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BEREZIN-TOEPLITZ QUANTIZATION OF NON-COMPACT MANIFOLDS

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ABSTRACT. We develop Berezin-Toeplitz quantization in a non-compact complex geometric setting. Let (X,Θ) be a Hermitian manifold, (L,h^L) a positive holomorphic line bundle, and (E,h^E) a holomorphic Hermitian vector bundle. Assuming that the Kodaira Laplacian on $(0,1)$-forms with values in $L^p\otimes E$ has a spectral gap growing linearly in p, we prove that the Bergman projection onto the L^2-holomorphic space $H^0_{(2)}(X,L^p\otimes E)$ enjoys the usual off-diagonal decay and admits a full asymptotic expansion on compact subsets as $p\to\infty$. As a consequence, for every smooth symbol $f\in\mathcal{C}^\infty_{\mathrm{const}}(X,\mathrm{End}(E))$ (constant outside a compact set), the associated Toeplitz operators $T_{f,p}=P_pfP_p$ form a closed algebra and satisfy a complete composition expansion, yielding a star-product on $\mathcal{C}^\infty_{\mathrm{const}}(X,\mathrm{End}(E))$ and the expected semiclassical commutator formula. We also give intrinsic criteria characterizing Toeplitz families with compactly supported kernels.

We then provide geometric conditions guaranteeing the spectral gap on large classes of non-compact manifolds, via fundamental L^2-estimates for $\bar\partial$ on complete Hermitian manifolds (including bounded-geometry complete Kähler manifolds, Kähler-Einstein manifolds, pseudoconvex/weakly 1-complete, and quasi-projective manifolds). Finally, for compactly supported bounded symbols, we prove a Szegő-type theorem describing the eigenvalue distribution of the compact Toeplitz operators $T_{f,p}$ as $p\to\infty$.

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1. INTRODUCTION

Berezin-Toeplitz quantization is one of the most concrete and flexible quantization procedures. On a compact Kähler manifold (X, ω) endowed with a prequantum line bundle (L, h^L) such that $\omega = \frac{\sqrt{-1}}{2\pi} R^L$, and a holomorphic Hermitian vector bundle (E, h^E), one considers the quantum spaces $H^0(X, L^p \otimes E)$, $p \in \mathbb{N}^*$, and the orthogonal Bergman projections P_p onto them. To every observable $f \in \mathscr{C}^\infty(X, \mathrm{End}(E))$, one associates the Toeplitz operator

$$T_{f,p} = P_p \, f \, P_p,$$

whose semiclassical behavior as $p \to +\infty$ encodes both the underlying geometry and the deformation of the commutative algebra of observables into a star-product. In the compact setting, this circle of ideas has a long history, from the microlocal approach of Boutet de Monvel–Guillemin [10] to the algebro-geometric and analytic developments of Bordemann–Meinrenken–Schlichenmaier, Schlichenmaier, and many others (see e.g. [8, 26, 62] and the references therein).

We have introduced in [47, 49] a different approach based on the existence of a full off-diagonal asymptotic expansion, of the Bergman kernel and its refinements, which, in turn, yield the Toeplitz calculus and the associated star-product. This method found several applications, cf. [2, 21, 22, 32, 33, 51, 53].

The goal of the present paper is to exhibit large classes of *non-compact* complex manifolds for which the Berezin–Toeplitz quantization package continues to hold, in a form suitable for geometric applications. Going beyond compactness raises two intertwined issues. First, the quantum spaces of L^2-holomorphic sections may be infinite-dimensional, and the Bergman projection need not enjoy global smoothing properties. Second, even when L is positive, the Bergman kernel asymptotics may fail without additional control at infinity. Our guiding principle is that, on non-compact manifolds, the Toeplitz calculus is available as soon as one can ensure a *spectral gap* for the Kodaira Laplacian in the high tensor power limit. This point of view was used in the analytic localization method developed in [47, 49] and underlies many vanishing theorems.

A Toeplitz package under a spectral gap. Let (X, Θ) be a Hermitian manifold, (L, h^L) a positive holomorphic line bundle, (E, h^E) a holomorphic Hermitian vector bundle, and set $\omega = \frac{\sqrt{-1}}{2\pi} R^L$. Assuming that the Kodaira Laplacian on $(0,1)$-forms with values in $L^p \otimes E$ has a *spectral gap* growing linearly in p (cf. (2.19)), we prove that the Bergman kernels of the L^2-holomorphic spaces $H^0_{(2)}(X, L^p \otimes E)$ admit a full asymptotic expansion on compact subsets, together with the usual off-diagonal exponential decay. Moreover, the full Berezin–Toeplitz calculus holds for the natural algebra $\mathscr{C}^\infty_{\mathrm{const}}(X, \mathrm{End}(E))$ of smooth endomorphism-valued functions that are constant outside a compact set. This is summarized in Theorem 2.7 (and its variants, including the compact case Theorem 2.8). In particular, the product and commutator expansions

$$T_{f,p} T_{g,p} \sim \sum_{r \geq 0} p^{-r} T_{C_r(f,g),p}, \qquad [T_{f,p}, T_{g,p}] \sim \frac{\sqrt{-1}}{p} T_{\{f,g\},p} + \cdots$$

hold in the appropriate sense, yielding a star-product on $\mathscr{C}^\infty_{\mathrm{const}}(X, \mathrm{End}(E))$; see Theorem 2.24. An essential result is an intrinsic criterion characterizing Toeplitz families with compactly supported kernels (Theorem 2.19), which also provides an algorithmic way to compute the coefficients $C_r(f, g)$.

Geometric conditions implying the spectral gap. A substantial part of the paper is devoted to verifying the spectral gap hypothesis in a wide range of non-compact situations. We first recall the *fundamental estimate* approach of [47, 48], which gives spectral gaps (and L^2-cohomology vanishing) under explicit curvature and torsion bounds on complete Hermitian manifolds. This is formulated here as Conditions 3.2 and 3.5, leading to Theorems 3.3, 3.6, and the resulting Toeplitz package Theorems 3.4, 3.7 for compactly supported (or constant outside a compact set) symbols. We then illustrate these criteria through a collection of standard non-compact geometries, showing that the Toeplitz package is applicable far beyond the compact case. Typical examples include:

Complete Kähler manifolds of bounded geometry. If (X, Θ) is complete with bounded geometry (uniform lower bound on the injectivity radius and uniform bounds on the curvature and its derivatives), and if (L, h^L) and (E, h^E) have bounded geometry, then the Berezin-Toeplitz quantization holds for observables whose derivatives are all bounded; cf. Theorem 3.21. In this setting, we mention the recent work [35], which investigates the exponential decay of the Bergman kernel and applies these results to the existence of Poincaré series.

Kähler–Einstein and negatively curved geometries. Many canonical complete metrics arising in complex geometry fall into the previous framework, such as complete Kähler–Einstein manifolds of negative Ricci curvature. In these settings, the positivity of the canonical line bundle combines with the global control provided by the metric to yield the desired spectral gap and hence the Bergman kernel expansion on compact subsets.

