theta} > 0$ *such that for all* $p \geqslant 1$ *and all* $(x, x') \in X \times X$ *with* $d(x, x') > bp^{-\theta/2}$,

$$\left| T_p(x, x') \right| \leqslant C_{b,l,\theta}\, p^{-l}. \tag{4.1}$$

(iii) *There exists a family of polynomials* $\left\{ \mathscr{Q}_{r,x_0} \in \operatorname{End}(E_{x_0})[Z, Z'] \right\}_{x_0 \in X}$ *such that*
 (a) *Each* $\mathscr{Q}_{r,x_0}$ *has the same parity as* r *and there exist* $b_1, b_0 \in \mathbb{N}$ *such that* $\deg \mathscr{Q}_r \leqslant b_1 r + b_0$,
 (b) *The family is smooth in* $x_0 \in X$ *and*
 (c) *For any* $k_0, m \in \mathbb{N}$, *there exists* $\theta_{k_0,m} \in (0, 1/2)$ *such that for any* $b > 0$, $p \in \mathbb{N}^*$, $x_0 \in X$ *and every* $Z, Z' \in T_{x_0}X$ *with* $|Z|, |Z'| < bp^{-\frac{1}{2}+\theta_{k_0,m}}$, *we have*

$$p^{-n} T_{p,x_0}(Z, Z') \cong \sum_{r=0}^{k_0} \left(\mathscr{Q}_{r,x_0} \mathscr{P}_{x_0} \right)(\sqrt{p} Z, \sqrt{p} Z') p^{-r/2} + \mathscr{O}_m(p^{-k_0/2}), \tag{4.2}$$

in the sense of Notation A for $k_0, m, \theta_{k_0,m}$.
Then $\{T_p\}$ *is a Toeplitz operator.*

Remark 4.2 By Lemmas 3.1 and 3.3, and by (1.11), (1.12) and the Sobolev inequality (cf. [10, (4.14)]), it follows that every Toeplitz operator in the sense of Definition 1.1 verifies the Conditions (i), (ii) and (iii) of Theorem 4.1.

We start the proof of Theorem 4.1. Let T_p^* be the adjoint of T_p. By writing

$$T_p = \frac{1}{2} \left(T_p + T_p^* \right) + \sqrt{-1} \frac{1}{2\sqrt{-1}} \left(T_p - T_p^* \right), \tag{4.3}$$

we may and will assume from now on that T_p is self-adjoint.

We will define inductively the sequence $(g_l)_{l \geqslant 0}$, $g_l \in \mathscr{C}^\infty(X, \operatorname{End}(E))$ such that

$$T_p = \sum_{l=0}^{q} P_{\mathcal{H}_p} g_l p^{-l} P_{\mathcal{H}_p} + \mathcal{O}(p^{-q-1}) \quad \text{for all } q \geqslant 0. \tag{4.4}$$

Moreover, we can make these g_l's to be self-adjoint.

Let us start with the case $q = 0$ of (4.4). For an arbitrary but fixed $x_0 \in X$, we set

$$g_0(x_0) = \mathscr{Q}_{0,x_0}(0, 0) \in \operatorname{End}(E_{x_0}). \tag{4.5}$$

We will show that

$$p^{-n}(T_p - T_{g_0,p})_{x_0}(Z, Z') \cong \mathscr{O}_m(p^{-1}), \tag{4.6}$$

which implies the case $q = 0$ of (4.4), namely,

$$T_p = P_{\mathcal{H}_p} g_0 P_{\mathcal{H}_p} + \mathcal{O}(p^{-1}). \tag{4.7}$$

The proof of (4.6)–(4.7) will be done in Propositions 4.3 and 4.9.

Proposition 4.3 *Under the conditions of Theorem 4.1, we have*

$$\mathscr{Q}_{0,x_0}(Z, Z') = \mathscr{Q}_{0,x_0}(0, 0) \in \mathrm{End}(E_{x_0}) \tag{4.8}$$

for all $x_0 \in X$ and all $Z, Z' \in T_{x_0}X$.

Proof The proof is divided in the series of Lemmas 4.4– 4.8. Our first observation is as follows. □

Lemma 4.4 $\mathscr{Q}_{0,x_0} \in \mathrm{End}(E_{x_0})[Z, Z']$ *and* $\mathscr{Q}_{0,x_0}$ *is a polynomial in* $z, \overline{z}'$.

Proof By (4.2), for $k_0 = 2$ there exists $\theta_3 \in (0, 1/2)$ such that for any $b > 0$ and every $Z, Z' \in T_{x_0}X$ with $|Z|, |Z'| < bp^{-\frac{1}{2}+\theta_3}$, we have

$$p^{-n} T_{p,x_0}(Z, Z') \cong \sum_{r=0}^{2} (\mathscr{Q}_{r,x_0} \mathscr{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z') p^{-r/2} + \mathscr{O}_m(p^{-1}). \tag{4.9}$$

By (3.8),

$$p^{-n} P_{\mathcal{H}_p,x_0}(Z, Z') \cong \sum_{r=0}^{2} (J_{r,x_0} \mathscr{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z') p^{-r/2} + \mathscr{O}_m(p^{-1}), \tag{4.10}$$

in the sense of Notation A with $\theta_4 = 1/4(n + m + 4)$. Combining (4.9) and (4.10), modeled the way we get (3.45) from (3.15) and (3.20), we obtain that for every $Z, Z' \in T_{x_0}X$ with $|Z|, |Z'| < bp^{-\frac{1}{2}+\theta_2}$, with θ_2 is given in (3.44) for some large k,

$$\begin{aligned} &p^{-n}(P_{\mathcal{H}_p} T_p P_{\mathcal{H}_p})_{x_0}(Z, Z') \\ &\cong \sum_{r=0}^{2} \sum_{r_1+r_2+r_3=r} \Big[(J_{r_1,x_0} \mathscr{P}_{x_0}) \circ (\mathscr{Q}_{r_2,x_0} \mathscr{P}_{x_0}) \circ (J_{r_3,x_0} \mathscr{P}_{x_0}) \Big] (\sqrt{p}Z, \sqrt{p}Z') p^{-r/2} \\ &\quad + \mathscr{O}_m(p^{-1}). \end{aligned} \tag{4.11}$$