Pseudoconvex domains and weighted Bergman spaces. For pseudoconvex domains (in particular, strongly pseudoconvex domains) endowed with a positive line bundle (for example, the trivial line bundle endowed with a strictly plurisubharmonic weight), we obtain the spectral gap for the Kodaira Laplacian with $\overline{\partial}$-Neumann boundary conditions. Our results then give a Toeplitz calculus, recovering and extending classical Toeplitz operators on Bergman-type spaces to a geometric line bundle setting.

Stein, 1-convex and weakly 1-complete manifolds. By using a smooth plurisubharmonic exhaustion that is strictly plurisubharmonic outside a compact set, one can build Hermitian metrics and weights with coercivity at infinity. This yields a spectral gap for the high-power Kodaira Laplacians and, consequently, Toeplitz quantization. This provides a flexible class of examples where the “quantum spaces” are naturally infinite-dimensional.

Quasi-projective manifolds and Poincaré-type ends. For $X = \overline{X} \backslash D$ with D being a normal crossings divisor in a compact manifold $\overline{X}$, one can work with complete Poincaré-type (or cusp-type) metrics near D and with line bundles whose curvature dominates the metric. We again derive a spectral gap and obtain Toeplitz asymptotics on compact subsets of X. This is particularly suited to arithmetic and locally symmetric situations, where finite-volume quotients naturally carry cusp geometries.

Let us mention the related work [29], where the authors develop Berezin–Toeplitz quantization with quantum spaces being the spectral spaces of the Kodaira Laplacian on the set where the curvature is positive; see also [34] for the symplectic case.

A Szegő-type theorem for compactly supported symbols. Finally, we prove a spectral asymptotics result for compact Toeplitz operators associated with bounded symbols of compact support. Under the hypotheses of Theorem 2.7, the operators $T_{f,p}$ are compact when f has compact support, and their eigenvalue distribution satisfies a Szegő-type law (Theorems 4.2), extending our non-compact framework to classical phenomena for Toeplitz quantization and Bergman spaces.

Organization of the paper. In Section 2, we set up the analytic framework and recall the Bergman kernel expansion and Toeplitz calculus under a spectral gap, proving, in particular, Theorem 2.7, as well as the Toeplitz criteria and product expansions. Section 3 is devoted to geometric conditions ensuring the spectral gap on non-compact manifolds and to a collection of examples. Section 4 establishes the Szegő-type eigenvalue asymptotics for compactly supported symbols.

2. THE BEREZIN-TOEPLITZ PACKAGE AND THE SPECTRAL GAP

2.1. **An abstract setting.** Let (X,J) be a complex manifold of complex dimension n, and let g^{TX} be a J-invariant Riemannian metric. Let Θ be the associated real $(1,1)$-form $\Theta(X,Y)=g^{TX}(JX,Y)$. The Riemannian volume form is $dv_X=\Theta^n/n!$. Let (F,h^F) be a holomorphic Hermitian vector bundle on X. On $\mathscr{C}^\infty_0(X,F)$, we consider the L^2 inner product

$$\langle s_1,s_2\rangle := \int_X \langle s_1(x),s_2(x)\rangle_F \, dv_X(x)\,, \tag{2.1}$$

where $\langle\cdot,\cdot\rangle_F$ is induced by h^F. The completion of $\mathscr{C}^\infty_0(X,F)$ with respect to (2.1) is denoted by $L^2(X,F)=L^2(X,F,dv_X,h^F)$. We consider the space of holomorphic L^2 sections:

$$H^0_{(2)}(X,F) \coloneqq H^0_{(2)}(X,F,dv_X,h^F)=\big\{s\in L^2(X,F,dv_X,h^F) : s \text{ is holomorphic}\big\}\,. \tag{2.2}$$

We deduce from the Cauchy estimates for holomorphic functions that for every compact set $K\subset X$ there exists $C_K>0$ such that for all $s\in H^0_{(2)}(X,F)$,

$$\sup_{x\in K}|s(x)|\leqslant C_K\|s\|_{L^2}\,. \quad \text{for all } s\in H^0_{(2)}(X,L^p\otimes E)\,. \tag{2.3}$$

This implies that $H^0_{(2)}(X,F)$ is a closed subspace of $L^2(X,F)$. Moreover, $H^0_{(2)}(X,F)$ is separable (cf. [69, p. 60]).

Definition 2.1. The *Bergman projection* is the orthogonal projection $P : L^2(X,F)\to H^0_{(2)}(X,F)$.

By (2.3), for a fixed $x\in X$, the evaluation functional $s\mapsto s(x)$ on $H^0_{(2)}(X,F)$ is continuous. By the Riesz representation theorem, there exists $P(x,\cdot)\in L^2(X,F_x\otimes F^*)$ such that

$$s(x)=\int_X P(x,x')s(x')dv_X(x')\,, \quad \text{for all } s\in H^0_{(2)}(X,F)\,. \tag{2.4}$$

Definition 2.2. The section $P(\cdot,\cdot)$ of $F\boxtimes F^*$ over $X\times X$ is called the *Bergman kernel* of $H^0_{(2)}(X,F)$.

Set $d:=\dim H^0_{(2)}(X,F)\in\mathbb{N}\cup\{\infty\}$. Let $\{s_i\}_{i=1}^d$ be any orthonormal basis of $H^0_{(2)}(X,F)$ with respect to the inner product (2.1). Using the estimate (2.3) we can show that

$$P(x,x')=\sum_{i=1}^d s_i(x)\otimes(s_i(x'))^*\in F_x\otimes F^*_{x'}\,, \tag{2.5}$$

where the right-hand side converges on every compact subset of X, together with all its derivatives (see e.g. [69, p. 62]). Thus $P(\cdot,\cdot)\in\mathscr{C}^\infty(X\times X,F\boxtimes F^*)$. It follows from (2.5) that $(Ps)(x)=\int_X P(x,x')s(x')dv_X(x')$, for all $s\in L^2(X,F)$, that is, $P(\cdot,\cdot)$ is the Schwartz kernel of the Bergman projection P.

Let L and E be two holomorphic vector bundles on X. We assume that L is a line bundle. The bundle E is an auxiliary twisting bundle. It is interesting to work with a twisting vector bundle E for several reasons. For example, when one has to deal with $(n,0)$-forms with values in $L^p:=L^{\otimes p}$ for $p\in\mathbb{N}^*$, one sets $E=\Lambda^n(T^{*(1,0)}X)$. From a physical point of view, the presence of E means a quantization of a system with several degrees of internal freedom. We fix Hermitian metrics h^L, h^E on L, E. We denote by $\mathscr{C}^\infty_b(X,\operatorname{End}(E))$ the space of smooth sections of $\operatorname{End}(E)$ whose all derivatives are bounded, cf. (3.21). The following definition introduces one of the main notions of this paper.

Definition 2.3. For a bounded section $f \in \mathscr{C}_b^\infty(X, \mathrm{End}(E))$, set

$$T_{f,p} : L^2(X, L^p \otimes E) \longrightarrow L^2(X, L^p \otimes E)\,, \quad T_{f,p} = P_p\, f\, P_p\,, \tag{2.6}$$

where the action of f is the fiberwise action of f. The map that associates $f \in \mathscr{C}_b^\infty(X, \mathrm{End}(E))$ with the family of bounded operators $\{T_{f,p}\}_p$ on $L^2(X, L^p \otimes E)$ is called the *Berezin-Toeplitz quantization*.