Since $P_{\mathcal{H}_p} T_p P_{\mathcal{H}_p} = T_p$, we deduce from (4.9) and (4.11) that

$$\mathscr{Q}_{0,x_0} \mathscr{P}_{x_0} = \mathscr{P}_{x_0} \circ (\mathscr{Q}_{0,x_0} \mathscr{P}_{x_0}) \circ \mathscr{P}_{x_0}. \tag{4.12}$$

By [22, (2.8)] and (4.12), we obtain

$$\mathscr{Q}_{0,x_0} \in \mathrm{End}(E_{x_0})[z, \overline{z}']. \tag{4.13}$$

The proof of Lemma 4.4 is complete. □

For simplicity we denote in the rest of the proof $F_x = \mathscr{Q}_{0,x} \in \mathrm{End}(E_x)$. Let $F_x = \sum_{i\geqslant 0} F_x^{(i)}$ be the decomposition of F_x in homogeneous polynomials $F_x^{(i)}$ of degree i. We will show $F_x^{(i)}$ vanish identically for $i > 0$, that is

$$F_x^{(i)}(z, \overline{z}') = 0 \quad \text{for all } i > 0 \text{ and } z, \overline{z}' \in \mathbb{C}. \tag{4.14}$$

The first step is to prove

$$F_x^{(i)}(0, \overline{z}') = 0 \quad \text{for all } i > 0 \text{ and } z' \in \mathbb{C}. \tag{4.15}$$

Since T_p is self-adjoint, then we have

$$F_x^{(i)}(z, \overline{z}') = \big(F_x^{(i)}(z', \overline{z})\big)^*. \tag{4.16}$$

Consider $0 < \theta_{k_0,m} < 1$ as in hypothesis (iii) (c) of Theorem 4.1. For $Z' \in \mathbb{R}^{2n} \simeq T_xX$ and $y = \exp_x^X(Z')$, set

$$\begin{aligned} F^{(i)}(x, y) &= F_x^{(i)}(0, \overline{z}') \in \mathrm{End}(E_x), \\ \tilde{F}^{(i)}(x, y) &= \big(F^{(i)}(y, x)\big)^* \in \mathrm{End}(E_y). \end{aligned} \tag{4.17}$$

$F^{(i)}$ and $\tilde{F}^{(i)}$ define smooth sections on a neighborhood of the diagonal of $X \times X$. Clearly, the $\tilde{F}^{(i)}(x, y)$'s need not be polynomials in z and $\overline{z}'$.

Since we wish to define global operators induced by these kernels, we use a cut-off function in the neighborhood of the diagonal. Pick a smooth function $\eta \in \mathscr{C}^\infty(\mathbb{R})$ such that

$$\eta(u) = 1 \text{ for } |u| \leqslant \varepsilon/2 \quad \text{and} \quad \eta(u) = 0 \text{ for } |u| \geqslant \varepsilon. \tag{4.18}$$

We denote by $F^{(i)}P_{\mathcal{H}_p}$ and $P_{\mathcal{H}_p}\tilde{F}^{(i)}$ the operators defined by the kernels

$$\eta(d(x, y))F^{(i)}(x, y)P_{\mathcal{H}_p}(x, y) \text{ and } \eta(d(x, y))P_{\mathcal{H}_p}(x, y)\tilde{F}^{(i)}(x, y) \tag{4.19}$$

with respect to $dv_X(y)$. Set

$$\mathscr{T}_p = T_p - \sum_{i\leqslant \deg F_x} (F^{(i)}P_{\mathcal{H}_p})p^{i/2}. \tag{4.20}$$

The operators $\mathscr{T}_p$ extend naturally to bounded operators on $L^2(X, L^p \otimes E)$.

From (4.2) and (4.20), we deduce that for any $k_0, m \in \mathbb{N}$, there exists $\theta_{k_0,m} \in (0, 1/2)$ such that for any $|Z'| < \varepsilon p^{-\frac{1}{2}+\theta_{k_0,m}}$, we have the following expansion in the normal coordinates around $x_0 \in X$ (which has to be understood in the sense of (3.7)):

$$p^{-n}\mathscr{T}_{p,x_0}(0, Z') \cong \sum_{r=1}^{k_0}(R_{r,x_0}\mathscr{P}_{x_0})(0, \sqrt{p}Z')p^{-r/2} + \mathscr{O}_m(p^{-k_0/2}), \tag{4.21}$$

for some polynomials R_{r,x_0} of the same parity as r. For simplicity let us define similarly to (4.17) the kernel

$$R_{r,p}(x, y) = p^n(R_{r,x}\mathscr{P}_x)(0, \sqrt{p}Z')\kappa_x^{-1/2}(Z')\eta(d(x, y)), \tag{4.22}$$

where $y = \exp_x^X(Z')$, and denote by $R_{r,p}$ the operator defined by this kernel.

Lemma 4.5 *For $k_0 \geqslant 2(n+1)$, there exists $C > 0$ such that for every $p \geqslant 1$ and $s \in L^2(X, L^p \otimes E)$, we have*

$$\begin{aligned} \big\|\mathscr{T}_p s\big\|_{L^2} &\leqslant Cp^{-1/2}\big\|s\big\|_{L^2}, \\ \big\|\mathscr{T}_p^* s\big\|_{L^2} &\leqslant Cp^{-1/2}\big\|s\big\|_{L^2}. \end{aligned} \tag{4.23}$$

Proof In order to use (4.21), we write

$$\big\|\mathscr{T}_p s\big\|_{L^2} \leqslant \Big\|\big(\mathscr{T}_p - \sum_{r=1}^{k_0} p^{-r/2}R_{r,p}\big)s\Big\|_{L^2} + \Big\|\sum_{r=1}^{k_0} p^{-r/2}R_{r,p}s\Big\|_{L^2}. \tag{4.24}$$

By the Cauchy–Schwarz inequality we have

$$\begin{aligned} &\Big\|\big(\mathscr{T}_p - \sum_{r=1}^{k_0} p^{-r/2}R_{r,p}\big)s\Big\|_{L^2}^2 \\ &\quad\leqslant \int_X \Big(\int_X \Big|\big(\mathscr{T}_p - \sum_{r=1}^{k_0} p^{-r/2}R_{r,p}\big)(x, y)\Big| dv_X(y)\Big) \\ &\qquad\times \Big(\int_X \Big|\big(\mathscr{T}_p - \sum_{r=1}^{k_0} p^{-r/2}R_{r,p}\big)(x, y)\Big|\big|s(y)\big|^2 dv_X(y)\Big) dv_X(x). \end{aligned} \tag{4.25}$$

By (2.1), (4.1), (4.19), (4.20) and (4.22), we obtain uniformly in $x \in X$,

$$\begin{aligned}\int_X \Big|\big(\mathscr{T}_p - \sum_{r=1}^{k_0} p^{-r/2} R_{r,p}\big)(x,y)\Big| \big|s(y)\big|^2 \mathrm{d}v_X(y) \\ \leqslant \int_{B^X(x,\frac{\varepsilon}{2}p^{-\frac{1}{2}+\theta_{k_0,m}})} \Big|\big(\mathscr{T}_p - \sum_{r=1}^{k_0} p^{-r/2} R_{r,p}\big)(x,y)\Big| \big|s(y)\big|^2 \mathrm{d}v_X(y) \\ + O(p^{-\infty}) \int_{X\setminus B^X(x,\frac{\varepsilon}{2}p^{-\frac{1}{2}+\theta_{k_0,m}})} |s(y)|^2 \mathrm{d}v_X(y),\end{aligned} \tag{4.26}$$

where $\theta_{k_0,m}$ is given by (4.21). By (3.6) and (4.21) we obtain

$$\begin{aligned}\int_{B^X(x,\frac{\varepsilon}{2}p^{-\frac{1}{2}+\theta_{k_0,m}})} \Big|\big(\mathscr{T}_p - \sum_{r=1}^{k_0} p^{-r/2} R_{r,p}\big)(x,y)\Big| \big|s(y)\big|^2 \mathrm{d}v_X(y) \\ = O(p^{-1}) \int_{B^X(x,\frac{\varepsilon}{2}p^{-\frac{1}{2}+\theta_{k_0,m}})} \big|s(y)\big|^2 \mathrm{d}v_X(y).\end{aligned} \tag{4.27}$$

In the same vein (by splitting the integral region as above), we obtain

$$\int_X \Big|\big(\mathscr{T}_p - \sum_{r=1}^{k_0} p^{-r/2} R_{r,p}\big)(x,y)\Big| \mathrm{d}v_X(y) = O(p^{-1}) + O(p^{-\infty}). \tag{4.28}$$

Combining (4.25)–(4.28) yields

$$\Big\| \Big(\mathscr{T}_p - \sum_{r=1}^{k_0} p^{-r/2} R_{r,p}\Big) s \Big\|_{L^2} \leqslant C p^{-1} \|s\|_{L^2}. \tag{4.29}$$

A similar proof as for (4.29) delivers for $s \in L^2(X, L^p \otimes E)$,

$$\|R_{r,p} s\|_{L^2} \leqslant C \|s\|_{L^2}, \tag{4.30}$$

which implies

$$\| \sum_{r=1}^{k_0} p^{-r/2} R_{r,p} s \|_{L^2} \leqslant C p^{-1/2} \|s\|_{L^2} \text{ for } s \in L^2(X, L^p \otimes E), \tag{4.31}$$

for some constant $C > 0$. Relations (4.29) and (4.31) entail the first inequality of (4.23), which is equivalent to the second of (4.23), by taking the adjoint. This completes the proof of Lemma 4.5. □

Let us consider the Taylor development of $\tilde{F}^{(i)}$ in normal coordinates around x with $y = \exp_x^X(Z')$:

$$\tilde{F}^{(i)}(x,y) = \sum_{|\alpha|\leqslant k} \frac{\partial^\alpha \tilde{F}^{(i)}}{\partial Z'^\alpha}(x,0)\frac{(\sqrt{p}Z')^\alpha}{\alpha!}p^{-|\alpha|/2} + O(|Z'|^{k+1}). \tag{4.32}$$

The next step of the proof of Proposition 4.3 is the following.

Lemma 4.6 *For every $j>0$, we have*

$$\frac{\partial^\alpha \tilde{F}^{(i)}}{\partial Z'^\alpha}(x,0) = 0 \text{ for } i-|\alpha| \geqslant j > 0. \tag{4.33}$$

Proof The definition (4.20) of $\mathscr{T}_p$ shows that

$$\mathscr{T}_p^* = T_p - \sum_{i\leqslant \deg F_x} p^{i/2}(P_{\mathcal{H}_p}\tilde{F}^{(i)}). \tag{4.34}$$

Let us develop the sum on the right-hand side. Considering the Taylor development (4.32) with the expansion (3.8) of the Bergman kernel we obtain

$$\begin{aligned}
&p^{-n}\big(P_{\mathcal{H}_p}\tilde{F}^{(i)}\big)_{x_0}(0,Z')\kappa^{1/2}(Z')\\
&\quad - \sum_{r,|\alpha|\leqslant k}\big(J_{r,x_0}\mathscr{P}_{x_0}\big)(0,\sqrt{p}Z')\frac{\partial^\alpha \tilde{F}^{(i)}}{\partial Z'^\alpha}(x_0,0)\frac{(\sqrt{p}Z')^\alpha}{\alpha!}p^{-\frac{|\alpha|+r}{2}}\\
&\;= \Big[p^{-n}P_{\mathcal{H}_p,x_0}(0,Z')\kappa^{1/2}(Z') - \sum_{r\leqslant k}\big(J_{r,x_0}\mathscr{P}_{x_0}\big)(0,\sqrt{p}Z')p^{-r/2}\Big]\tilde{F}^{(i)}_{x_0}(0,Z')\\
&\quad + \sum_{r\leqslant k}\big(J_{r,x_0}\mathscr{P}_{x_0}\big)(0,\sqrt{p}Z')p^{-r/2}\Big[\tilde{F}^{(i)}_{x_0}(0,Z')\\
&\quad - \sum_{|\alpha|\leqslant k}\frac{\partial^\alpha \tilde{F}^{(i)}}{\partial Z'^\alpha}(x_0,0)\frac{(\sqrt{p}Z')^\alpha}{\alpha!}p^{-\frac{|\alpha|}{2}}\Big].
\end{aligned} \tag{4.35}$$