Note that $T_{f,p}$ is a Carleman operator with a smooth integral kernel given by

$$T_{f,p}(x, x') = \int_X P_p(x, x'') f(x'') P_p(x'', x')\, dv_X(x'')\,. \tag{2.7}$$

For two arbitrary bounded sections $f, g \in \mathscr{C}_b^\infty(X, \mathrm{End}(E))$, it is easy to see that the composition $T_{f,p}T_{g,p}$ is not of the form $T_{fg,p}$ in general. But we shall show that we have $T_{f,p}T_{g,p} \sim T_{fg,p}$ asymptotically for $p \to \infty$. In order to explain this, we introduce the following more general notion of a Toeplitz operator.

Definition 2.4. A *Toeplitz operator* is a sequence $\{T_p\}_{p\in\mathbb{N}}$ of linear operators

$$T_p : L^2(X, L^p \otimes E) \longrightarrow L^2(X, L^p \otimes E) \tag{2.8}$$

Verifying $T_p = P_p\, T_p\, P_p$, such that there exists a sequence $g_\ell \in \mathscr{C}_b^\infty(X, \mathrm{End}(E))$ such that for any $k \geqslant 0$, there exists $C_k > 0$ with

$$\Big\| T_p - \sum_{\ell=0}^{k} T_{g_\ell, p}\, p^{-\ell} \Big\| \leqslant C_k\, p^{-k-1} \quad \text{for any } p \in \mathbb{N}^*, \tag{2.9}$$

where $\|\cdot\|$ denotes the operator norm on the space of bounded operators. The section g_0 is called the *principal symbol* of $\{T_p\}$.

We express (2.9) symbolically by

$$T_p = \sum_{\ell=0}^{k} T_{g_\ell, p}\, p^{-\ell} + \mathscr{O}(p^{-k-1})\,, \;\; p \to \infty. \tag{2.10}$$

If (2.9) holds for any $k \in \mathbb{N}$, then we write (2.10) with $k = +\infty$. The Poisson bracket $\{f, g\}$ on $(X, 2\pi\omega)$ is defined by: for $f, g \in \mathscr{C}^\infty(X)$, if ξ_f denotes the Hamiltonian vector field generated by f, which is defined by $2\pi i_{\xi_f}\omega = df$, then

$$\{f, g\} = \xi_f(dg). \tag{2.11}$$

Endowed with the Poisson bracket, the algebra $\mathscr{C}^\infty(X)$ becomes a Lie algebra.

One of our goals is to show that $T_{f,p}T_{g,p}$ is a Toeplitz operator in the sense of Definition 2.4. This will be achieved by using the asymptotic expansions of the Bergman kernel and the kernels of the Toeplitz operators.

Definition 2.5. Let (X, Θ) be a Hermitian manifold, and let (L, h^L) and (E, h^E) be holomorphic Hermitian vector bundles on X of rank one and r, respectively. We assume that (L, h^L) is positive and denote by $\omega = \frac{\sqrt{-1}}{2\pi} R^L$ the Kähler metric induced by the curvature of (L, h^L). Let $\mathscr{A}$ be a $\mathbb{C}$-subalgebra of $\mathscr{C}^\infty(X, \mathrm{End}(E))$ such that the subalgebra

$$\mathscr{A}_{\mathbb{C}} := \{f \in \mathscr{A} : \text{there exists } \widetilde{f} \in \mathscr{C}^\infty(X) \text{ with } f = \widetilde{f}\, \mathrm{Id}_E\}$$

is a Lie subalgebra of $(\mathscr{C}^\infty(X), \{\cdot,\cdot\})$, where $\{\cdot,\cdot\}$ is the Poisson bracket on $(X, 2\pi\omega)$.

We say that the Berezin-Toeplitz package holds for the Kähler manifold (X, ω) and algebra $\mathscr{A}$ with quantum spaces $H^0_{(2)}(X, L^p \otimes E)$ if the following statements hold:

(i) For any $f, g \in \mathscr{A}$, the composition $T_{f,p}T_{g,p}$ admits the asymptotic expansion

$$T_{f,p}T_{g,p} = \sum_{r=0}^{\infty} p^{-r} T_{C_r(f,g),p} + \mathscr{O}(p^{-\infty})\,, \;\; p \to \infty \tag{2.12}$$

in the sense of (2.10), where C_r are bi-differential operators, $C_0(f,g)=fg$ and $C_r(f,g)\in\mathscr{C}^\infty(X,\operatorname{End}(E))$, especially, $\operatorname{supp}(C_r(f,g))\subset\operatorname{supp}(f)\cap\operatorname{supp}(g)$.
(ii) If $f,g\in\mathscr{A}_{\mathbb{C}}$, then we have

$$[T_{f,p}\,,T_{g,p}]=\frac{\sqrt{-1}}{p}T_{\{f,g\},p}+\mathcal{O}(p^{-2})\,,\;\;p\to\infty. \tag{2.13}$$

(iii) For every $f\in\mathscr{A}$, let us denote by $\|f\|_\infty:=\sup\{|f(x)(u)|_{h^E}/|u|_{h^E}:x\in X,0\neq u\in E_x\}$ Then for any $f\in\mathscr{A}$, there exists $C>0$ such that the norm of $T_{f,p}$ satisfies

$$\|f\|_\infty-\frac{C}{\sqrt{p}}\leq\|T_{f,p}\|\leq\|f\|_\infty\,,\qquad\lim_{p\to\infty}\|T_{f,p}\|=\|f\|_\infty. \tag{2.14}$$

(iv) The coefficients $C_r(f,g)$ can be algorithmically computed in terms of the geometric data Θ,h^L,h^E.

Let $\mathscr{C}^\infty_{\mathrm{const}}(X,\operatorname{End}(E))$ denote the algebra of smooth sections of X that are constant outside a compact set; that is, $f\in\mathscr{C}^\infty_{\mathrm{const}}(X,\operatorname{End}(E))$ if there exists a compact set $K\subset X$ such that $f=C\operatorname{Id}_{E_x}$ for all $x\in X\setminus K$. For any $f\in\mathscr{C}^\infty_{\mathrm{const}}(X,\operatorname{End}(E))$, we consider the Toeplitz operator $T_{f,p}$ as in (2.6):

$$T_{f,p}:L^2(X,L^p\otimes E)\longrightarrow L^2(X,L^p\otimes E)\,,\quad T_{f,p}=P_p\,f\,P_p\,. \tag{2.15}$$

Let $\Omega^{0,\bullet}(X,F)$ be the space of $(0,q)$-forms over X with values in F. We denote by $\Omega^{0,\bullet}_0(X,F)$ the subspace of $\Omega^{0,\bullet}(X,F)$ consisting of elements with compact support.