By (3.8), (3.32), (4.32) and $\deg J_{r,x_0} \leqslant 3r$, we obtain for $k \geqslant \deg F_x + 1$ and $m \in \mathbb{N}$, there exists $\theta_{k,m} \in (0,1)$ such that for any $Z' \in T_{x_0}X$ with $|Z'| \leqslant bp^{-\frac{1}{2}+\theta_{k,m}}$, we have

$$\begin{aligned}
&p^{-n}\sum_i \big(P_{\mathcal{H}_p}\tilde{F}^{(i)}\big)_{x_0}(0,Z')p^{i/2}\\
&\quad\cong \sum_i\sum_{|\alpha|,r\leqslant k}\big(J_{r,x_0}\mathscr{P}_{x_0}\big)(0,\sqrt{p}Z')\frac{\partial^\alpha \tilde{F}^{(i)}}{\partial Z'^\alpha}(x_0,0)\frac{(\sqrt{p}Z')^\alpha}{\alpha!}p^{(i-|\alpha|-r)/2}\\
&\qquad + \mathscr{O}_m(p^{(\deg F-k)/2}).
\end{aligned} \tag{4.36}$$

Having in mind the second inequality of (4.23), this is only possible if for every $j>0$ the coefficients of $p^{j/2}$ on the right-hand side of (4.36) vanish. Thus, we have for every $j>0$:

$$\sum_{l=j}^{\deg F_x} \sum_{|\alpha|+r=l-j} J_{r,x_0}(0, \sqrt{p}Z') \frac{\partial^\alpha \tilde{F}^{(l)}}{\partial Z'^\alpha}(x_0, 0) \frac{(\sqrt{p}Z')^\alpha}{\alpha!} = 0. \tag{4.37}$$

From (4.37), we will prove by recurrence that for any $j > 0$, (4.33) holds. As the first step of the recurrence let us take $j = \deg F_x$ in (4.37). Since $J_{0,x_0} = \mathrm{Id}_{E_{x_0}}$, then we get immediately $\tilde{F}^{(\deg F_x)}(x_0, 0) = 0$. Hence (4.33) holds for $j = \deg F_x$. Assume that (4.33) holds for $j > j_0 > 0$. Then for $j = j_0$, the coefficient with $r > 0$ in (4.37) is zero. Since $J_{0,x_0} = \mathrm{Id}_{E_{x_0}}$, then (4.37) reads

$$\sum_\alpha \frac{\partial^\alpha \tilde{F}^{(j_0+|\alpha|)}}{\partial Z'^\alpha}(x_0, 0) \frac{(\sqrt{p}Z')^\alpha}{\alpha!} = 0, \tag{4.38}$$

which entails (4.33) for $j = j_0$. The proof of (4.33) is complete. □

Lemma 4.7 *For $i > 0$, we have*

$$\frac{\partial^\alpha F_x^{(i)}}{\partial \overline{z}'^\alpha}(0, 0) = 0, \quad |\alpha| \leqslant i. \tag{4.39}$$

Therefore, $F_x^{(i)}(0, \overline{z}') = 0$ for all $i > 0$ and $z' \in \mathbb{C}$, i.e., (4.15) holds true. Moreover,

$$F_x^{(i)}(z, 0) = 0 \text{ for all } i > 0 \text{ and all } z \in \mathbb{C}. \tag{4.40}$$

Proof Let us start with some preliminary observations. In view of (4.23), (4.33), and (4.36), a comparison of the coefficients of p^0 in (4.9) and (4.34) yields

$$\tilde{F}^{(i)}(x, Z') = F_x^{(i)}(0, \overline{z}') + O(|Z'|^{i+1}). \tag{4.41}$$

Using the definition (4.17) of $\tilde{F}^{(i)}(x, Z')$ and taking the adjoint of (4.41), we get

$$F^{(i)}(Z', x) = \left(F_x^{(i)}(0, \overline{z}')\right)^* + O(|Z'|^{i+1}), \tag{4.42}$$

which implies

$$\frac{\partial^\alpha}{\partial z^\alpha} F^{(i)}(\cdot, x)\Big|_x = \left(\frac{\partial^\alpha}{\partial \overline{z}'^\alpha} F_x^{(i)}(0, \overline{z}')\right)^* \text{ for } |\alpha| \leqslant i. \tag{4.43}$$

In order to prove the Lemma, it suffices to show that

$$\frac{\partial^\alpha}{\partial z^\alpha} F^{(i)}(\cdot, x)\Big|_x = 0 \text{ for } |\alpha| \leqslant i. \tag{4.44}$$

We prove this by induction over $|\alpha|$. For $|\alpha| = 0$, it is obvious that $F^{(i)}(0, x) = 0$, since $F^{(i)}(x, \overline{z}')$ is a homogeneous polynomial of degree $i > 0$. For the induction step, let $j_X : X \to X \times X$ be the diagonal injection. By Lemma 4.4 and the definition (4.17) of $F^{(i)}(x, y)$, we have

$$\frac{\partial}{\partial z'_j} F^i(x, y) = 0 \text{ near } j_X(X), \tag{4.45}$$

where $y = \exp_x^X(Z')$. Assume now that $\alpha \in \mathbb{N}^n$ and (4.44) holds for $|\alpha| - 1$. Consider j with $\alpha_j > 0$ and set

$$\alpha' = (\alpha_1, \dots, \alpha_j - 1, \dots, \alpha_n). \tag{4.46}$$

Taking the derivative of (4.17) and using the induction hypothesis and (4.45), we have

$$\frac{\partial^\alpha}{\partial z^\alpha} F^{(i)}(\cdot, x)\bigg|_x = \frac{\partial}{\partial z_j} j_X^* \left(\frac{\partial^{\alpha'}}{\partial z^{\alpha'}} F^{(i)}\right)\bigg|_x - \frac{\partial^{\alpha'}}{\partial z^{\alpha'}} \frac{\partial}{\partial z'_j} F^{(i)}(\cdot, \cdot)\bigg|_{0,0} = 0. \tag{4.47}$$

Thus, (4.39) is proved. The identity (4.15) follows too, since it is equivalent to (4.39). Furthermore, (4.40) follows from (4.15) and (4.16). This finishes the proof of Lemma 4.7. □

Lemma 4.8 *We have* $F_x^{(i)}(z, \overline{z}') = 0$ *for all* $i > 0$ *and* $z, z' \in \mathbb{C}^n$.

Proof Let us consider the operator

$$\frac{1}{\sqrt{p}} P_{\mathcal{H}_p} \big(\nabla^{L^p \otimes E}_{X,x} T_p\big) P_{\mathcal{H}_p} \text{ with } X \in \mathscr{C}^\infty(X, TX),\, X(x_0) = \frac{\partial}{\partial z_j} + \frac{\partial}{\partial \overline{z}_j}. \tag{4.48}$$

By Remark 3.4, the leading term of its asymptotic expansion as in (3.7) is

$$\Big(\frac{\partial}{\partial z_j} F_{x_0}\Big)(\sqrt{p} z, \sqrt{p} \overline{z}') \mathscr{P}_{x_0}(\sqrt{p} Z, \sqrt{p} Z'). \tag{4.49}$$

By (4.15) and (4.40), $(\frac{\partial}{\partial z_j} F_{x_0})(z, \overline{z}')$ is an odd polynomial in $z, \overline{z}'$ whose constant term vanishes. We reiterate the argument from (4.20)–(4.43) by replacing the operator T_p with the operator (4.48); we get for $i > 0$,

$$\frac{\partial}{\partial z_j} F_x^{(i)}(0, \overline{z}') = 0. \tag{4.50}$$

By (4.16) and (4.50),

$$\frac{\partial}{\partial \overline{z}'_j} F_x^{(i)}(z, 0) = 0. \tag{4.51}$$

By continuing this process, we show that for all $i > 0$, $\alpha \in \mathbb{Z}^n$, $z, z' \in \mathbb{C}^n$,

$$\frac{\partial^\alpha}{\partial z^\alpha} F_x^{(i)}(0, \overline{z}') = \frac{\partial^\alpha}{\partial \overline{z}'^\alpha} F_x^{(i)}(z, 0) = 0. \tag{4.52}$$

Thus, Lemma 4.8 is proved and (4.14) holds true. □

Lemma 4.8 finishes the proof of Proposition 4.3. □

We come now to the proof of the first induction step leading to (4.4).

Proposition 4.9 *We have*

$$p^{-n}(T_p - T_{g_0,p})_{x_0}(Z, Z') \cong \mathscr{O}_m(p^{-1}) \tag{4.53}$$

in the sense of Notation A for some $\theta_m \in (0, 1/2)$. *Consequently,*

$$T_p = P_{\mathcal{H}_p} g_0 P_{\mathcal{H}_p} + \mathcal{O}(p^{-1}), \tag{4.54}$$

i.e., relation (4.7) *holds true in the sense of* (1.13).

Proof Let us compare the asymptotic expansion of T_p and $T_{g_0,p} = P_{\mathcal{H}_p} g_0 P_{\mathcal{H}_p}$. Using the Notation A, the expansion (3.10) (for $k_0 = 2$) reads for $\theta_m = 1/4(n+k+m+2)$,

$$\begin{aligned} & p^{-n} T_{g_0,p,x_0}(Z, Z') \\ & \quad \cong \big(g_0(x_0)\mathscr{P}_{x_0} + Q_{1,x_0}(g_0)\mathscr{P}_{x_0} p^{-1/2} + Q_{2,x_0}(g_0)\mathscr{P}_{x_0} p^{-1}\big)(\sqrt{p}Z, \sqrt{p}Z') \\ & \qquad + \mathscr{O}_m(p^{-1}), \end{aligned} \tag{4.55}$$

since $Q_{0,x_0}(g_0) = g_0(x_0)\,\mathrm{Id}_{E_{x_0}}$ by (3.13). The expansion (4.2) (also for $k_0 = 2$) takes the form for θ_m in (4.2),

$$\begin{aligned} & p^{-n} T_{p,x_0}(Z, Z') \\ & \quad \cong \big(g_0(x_0)\mathscr{P}_{x_0} + \mathscr{Q}_{1,x_0}\mathscr{P}_{x_0} p^{-1/2} + \mathscr{Q}_{2,x_0}\mathscr{P}_{x_0} p^{-1}\big)(\sqrt{p}Z, \sqrt{p}Z') + \mathscr{O}_m(p^{-1}), \end{aligned} \tag{4.56}$$

where we have used Proposition 4.3 and the definition (4.5) of g_0. Thus subtracting (4.55) from (4.56) we obtain for some $\theta_m \in (0, 1/2)$,

$$\begin{aligned} & p^{-n}(T_p - T_{g_0,p})_{x_0}(Z, Z') \\ & \quad \cong \big((\mathscr{Q}_{1,x_0} - Q_{1,x_0}(g_0))\mathscr{P}_{x_0}\big)(\sqrt{p}Z, \sqrt{p}Z')p^{-1/2} + \mathscr{O}_m(p^{-1}). \end{aligned} \tag{4.57}$$

Thus, it suffices to prove the following result.