The Dolbeault operator acting on sections of the holomorphic vector bundle F gives rise to the Dolbeault complex $\big(\Omega^{0,\bullet}(X,F),\overline{\partial}^F\big)$. We denote by $\overline{\partial}^{F,*}$ the formal adjoint of $\overline{\partial}^F$ with respect to the L^2-inner product (2.1). Set

$$D=\sqrt{2}\big(\overline{\partial}^F+\overline{\partial}^{F,*}\big)\,,\quad\square^F=\tfrac{1}{2}D^2=\overline{\partial}^F\,\overline{\partial}^{F,*}+\overline{\partial}^{F,*}\,\overline{\partial}^F. \tag{2.16}$$

The operator $\square^F$ is called the *Kodaira Laplacian*. It acts on $\Omega^{0,\bullet}(X,F)$ and preserves its $\mathbb{Z}$-grading. Let us denote by $\Omega^{0,q}_{(2)}(X,F):=L^2(X,\Lambda^q(T^{*(0,1)}X)\otimes F)$.

In the following, we consider the maximal extensions of the operator $\overline{\partial}^F$ in the L^2-spaces and denote by $\overline{\partial}^{F,*}$ its Hilbert-space adjoint. The operator defined by

$$\begin{aligned}&\operatorname{Dom}(\square^F)=\big\{u\in\operatorname{Dom}(\overline{\partial}^F)\cap\operatorname{Dom}(\overline{\partial}^{F,*})\colon\overline{\partial}^F u\in\operatorname{Dom}(\overline{\partial}^{F,*})\,,\;\overline{\partial}^{F,*}u\in\operatorname{Dom}(\overline{\partial}^F)\big\}\,,\\&\qquad\square^F u=\overline{\partial}^F\overline{\partial}^{F,*}u+\overline{\partial}^{F,*}\overline{\partial}^F u\;\;\text{for }u\in\operatorname{Dom}(\square^F)\,.\end{aligned} \tag{2.17}$$

is a self-adjoint extension of the Kodaira Laplacian, called the Gaffney extension (see [47, Proposition 3.1.2]). The quadratic form associated with $\square^F$ is the form Q given by

$$\begin{aligned}&\operatorname{Dom}(Q):=\operatorname{Dom}(\overline{\partial}^F)\cap\operatorname{Dom}(\overline{\partial}^{F,*}),\\&Q(s_1,s_2)=\langle\overline{\partial}^F s_1,\overline{\partial}^F s_2\rangle+\langle\overline{\partial}^{F,*}s_1,\overline{\partial}^{F,*}s_2\rangle\,,\quad\text{for }s_1,s_2\in\operatorname{Dom}(Q).\end{aligned} \tag{2.18}$$

In our situation, we consider $F=L^p\otimes E$ the operators from (2.16) and their extensions by D_p and $\square_p$.

Definition 2.6 (spectral gap). Let (X,Θ) be a Hermitian manifold, let (L,h^L) and (E,h^E) be holomorphic Hermitian vector bundles on X of rank one and r, respectively. We say that the Kodaira-Laplacian has a *spectral gap* if there exist C_0, $C_L>0$ such that for any $p\in\mathbb{N}^*$,

$$\begin{aligned}&\|D_ps\|^2_{L^2}\geqslant(2C_0p-C_L)\|s\|^2_{L^2}\,,\\&s\in\operatorname{Dom}\left(\overline{\partial}^{L^p\otimes E}\right)\cap\operatorname{Dom}\left(\overline{\partial}^{L^p\otimes E,*}\right)\cap\Omega^{0,1}_{(2)}(X,L^p\otimes E).\end{aligned} \tag{2.19}$$

The following result generalizes the expansion of the Bergman kernel [47, Theorem 4.1.1] and the Berezin-Toeplitz package [47, Theorem 7.4.1], [49, Theorem 1.1] from the case of compact manifolds in the situation where a spectral gap exists.

Theorem 2.7. *Let* (X,Θ) *be a Hermitian manifold, let* (L,h^L) *and* (E,h^E) *be holomorphic Hermitian vector bundles on* X *of rank one and* r*, respectively. We assume that* (L,h^L) *is positive and denote by* $\omega=\frac{\sqrt{-1}}{2\pi}R^L$*. Assume that the Kodaira-Laplacian possesses a spectral gap, as stated in* (2.19)*. Then we have the following two statements:*
(1) The Bergman kernel asymptotics for $H^0_{(2)}(X,L^p\otimes E)$ *holds on compact sets of* X*. More precisely, there exist coefficients* $\boldsymbol{b}_r\in\mathscr{C}^\infty(X,\operatorname{End}(E))$*,* $r\in\mathbb{N}$*, such that for any compact set* $K\subset X$ *and any* $k,l\in\mathbb{N}$*, there exists* $C_{k,l,K}>0$ *such that*

$$\Big|\frac{1}{p^n}P_p(x,x)-\sum_{r=0}^{k}\boldsymbol{b}_r(x)p^{-r}\Big|_{\mathscr{C}^l(K)}\leqslant C_{k,l,K}\,p^{-k-1}, \tag{2.20}$$

where $\boldsymbol{b}_0=\det(\frac{\dot{R}^L}{2\pi})\operatorname{Id}_E$ *and* $\dot{R}^L\in\operatorname{End}(T^{(1,0)}X)$ *is defined by for* $W,Y\in T^{(1,0)}X$*,*

$$R^L(W,\overline{Y})=\langle\dot{R}^LW,\overline{Y}\rangle. \tag{2.21}$$

(2) The Berezin-Toeplitz quantization package holds for the Kähler manifold (X,ω)*, the algebra* $\mathscr{C}^\infty_{\rm const}(X,\operatorname{End}(E))$*, and quantum spaces* $H^0_{(2)}(X,L^p\otimes E)$ *in the sense of Definition* 2.5 *with* $C_0(f,g)=fg$*.*

The abstract setting above holds for compact Hermitian manifolds; see [50, Section 2.7].

Theorem 2.8. *Let* (X,Θ) *be a compact Hermitian manifold of dimension* n*, and* (L,h^L) *be a positive line bundle. We set* $\omega=\frac{\sqrt{-1}}{2\pi}R^L$*. Let* (E,h^E) *be a holomorphic Hermitian vector bundle on* X*. Then we have the following two statements:*

(1) The Bergman kernel asymptotics for $H^0(X,L^p\otimes E)$ *hold on* X*.*

(2) The Berezin-Toeplitz quantization package holds for the Kähler manifold (X,ω)*, the algebra* $\mathscr{C}^\infty(X,\operatorname{End}(E))$*, and quantum spaces* $H^0(X,L^p\otimes E)$*.*

Remark 2.9. Relations (2.13) and (2.14) were first proved in some special cases: in [38] for Riemann surfaces, in [14] for $\mathbb{C}^n$, and in [9] for bounded symmetric domains in $\mathbb{C}^n$, by using explicit calculations. Then Bordemann, Meinrenken, and Schlichenmaier [8] treated the case of a compact Kähler manifold (with $E=\mathbb{C}$) using the theory of Toeplitz structures (generalized Szegő operators) by Boutet de Monvel and Guillemin [10]. Moreover, Schlichenmaier [62] (cf. also [12, 37]) continued this train of thought and showed that for any $f,g\in\mathscr{C}^\infty(X)$, the product $T_{f,p}\,T_{g,p}$ has an asymptotic expansion (2.12) and constructed an associative star product geometrically.