Lemma 4.10

$$F_{1,x} := \mathscr{Q}_{1,x} - Q_{1,x}(g_0) \equiv 0. \tag{4.58}$$

Proof We note first that $F_{1,x}$ is an odd polynomial in z and $\overline{z}'$; we verify this statement as in Lemma 4.4. Thus the constant term of $F_{1,x}$ vanishes. To show that the rest of the term vanish, we consider the decomposition $F_{1,x} = \sum_{i\geqslant 0} F_{1,x}^{(i)}$ in homogeneous polynomials $F_{1,x}^{(i)}$ of degree i. To prove (4.58), it suffices to show that

$$F_{1,x}^{(i)}(z,\overline{z}') = 0 \text{ for all } i > 0 \text{ and } z, z' \in \mathbb{C}^n. \tag{4.59}$$

The proof of (4.59) is similar to that of (4.14). Namely, we define as in (4.17) the operator $F_1^{(i)}$, by replacing $F_x^{(i)}(0,\overline{z}')$ by $F_{1,x}^{(i)}(0,\overline{z}')$, and we set (analogue to (4.20))

$$\mathscr{T}_{p,1} = T_p - P_{\mathcal{H}_p} g_0 P_{\mathcal{H}_p} - \sum_{i\leqslant \deg F_1} (F_1^{(i)} P_{\mathcal{H}_p}) p^{(i-1)/2}. \tag{4.60}$$

Due to (3.10) and (4.2), there exist polynomials $\tilde{R}_{r,x_0} \in \mathbb{C}[Z, Z']$ of the same parity as r such that the following expansion in the normal coordinates around $x_0 \in X$ holds for any $k_0 \geqslant 2$:

$$p^{-n}\mathscr{T}_{p,1,x_0}(0,Z') \cong \sum_{r=2}^{k_0} (\tilde{R}_{r,x_0}\mathscr{P}_{x_0})(0,\sqrt{p}Z')p^{-r/2} + \mathscr{O}_m(p^{-k_0/2}), \tag{4.61}$$

in the sense of Notation A with $\theta_{k_0,m}$ the minimum of $\theta_{k_0,m}$ in (3.10) and $\theta_{k_0,m}$ in (4.2). This is an analogue of (4.21). Now we can repeat with obvious modifications the proof of (4.14) and obtain the analogue of (4.14) with F_x replaced by $F_{1,x}$. This completes the proof of Lemma 4.10. □

Lemma 4.10 and the expansion (4.57) imply immediately Proposition 4.9. □

Proof of Theorem 4.1 Proposition 4.9 shows that the asymptotic expansion (4.4) of T_p holds for $q = 0$. Moreover, if T_p is self-adjoint, then from (4.56), g_0 is also self-adjoint. We show inductively that (4.4) holds for every $q \in \mathbb{N}$. To prove (4.4) for $q = 1$ let us consider the operator $p(T_p - P_{\mathcal{H}_p} g_0 P_{\mathcal{H}_p})$. We have to show now that $p(T_p - P_{\mathcal{H}_p} g_0 P_{\mathcal{H}_p})$ satisfies the hypotheses of Theorem 4.1. Due to Lemma 3.1 and Theorem 4.1 (ii), the first two conditions are easily verified. To prove the third, just subtract the asymptotics of $T_{p,x_0}(Z,Z')$ (given by (4.2)) and $T_{g_0,p,x_0}(Z,Z')$ (given by (3.10)). Taking into account Proposition 4.3 and (4.58) the coefficients of p^0 and $p^{-1/2}$ in the difference vanish, which yields the desired conclusion.

Propositions 4.3 and 4.9 applied to $p(T_p - P_{\mathcal{H}_p} g_0 P_{\mathcal{H}_p})$ yield $g_1 \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ such that (4.4) holds true for $q = 1$.

We continue in this way the induction process to get (4.4) for any q. This completes the proof of Theorem 4.1. □

5 Algebra of Toeplitz Operators

The Poisson bracket $\{\cdot,\cdot\}$ on $(X, 2\pi\omega)$ is defined as follows. For $f, g \in \mathscr{C}^\infty(X)$, let ξ_f be the Hamiltonian vector field generated by f, which is defined by $2\pi i_{\xi_f}\omega = df$. Then

$$\{f,g\} = \xi_f(dg). \tag{5.1}$$

Proof of Theorem 1.2 First, it is obvious that $P_{\mathcal{H}_p} T_{f,p} T_{g,p} P_{\mathcal{H}_p} = T_{f,p} T_{g,p}$. To prove (4.1), note that from Lemma 3.1 and (3.10), we know that for any $k \in \mathbb{N}$ there exist $C_k > 0$ and $M_k > 0$ such that for all $(x,x') \in X \times X$,

$$\left| T_{f,p}(x,x') \right|_{\mathscr{C}^k(X\times X)} \leqslant C_k p^{M_k}. \tag{5.2}$$

For any $b > 0$ and $0 < \theta < 1$, if $d(x,x') > bp^{-\theta/2}$, then

$$T_{f,p}T_{g,p}(x,x') = \left(\int_{B^X(x,\frac{b}{2}p^{-\theta/2})} + \int_{X\setminus B^X(x,\frac{b}{2}p^{-\theta/2})} \right) T_{f,p}(x,x'')T_{g,p}(x'',x')\mathrm{d}v_X(x''). \tag{5.3}$$

Then (4.1) follows from (3.2), (3.5), (5.2) and (5.3). Like (3.18), for $|Z|, |Z'| < \frac{b}{2}p^{-\theta/2}$, we have

$$\begin{aligned}(T_{f,p}T_{g,p})_{x_0}(Z,Z') \\ = \int_{|Z''|<bp^{-\theta_1/2}} T_{f,p,x_0}(z,z'')T_{g,p,x_0}(Z'',Z')\kappa_{x_0}(Z'')\mathrm{d}v_{TX}(Z'') + \mathscr{O}_m(p^{-\infty}).\end{aligned} \tag{5.4}$$

By Lemmas 3.1 and 3.3 and (5.4), we deduce as we obtain (3.45) from Proposition 2.1, (3.15) and (3.18) in the proof of Lemma 3.3 that for $|Z|, |Z'| < \frac{b}{2}p^{-\theta/2}$, we have

$$\begin{aligned}p^{-n}(T_{f,p}T_{g,p})_{x_0}(Z,Z') \cong \sum_{r=0}^{k_0} \left(Q_{r,x_0}(f,g)\mathscr{P}_{x_0}\right)(\sqrt{p}Z, \sqrt{p}Z')p^{-r/2} \\ + \mathscr{O}_m(p^{-k_0/2}),\end{aligned} \tag{5.5}$$

with

$$Q_{r,x_0}(f,g) = \sum_{r_1+r_2=r} \mathscr{K}[Q_{r_1,x_0}(f), Q_{r_2,x_0}(g)]. \tag{5.6}$$

Thus, $T_{f,p}T_{g,p}$ is a Toeplitz operator by Theorem 4.1. Moreover, it follows form the proofs of Lemma 3.3 and Theorem 4.1 that $g_l = C_l(f,g)$, where C_l are bidifferential operators.

The rest of the proof of Theorem 1.2 is exactly the same as that of [22, Theorem 1.1] and we omit it here. This finishes the proof of Theorem 1.2. □

Proof of Theorem 1.4 Take a point $x_0 \in X$ and $u_0 \in E_{x_0}$ with $|u_0|_{h^E} = 1$ such that $|f(x_0)(u_0)| = \|f\|_\infty$. Recall that we trivialized the bundles L, E in normal coordinates near x_0, and e_L is the unit frame of L which trivializes L. Moreover, in these normal coordinates, u_0 is a trivial section of E. Considering the sequence of sections $S^p_{x_0} = p^{-n/2} P_{\mathcal{H}_p}(e_L^{\otimes p} \otimes u_0)$, we have by (3.8),

$$\left\| T_{f,\,p}\, S^p_{x_0} - f(x_0) S^p_{x_0} \right\|_{L^2} \leqslant \frac{C}{\sqrt{p}} \| S^p_{x_0} \|_{L^2}\,, \tag{5.7}$$

which immediately implies (1.19). □

6 Proof of Theorem 1.5

In this section, we show how to adapt the results of [14] in order to give a proof to Theorem 1.5, that is the computation of the coefficient $C_1(f, g)$ of Theorem 1.2.

Fix $x_0 \in X$ and $\varepsilon \in (0, a^X/4)$. It is shown in [21, Theorem 1.4] that the restriction on $B^X(x_0, \varepsilon)$ of the operator $\Delta_{p,\Phi}$ defined in (1.4) is equal, through the trivializations given in Sect. 2 and after a convenient rescaling in $\sqrt{p} := 1/t$, to an operator $\mathscr{L}_t$ on $B^{T_{x_0}X}(0, \varepsilon/t)$ satisfying

$$\mathscr{L}_t = \mathscr{L}_0 + \sum_{r=1}^{m} t^r \mathcal{O}_r + \mathscr{O}(t^{m+1}), \tag{6.1}$$

for any $m \in \mathbb{N}$, where $\{\mathcal{O}_r\}_{r\in\mathbb{N}}$ is a family of differential operators of order equal or less than 2, with coefficients explicitly computable in terms of local data, and where the differential operator $\mathscr{O}(t^{m+1})$ has its coefficients and their derivatives up to order k dominated by $C_k t^{m+1}$ for any $k \in \mathbb{N}$ and $C_k > 0$.

Moreover, as explained in [21, §1.4], the differential operator $\mathscr{L}_0$ acts on the scalar part of smooth functions on $\mathbb{R}^{2n}$ with values in E_{x_0}, and the spectrum of its restriction to $L^2(\mathbb{R}^{2n})$ is given by $\{4\pi n \mid n \in \mathbb{N}\}$. Furthermore, the kernel of the orthogonal projection P from $L^2(\mathbb{R}^{2n}, E_{x_0})$ to $\mathrm{Ker}(\mathscr{L}_0)$ is given by $\mathscr{F}_{0,x_0}(Z, Z') = \mathscr{P}(Z, Z')\, \mathrm{Id}_{E_{x_0}}$ as in (2.5). We write $P^\perp = \mathrm{Id} - P$, and define the operator $(\mathscr{L}_0)^{-1} P^\perp$ by inverting the positive eigenvalues of $\mathscr{L}_0|_{L^2(\mathbb{R}^{2n})}$.

As shown in [21], there is a direct method to compute the family $\{\mathscr{F}_{r,x_0}(Z, Z')\}_{r\in\mathbb{N}}$ defined in Theorem 2.2, using (6.1). The following lemma, which has been established in [21, Theorem 1.16, (1.30), (1.111)], gives the first elements of this family.

Lemma 6.1 *For any $r \in \mathbb{N}$, let $\mathscr{F}_{r,x_0}$ be the operator associated to the kernel $\mathscr{F}_{r,x_0}(Z, Z')$, and let the differential operators $\mathscr{O}_1$ and $\mathscr{O}_2$ be as in (6.1). Then the following formulas hold:*

$$\begin{aligned}
\mathscr{F}_{1,x_0} &= -(\mathscr{L}_0)^{-1} P^\perp \mathscr{O}_1 P - P \mathscr{O}_1 (\mathscr{L}_0)^{-1} P^\perp, \\
\mathscr{F}_{2,x_0} &= (\mathscr{L}_0)^{-1} P^\perp \mathscr{O}_1 (\mathscr{L}_0)^{-1} P^\perp \mathscr{O}_1 P - (\mathscr{L}_0)^{-1} P^\perp \mathscr{O}_2 P \\
&\quad + P \mathscr{O}_1 (\mathscr{L}_0)^{-1} P^\perp \mathscr{O}_1 (\mathscr{L}_0)^{-1} P^\perp - P \mathscr{O}_2 (\mathscr{L}_0)^{-1} P^\perp \\
&\quad + (\mathscr{L}_0)^{-1} P^\perp \mathscr{O}_1 P \mathscr{O}_1 (\mathscr{L}_0)^{-1} P^\perp - P \mathscr{O}_1 (\mathscr{L}_0)^{-2} P^\perp \mathscr{O}_1 P.
\end{aligned} \tag{6.2}$$

Moreover, $\mathscr{O}_1$ commutes with any $A \in \mathrm{End}(E_{x_0})$, and we have the formula

$$P\mathscr{O}_1 P = 0. \tag{6.3}$$

In particular, $\mathscr{F}_{0,x_0}$ and $\mathscr{F}_{1,x_0}$ commute with any $A \in \mathrm{End}(E_{x_0})$.