2.2. Model situation: Bergman kernel on $\mathbb{C}^n$.

In this section, we introduce the model operator $\mathscr{L}$, a Kodaira-Laplace operator acting on $\mathbb{C}^n$, and describe its spectrum. We formulate the expansion of Bergman and Toeplitz kernels in terms of the Schwartz kernel associated with the projection onto the kernel of $\mathscr{L}$. Our analysis is based on the Fourier expansion with respect to the eigenfunctions of this model operator.

Write $\mathbb{C}^n\simeq\mathbb{R}^{2n}$ with real coordinates $Z=(Z_1,\dots,Z_{2n})$ and complex coordinates $z_j=Z_{2j-1}+\sqrt{-1}Z_{2j}$. Equip $\mathbb{C}^n$ with the Euclidean metric and the Kähler form

$$\omega=\frac{\sqrt{-1}}{2}\sum_{j=1}^{n}dz_j\wedge d\overline{z}_j\,.$$

Its volume form is the Euclidean volume form $dZ=dZ_1\cdots dZ_{2n}$. Let $(L^2(\mathbb{R}^{2n}),\|\cdot\|_{L^2})$ be the space of square integrable functions on $\mathbb{R}^{2n}$ with respect to the Lebesgue measure.

Let $0 < a_1 \leq a_2 \leq \ldots \leq a_n$. Let $L = \mathbb{C}$ be the trivial holomorphic line bundle on $\mathbb{C}^n$ with canonical section $\mathbf{1}$, endowed with the Hermitian metric

$$|\mathbf{1}|_{h^L}(z) = \exp\Big(-\frac{1}{4}\sum_{j=1}^{n} a_j|z_j|^2\Big) =: \rho(Z). \tag{2.22}$$

Thus $H^0_{(2)}(\mathbb{C}^n, L)$ identifies with the Segal–Bargmann space of holomorphic functions square-integrable with respect to $\rho\, dZ$. It is well-known that $\{z^\beta : \beta \in \mathbb{N}^n\}$ forms an orthogonal basis of this space.

We introduce the operators of creation

$$b_i = -2\frac{\partial}{\partial z_i} + \frac{a_i}{2}\,\overline{z}_i,$$

and annihilation

$$b_i^+ = 2\frac{\partial}{\partial \overline{z}_i} + \frac{a_i}{2}\, z_i,$$

and the model operator (complex harmonic oscillator)

$$\mathscr{L} = \sum_{i=1}^{n} b_i\, b_i^+. \tag{2.23}$$

The operator $\mathscr{L}$ is the $(0,0)$-part of the Kodaira Laplacian on $(\mathbb{C}^n, L)$ after conjugation by ρ, cf. [47, 48]. It acts as a densely defined self-adjoint operator on $(L^2(\mathbb{R}^{2n}), \|\cdot\|_{L^2})$.

Theorem 2.10 ([47, Theorem 4.1.20], [48, Theorem 1.15]). *The spectrum of $\mathscr{L}$ on $L^2(\mathbb{R}^{2n})$ is given by*

$$\operatorname{Spec}(\mathscr{L}) = \Big\{2\sum_{i=1}^{n} a_i\alpha_i \,:\, \alpha \in \mathbb{N}^n\Big\}. \tag{2.24}$$

Each $\lambda \in \operatorname{Spec}(\mathscr{L})$ is an eigenvalue of infinite multiplicity and an orthogonal basis of the eigenspace of $\lambda = 2\sum_{i=1}^{n} \alpha_i a_i$ is given by

$$B_\lambda = \Big\{b^\alpha\big(z^\beta \exp\big(-\frac{1}{4}\sum_{i=1}^{n} a_i|z_i|^2\big)\big)\,, \quad \textit{with } \beta \in \mathbb{N}^n\Big\}. \tag{2.25}$$

where $b^\alpha := b_1^{\alpha_1}\cdots b_n^{\alpha_n}$. Moreover, $\bigcup\{B_\lambda : \lambda \in \operatorname{Spec}(\mathscr{L})\}$ forms a complete orthogonal basis of $L^2(\mathbb{R}^{2n})$. In particular, an orthonormal basis of

$$\varphi_\beta(z) = \left(\frac{a^\beta}{(2\pi)^n 2^{|\beta|}\beta!}\prod_{i=1}^{n} a_i\right)^{1/2} z^\beta \exp\Big(-\frac{1}{4}\sum_{j=1}^{n} a_j|z_j|^2\Big), \quad \beta \in \mathbb{N}^n. \tag{2.26}$$

Let $\mathscr{P} : L^2(\mathbb{R}^{2n}) \to \operatorname{Ker}(\mathscr{L})$ be the orthogonal projection and $\mathscr{P}(Z, Z')$ its Schwartz kernel (with respect to dZ'). Summing (2.26) yields

$$\mathscr{P}(Z, Z') = \prod_{i=1}^{n}\frac{a_i}{2\pi}\ \exp\Big(-\frac{1}{4}\sum_{i=1}^{n} a_i\big(|z_i|^2 + |z_i'|^2 - 2z_i\overline{z}_i'\big)\Big). \tag{2.27}$$

2.3. **Asymptotic expansion of Bergman kernel.** Let (X, Θ) be a Hermitian manifold, and let (L, h^L) and (E, h^E) be holomorphic Hermitian vector bundles on X of rank one and r, respectively. We use the identifications and notations to state the asymptotics.

Normal coordinates. For $x \in X$, let a_x^X be the injectivity radius of (X, g^{TX}) at x. Denote by $B^X(x, \varepsilon)$ and $B^{T_xX}(0, \varepsilon)$ the open balls in X and T_xX, respectively. We identify them via the exponential map $Z \mapsto \exp_x^X(Z)$ for $\varepsilon \leq a_x^X$. For any subset $Y \subset X$, we set $a^Y = \inf_{x\in X} a_x^X$. Throughout the paper, $\varepsilon \in]0, a_x^X/4[$. Let $d(\cdot,\cdot)$ denote the Riemannian distance function associated with the Riemannian manifold (X, g^{TX}).

Basic trivialization. Fix $x_0 \in X$. For $Z \in B^{T_{x_0}X}(0,\varepsilon)$, we identify (L_Z, h^L_Z) and (E_Z, h^E_Z) with $(L_{x_0}, h^L_{x_0})$ and $(E_{x_0}, h^E_{x_0})$ by parallel transport along $\gamma_Z : [0,1] \ni u \longmapsto \exp^X_{x_0}(uZ)$, and similarly for $L^p \otimes E$ using $\nabla^{L^p\otimes E}$. With this identification, a function $f \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ corresponds to $f_{x_0} : B^{T_{x_0}X}(0,\varepsilon) \to \mathrm{End}(E_{x_0})$, $f_{x_0}(Z) = f \circ \exp^X_{x_0}(Z)$, and the Bergman kernel $P_p(x,x')$ induces a family of smooth sections $(Z,Z') \longmapsto P_{p,x_0}(Z,Z') \in \mathrm{End}(E_{x_0})$, $|Z|, |Z'| < \varepsilon$, depending smoothly on x_0.

Coordinates on $T_{x_0}X$. Let $\{w_i\}_{i=1}^n$ be an orthonormal basis of $T^{(1,0)}_{x_0}X$ and set $e_{2j-1} = \frac{1}{\sqrt{2}}(w_j + \overline{w}_j)$, $e_{2j} = \frac{\sqrt{-1}}{\sqrt{2}}(w_j - \overline{w}_j)$. We use real coordinates $Z = (Z_1, \dots, Z_{2n})$ on $T_{x_0}X \simeq \mathbb{R}^{2n}$ via $(Z_1,\dots,Z_{2n}) \longmapsto \sum_i Z_i e_i \in T_{x_0}X$, and complex coordinates $z = (z_1,\dots,z_n)$ on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$.

Volume form on $T_{x_0}X$. Let dv_{TX} be the Riemannian volume form on $(T_{x_0}X, g^{T_{x_0}X})$. Then there exists a smooth positive function κ_{x_0} such that

$$dv_X(Z) = \kappa_{x_0}(Z)\, dv_{TX}(Z), \qquad \kappa_{x_0}(0) = 1. \tag{2.28}$$

Sequences of operators. Let $\Theta_p : L^2(X, L^p \otimes E) \to L^2(X, L^p \otimes E)$ be a sequence of continuous operators with smooth kernel $\Theta_p(\cdot,\cdot)$ with respect to dv_X (e.g. $\Theta_p = T_{f,p}$). In the basic trivialization we write the corresponding kernels as $\Theta_{p,x_0}(Z,Z')$. We say that $\Theta_{p,x_0}(Z,Z') = \mathscr{O}(p^{-\infty})$ if for any $l, m \in \mathbb{N}$ there exists $C_{l,m} > 0$ such that $|\Theta_{p,x_0}(Z,Z')|_{\mathscr{C}^m(X)} \le C_{l,m} p^{-l}$. The asymptotics will be expressed in terms of the model Bergman kernel $\mathscr{P}_{x_0}$ of the operator $\mathscr{L}$ on $T_{x_0}X \simeq \mathbb{R}^{2n}$ (cf. (2.27)).

Notation 2.11. Fix $k \in \mathbb{N}$ and let $\{Q_{r,x_0} \in \mathrm{End}(E)_{x_0}[Z,Z'] :\ 0 \le r \le k,\ x_0 \in X\}$ be a family of polynomials in Z, Z', smooth in x_0. Let $K \subset X$ be compact and $\varepsilon' \in]0, a^K[$. We write

$$p^{-n}\Theta_{p,x_0}(Z,Z') \cong \sum_{r=0}^{k} (Q_{r,x_0}\mathscr{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z')\, p^{-r/2} + \mathscr{O}(p^{-(k+1)/2}) \tag{2.29}$$

on $\{(Z,Z') \in TX \times_K TX :\ |Z|, |Z'| < \varepsilon'\}$ if there exist $C_0 > 0$ and a decomposition

$$\begin{aligned} p^{-n}\Theta_{p,x_0}(Z,Z') - \sum_{r=0}^{k} (Q_{r,x_0}\mathscr{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z')\, \kappa_{x_0}^{-1/2}(Z)\kappa_{x_0}^{-1/2}(Z')\, p^{-r/2} \\ = \Psi_{p,k,x_0}(Z,Z') + \mathscr{O}(p^{-\infty}), \end{aligned} \tag{2.30}$$

where for every $l \in \mathbb{N}$ there exist $C_{k,l} > 0$, $M > 0$ such that for all $p \in \mathbb{N}^*$,

$$|\Psi_{p,k,x_0}(Z,Z')|_{\mathscr{C}^l(X)} \le C_{k,l}\, p^{-(k+1)/2}\, (1 + \sqrt{p}|Z| + \sqrt{p}|Z'|)^M\, e^{-C_0\sqrt{p}\,|Z-Z'|},$$

on $\{(Z,Z') \in TX \times_K TX :\ |Z|, |Z'| < \varepsilon'\}$.

The sequence P_p. Let $K \subset X$ be compact and $\varepsilon \in]0, a^K[$. Choose $\boldsymbol{f}_\varepsilon : \mathbb{R} \to [0,1]$ smooth, even, with $\boldsymbol{f}_\varepsilon(v) = 1$ for $|v| \le \varepsilon/2$ and $\boldsymbol{f}_\varepsilon(v) = 0$ for $|v| \ge \varepsilon$, and set

$$F_\varepsilon(a) = \Big(\int_{-\infty}^{+\infty} \boldsymbol{f}_\varepsilon(v)\, dv\Big)^{-1} \int_{-\infty}^{+\infty} e^{iva}\, \boldsymbol{f}_\varepsilon(v)\, dv. \tag{2.31}$$

Then $F_\varepsilon \in \mathcal{S}(\mathbb{R})$ is even and $F_\varepsilon(0) = 1$. We first record the far off-diagonal decay.

Theorem 2.12 (Off-diagonal expansion)**.** *Assume that the spectral gap condition* (2.19) *holds. Then for any compact set* $K \subset X$*, for any* $\ell, m \in \mathbb{N}$ *and* $\varepsilon \in]0, a^K[$*, there exists a positive constant* $C_{K,\ell,m,\varepsilon} > 0$ *such that for any* $p \geqslant 1$ *and* $x, x' \in K$*, the following estimate holds:*

$$|F_\varepsilon(D_p)(x,x') - P_p(x,x')|_{\mathscr{C}^m(K\times K)} \leqslant C_{K,\ell,m,\varepsilon}\, p^{-\ell}. \tag{2.32}$$

Especially, by setting $D_\varepsilon = \{(x,x') \in X \times X : d(x,x') > \varepsilon\}$, where $d(\cdot,\cdot)$ is the Riemannian distance on (X, g^{TX}), we have

$$|P_p(x,x')|_{\mathscr{C}^m(K\times K\setminus D_\varepsilon)} \leqslant C_{K,l,m,\varepsilon}\, p^{-\ell}\,. \tag{2.33}$$

The $\mathscr{C}^m$ norm in (2.32) and (2.33) is induced by ∇^L, ∇^E, h^L, h^E, and g^{TX}.

Proof. For $a \in \mathbb{R}$, set $\phi_p(a) = \mathbb{1}_{[\sqrt{p\mu_0},+\infty[}(|a|)\, F_\varepsilon(a)$. By (2.19), for p large enough,

$$F_\varepsilon(D_p) - P_p = \phi_p(D_p). \tag{2.34}$$

By (2.31), for any $m \in \mathbb{N}$ there exists $C_m > 0$ such that $\sup_{a\in\mathbb{R}} |a|^m |F_\varepsilon(a)| \leq C_m$. Using elliptic estimates for D_p^2 as in [47] (cf. [47, Lemma 1.6.2]) and Sobolev norms defined on a finite cover of K by geodesic balls, one obtains: for $l, m' \in \mathbb{N}$ there exists $C_{l,m'} > 0$ such that for $p \geq 1$, $\|D_p^{m'}\phi_p(D_p)Qs\|_{L^2} \leq C_{l,m'}\, p^{-l+2m}\, \|s\|_{L^2}$, for any differential operator Q of order m with compact support in a chart and scalar principal symbol. Combining this with the elliptic estimates (as in [47]) yields, for differential operators P, Q of orders m', m with compact supports in charts, $\|P\phi_p(D_p)Qs\|_{L^2} \leq C_l\, p^{-l}\, \|s\|_{L^2}$. By the Sobolev inequality and (2.34) we obtain (2.32). Finally, by finite propagation speed [47, Theorem D.2.1], $F_\varepsilon(D_p)(x,x')$ depends only on the restriction of D_p to $B^X(x,\varepsilon)$ and vanishes if $d(x,x') \geq \varepsilon$, hence (2.33). □