Lemma 6.1 corresponds to [14, Lemma 3.3], and the following technical Lemma corresponds to [14, Lemma 3.5]. It was essentially proved in [21, (2.25)].

Lemma 6.2 *The following formulas hold:*

$$\begin{aligned}
&(P\mathscr{O}_1(\mathscr{L}_0)^{-1}P^{\perp})(0,Z') = 0,\\
&(P\mathscr{O}_1(\mathscr{L}_0)^{-1}P^{\perp})(Z,0) = 0,\\
&((\mathscr{L}_0)^{-1}P^{\perp}\mathscr{O}_1 P)(Z,0) = 0.
\end{aligned} \tag{6.4}$$

The result of Lemma 6.2 is a simple computation from the first line of [21, (2.25)], using [21, (1.98), (1.99)] and recalling the formula $T^*(Z,Z') = T(Z',Z)^*$ for the kernel of the dual T^* of an operator T. In fact, all the kernels associated to the situation in this paper are the degree-0 part of the kernels of the corresponding situation in [14]. Lemma 6.2 is then an expression of the fact that the corresponding formulas in [14, Lemma 3.5] have vanishing degree-0 part.

Now, from the proof of Theorem 4.1, the following formula holds,

$$C_1(f,g)(x_0) = Q_{2,x_0}(f,g)(0,0) - Q_{2,x_0}(fg)(0,0), \tag{6.5}$$

where the coefficients $Q_{2,x_0}(f,g)$ and $Q_{2,x_0}(fg)$ have been defined in Lemma 3.3 and (5.6) respectively. Note that the formula (6.5) is actually simpler than the one given in [22, (4.82)], due to the fact that we only need to consider the degree-0 part. The following Proposition corresponds to [14, (3.19)], and is easily seen to imply Theorem 1.5 in the trivialization described in Sect. 2.

Proposition 6.3 *Assume that $g^{TX}(\cdot,\cdot) = \omega(\cdot, J\cdot)$. Then in the complex coordinates of $Z \in \mathbb{R}^{2n} \simeq (T_{x_0}X, J)$ as in Sect. 2, the following formula holds,*

$$Q_{2,x_0}(f,g)(0,0) - Q_{2,x_0}(fg)(0,0) = -\frac{1}{\pi}\sum_{j=1}^{n}\frac{\partial f_{x_0}}{\partial z_j}(0)\frac{\partial g_{x_0}}{\partial \bar{z}_j}(0). \tag{6.6}$$

Proof By (2.11) and following [22, Lemma 2.2, Example 2.3], the kernel calculus of [20, § 7.1] as described in [14, § 2.3] is still valid. Note that the assumption $g^{TX}(\cdot,\cdot) = \omega(\cdot, J\cdot)$ is equivalent to $a_j = 2\pi$ in (2.4), for all $1 \le j \le n$.

Recall the formulas (3.12) and (5.6) with $r = 2$ for the second and the first term of (6.5) respectively. Furthermore, by Lemma 6.2, by (3.12) with $r = 1$ and as in the proof of [14, Lemma 3], the following formula still holds,

$$Q_{1,x_0}(f) = f(x_0)J_{1,x_0} + \mathscr{K}\left[J_{1,x_0}, \sum_{j=1}^{2n}\frac{\partial f_{x_0}}{\partial Z_j}(0)Z_j J_{0,x_0}\right]. \tag{6.7}$$

Then the computations of [14, § 3.2] go through, and even simplify due to Lemma 6.2. In particular, writing Z_j, Z'_j for the operators of scalar multiplication by Z_j, Z'_j in $\mathrm{End}(E_{x_0})[Z, Z']$ for all $1 \leq j \leq 2n$, by (2.5), (2.11), (6.2) and the first line of (6.4), we have as in [14, (3.46), (3.49)],

$$\begin{aligned}
&\mathscr{K}\big[J_{0,x_0}, \mathscr{K}\big[J_{1,x_0}, Z_j J_{0,x_0}\big]\big](0,0)\\
&\quad = (\mathscr{F}_{0,x_0}\mathscr{F}_{1,x_0} Z_j \mathscr{F}_{0,x_0})(0,0)\\
&\quad = -(P\mathscr{O}_1(\mathscr{L}_0)^{-1}P^{\perp}Z_j P)(0,0)\\
&\quad = -\int_{\mathbb{R}^{2n}} (P\mathscr{O}_1(\mathscr{L}_0)^{-1}P^{\perp})(0,Z)Z_j\mathscr{P}(Z,0)dZ = 0.
\end{aligned} \tag{6.8}$$

By (6.2) and the second line of (6.4), we have as in [14, (3.53)],

$$\begin{aligned}
&\mathscr{K}\big[Z'_j J_{0,x_0}, \mathscr{K}\big[J_{1,x_0}, J_{0,x_0}\big]\big](0,0)\\
&= (Z'_j\mathscr{F}_{0,x_0}\mathscr{F}_{1,x_0}\mathscr{F}_{0,x_0})(0,0)\\
&= -(Z'_j P(\mathscr{L}_0)^{-1}P^{\perp}\mathscr{O}_1 P)(0,0)\\
&= -\int_{\mathbb{R}^{2n}} Z'_j\mathscr{P}(0,Z')((\mathscr{L}_0)^{-1}P^{\perp}\mathscr{O}_1 P)(Z',0)dZ' = 0.
\end{aligned} \tag{6.9}$$

Finally, writing z_j for the operator of multiplication by z_j in $\mathrm{End}(E_{x_0})[Z, Z']$ for all $1 \leq j \leq n$, by (6.2) and the last line of (6.4), we have as in [14, (3.65)],

$$\begin{aligned}
P(z_j(\mathscr{L}_0)^{-1}P^{\perp}\mathscr{O}_1 P)(0,0) &= \int_{\mathbb{R}^{2n}} \mathscr{P}(0,Z)z_j((\mathscr{L}_0)^{-1}P^{\perp}\mathscr{O}_1 P)(Z,0)dZ\\
&= 0.
\end{aligned} \tag{6.10}$$

Then (6.6) is precisely [14, (3.78)]. $\square$

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