Theorem 2.13 (Near off-diagonal expansion). *There exists a family of polynomials in Z, Z' with the same parity as r,*

$$\{J_{r,x_0} \in \operatorname{End}(E)_{x_0}[Z,Z'] : r \in \mathbb{N}, x_0 \in X\},$$

with the following property. Assuming that the spectral gap condition (2.19) holds, then for any compact set $K \subset X$, for any $k \in \mathbb{N}$, and any $\varepsilon \in]0, a^K/4[$, we have

$$p^{-n} P_{p,x_0}(Z,Z') \cong \sum_{r=0}^{k} (J_{r,x_0}\mathscr{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z')p^{-\frac{r}{2}} + \mathcal{O}(p^{-\frac{k+1}{2}})\,, \tag{2.35}$$

on the set $\{(Z,Z') \in TX \times_K TX : |Z|, |Z'| < 2\varepsilon\}$, in the sense of Notation 2.11.

Proof. We only sketch the proof; details are in [16, Proposition 4.1, Theorem 4.18′] and [47, Proposition 4.1.1, Theorem 4.1.24]. Although X is assumed compact in [16], the arguments use only the spectral gap, so the localization applies verbatim here. One pulls back the geometric data by the exponential map to $T_{x_0}X \simeq \mathbb{R}^{2n}$, extends them suitably, and compares with the model kernel of $\mathscr{L}$ from Section 2.2. The conclusion follows from the spectral gap, the rescaling procedure, and functional analytic techniques inspired by Bismut–Lebeau [7, §11]. □

Setting $\boldsymbol{b}_r(x_0) = (J_{2r,x_0}\mathscr{P}_{x_0})(0,0)$, (2.35) implies the diagonal expansion:

Theorem 2.14 (On-diagonal expansion). *Assume that the spectral gap condition (2.19) holds. Then, for any compact set $K \subset X$, and for any k, $\ell \in \mathbb{N}$, there exists a positive constant $C_{K,k,\ell} > 0$ such that for any $p \geqslant 1$, the following estimate holds:*

$$\Big|P_p(x,x) - \sum_{r=0}^{k} \boldsymbol{b}_r(x)p^{n-r}\Big|_{\mathscr{C}^\ell(K)} \leqslant C_{K,k,\ell}\, p^{n-k-1}\,, \tag{2.36}$$

where $\boldsymbol{b}_0 = \det(\frac{\dot{R}^L}{2\pi})\operatorname{Id}_E$ and

$$\boldsymbol{b}_1 = \frac{1}{8\pi}\det\Big(\frac{\dot{R}^L}{2\pi}\Big)\Big[r^X_\omega - 2\Delta_\omega\Big(\log(\det(\dot{R}^L))\Big) + 4\sqrt{-1}\Lambda_\omega(R^E)\Big], \tag{2.37}$$

where r^X_ω, Δ_ω are the scalar curvature and the Bochner Laplacian associated with g^{TX}_ω.

The coefficients $\boldsymbol{b}_r$ can be computed from J_{r,x_0} and equivalently from the operators $\mathscr{F}_{r,x_0}$ with kernels

$$\mathscr{F}_{r,\,x_0}(Z,Z') = J_{r,\,x_0}(Z,Z')\,\mathscr{P}(Z,Z') \tag{2.38}$$

with respect to dZ'. Following [47, 48], one rescales the Kodaira-Laplacian and performs resolvent analysis. For $s \in \mathscr{C}^\infty(\mathbb{R}^{2n}, E_{x_0})$, $|Z| \le 2\varepsilon$, and $t = 1/\sqrt{p}$, set

$$(S_t s)(Z) := s(Z/t), \qquad \mathscr{L}_t := S_t^{-1}\kappa^{1/2}\,t^2(2\Box_p)\kappa^{-1/2}S_t. \tag{2.39}$$

Then [47, Theorem 4.1.7] gives second order operators $\mathcal{O}_r$ such that, as $t \to 0$,

$$\mathscr{L}_t = \mathscr{L}_0 + \sum_{r=1}^{m} t^r\mathcal{O}_r + \mathscr{O}(t^{m+1}), \tag{2.40}$$

and by [47, Theorems 4.1.21, 4.1.25], $\mathscr{L}_0 = \sum_j b_j b_j^+ = \mathscr{L}$.

Resolvent analysis. Define recursively $f_r(\lambda) \in \operatorname{End}(L^2(\mathbb{R}^{2n}, E_{x_0}))$ by

$$f_0(\lambda) = (\lambda - \mathscr{L}_0)^{-1}, \qquad f_r(\lambda) = (\lambda - \mathscr{L}_0)^{-1}\sum_{j=1}^{r}\mathcal{O}_j\,f_{r-j}(\lambda). \tag{2.41}$$

Let δ denote the positively oriented circle centered at the origin with a sufficiently small radius. Then [48, (1.110)] (cf. also [47, (4.1.91)]) gives

$$\mathscr{F}_{r,\,x_0} = \frac{1}{2\pi\sqrt{-1}}\int_\delta f_r(\lambda)\,d\lambda. \tag{2.42}$$

Assume now that $\omega = \Theta$. Since $\operatorname{Spec}(\mathscr{L})$ is explicit, one obtains (with $\mathscr{P}^\perp = \operatorname{Id} - \mathscr{P}$)

$$\mathscr{F}_{0,\,x_0} = \mathscr{P}, \quad \mathscr{F}_{1,\,x_0} = 0, \qquad \mathscr{F}_{2,\,x_0} = -\mathscr{L}^{-1}\mathscr{P}^\perp\mathcal{O}_2\mathscr{P} - \mathscr{P}\mathcal{O}_2\mathscr{L}^{-1}\mathscr{P}^\perp, \tag{2.43}$$

In particular,

$$J_{0,\,x_0} = 1, \qquad J_{1,\,x_0} = 0. \tag{2.44}$$

The coefficients $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ were computed by Lu [45] (for $E = \mathbb{C}$), X. Wang [68], and L. Wang [67], in various generalities, using peak sections and Hörmander's L^2 $\bar\partial$-method as in [66]. Dai–Liu–Ma computed $\boldsymbol{b}_1$ via the heat kernel [16, §5.1]; see also [48, §2], [46, §2] and [47, §4.1.8, §8.3.4] for the symplectic case. A new method for $\boldsymbol{b}_2$ was given in [51].

2.4. **Calculus and expansion of Toeplitz kernels.** We derive the calculus of Toeplitz kernels from the Bergman kernel expansion (2.35) together with the Taylor expansion of the symbol. This reduces the problem to a kernel calculus on $\mathbb{C}^n$ for kernels of the form $F\mathscr{P}$, where F is a polynomial and $\mathscr{P}$ is the model Bergman kernel (2.27). For $F \in \mathbb{C}[Z,Z']$ we denote by $F\mathscr{P}$ the operator on $L^2(\mathbb{R}^{2n})$ with kernel $F(Z,Z')\mathscr{P}(Z,Z')$ with respect to the volume form dZ (cf. (2.7)). The following lemma [47, Lemma 7.1.1] summarizes the corresponding kernel calculus.

Lemma 2.15. *For any $F, G \in \mathbb{C}[Z,Z']$ there exists a polynomial $\mathscr{K}[F,G] \in \mathbb{C}[Z,Z']$ with degree $\deg\mathscr{K}[F,G]$ of the same parity as $\deg F + \deg G$, such that*

$$((F\mathscr{P})\circ(G\mathscr{P}))(Z,Z') = \mathscr{K}[F,G](Z,Z')\mathscr{P}(Z,Z'). \tag{2.45}$$

For $F \in \mathbb{C}[Z,Z']$ we denote by $(F\mathscr{P})_p$ the operator with kernel $p^n(F\mathscr{P})(\sqrt{p}Z, \sqrt{p}Z')$, i.e.

$$((F\mathscr{P})_p\varphi)(Z) = \int_{\mathbb{R}^{2n}} p^n(F\mathscr{P})(\sqrt{p}Z, \sqrt{p}Z')\varphi(Z')\,dZ', \qquad \varphi \in L^2(\mathbb{R}^{2n}).$$

Then a change of variables gives, for $F, G \in \mathbb{C}[Z,Z']$,

$$((F\mathscr{P})_p \circ (G\mathscr{P})_p)(Z,Z') = p^n((F\mathscr{P})\circ(G\mathscr{P}))(\sqrt{p}Z, \sqrt{p}Z'). \tag{2.46}$$

We now turn to Toeplitz operators $T_{f,p} = P_p f P_p$ with $f \in \mathscr{C}^\infty_{\rm const}(X, {\rm End}(E))$. As a first step, we show that their kernels decay rapidly away from the diagonal.

Lemma 2.16 ([49, Lemma 4.2]). *Assume that the spectral gap condition (2.19) holds and let $f \in \mathscr{C}^\infty_{\rm const}(X, {\rm End}(E))$. Then for any compact set $K \subset X$, for any $l, m \in \mathbb{N}$ and $\varepsilon \in]0, a^K[$, there exists $C_{K,l,m,\varepsilon} > 0$ such that for all $p \geqslant 1$,*

$$|T_{f,p}(x,x')|_{\mathscr{C}^m(K\times K\setminus D_\varepsilon)} \leqslant C_{K,l,m,\varepsilon}\, p^{-l} \tag{2.47}$$

where $D_\varepsilon = \{(x,x') \in X \times X : d(x,x') < \varepsilon\}$ and the $\mathscr{C}^m$-norm is induced by ∇^L, ∇^E and h^L, h^E, g^{TX}.

Proof. By (2.7) it suffices to combine the far off-diagonal estimate (2.33) for P_p with the fact that on $K \times K$ the kernel $P_p(x,x')$ has at most polynomial growth in p in $\mathscr{C}^m$ (which follows from (2.35)). This yields (2.47). □

The near off-diagonal expansion (2.35) and Lemma 2.15 imply the corresponding expansion for Toeplitz kernels (cf. [49, Lemma 4.6], [47, Lemma 7.2.4]).

Theorem 2.17. *Assume that the spectral gap condition (2.19) holds. Let $f \in \mathscr{C}^\infty_{\rm const}(X, {\rm End}(E))$. There exists a family*

$$\{Q_{r,x_0}(f) \in {\rm End}(E)_{x_0}[Z, Z'] : r \in \mathbb{N},\, x_0 \in X\}\,,$$

depending smoothly on the parameter $x_0 \in X$, where $Q_{r,x_0}(f)$ are polynomials with the same parity as r with the following property. For any compact set $K \subset X$, for any $k \in \mathbb{N}$, and any $\varepsilon \in]0, a^K/4[$, we have

$$p^{-n} T_{f,p,x_0}(Z,Z') \cong \sum_{r=0}^{k} (Q_{r,x_0}(f)\mathscr{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z')p^{-r/2} + \mathcal{O}(p^{-(k+1)/2})\,, \tag{2.48}$$

on the set $\{(Z,Z') \in TX \times_K TX : |Z|, |Z'| < 2\varepsilon\}$, in the sense of Notation 2.11. Moreover, $Q_{r,x_0}(f)$ are expressed by

$$Q_{r,x_0}(f) = \sum_{r_1+r_2+|\alpha|=r} \mathscr{K}\left[J_{r_1,x_0}\,,\ \frac{\partial^\alpha f_{x_0}}{\partial Z^\alpha}(0)\frac{Z^\alpha}{\alpha!}J_{r_2,x_0}\right]. \tag{2.49}$$

where $\mathscr{K}[\cdot,\cdot]$ was introduced in (2.45). We have for any $x_0 \in X$,

$$Q_{0,x_0}(f) = f(x_0) \in {\rm End}(E_{x_0}). \tag{2.50}$$

Proof. By (2.7) and Lemma 2.16, for $|Z|, |Z'| < \varepsilon/2$ the kernel $T_{f,p,x_0}(Z,Z')$ is determined up to $\mathscr{O}(p^{-\infty})$ by the restriction of P_p and f to a fixed neighborhood of x_0. Choose $\rho \in \mathscr{C}^\infty(\mathbb{R})$ even, with $\rho(v) = 1$ if $|v| < 2$, $\rho(v) = 0$ if $|v| > 4$. Then for $|Z|, |Z'| < \varepsilon/2$,

$$T_{f,p,x_0}(Z,Z') = \int_{T_{x_0}X} P_{p,x_0}(Z,Z'')\rho\left(\tfrac{2}{\varepsilon}|Z''|\right) f_{x_0}(Z'')P_{p,x_0}(Z'',Z')\kappa_{x_0}(Z'')\,dv_{TX}(Z'') + \mathscr{O}(p^{-\infty})\,. \tag{2.51}$$

Expand f_{x_0} at 0 and rescale:

$$\begin{aligned} f_{x_0}(Z) &= \sum_{|\alpha|\leqslant k} \frac{\partial^\alpha f_{x_0}}{\partial Z^\alpha}(0)\frac{Z^\alpha}{\alpha!} + \mathscr{O}(|Z|^{k+1}) \\ &= \sum_{|\alpha|\leqslant k} p^{-|\alpha|/2}\frac{\partial^\alpha f_{x_0}}{\partial Z^\alpha}(0)\frac{(\sqrt{p}Z)^\alpha}{\alpha!} + p^{-\frac{k+1}{2}}\mathscr{O}(|\sqrt{p}Z|^{k+1}). \end{aligned} \tag{2.52}$$

Insert (2.52) and the Bergman kernel expansion (2.35) into (2.51), taking into account the κ_{x_0}-factors in (2.30). After the change of variables $\sqrt{p}\,Z'' = W$, the resulting composition reduces to the model calculus (2.45), (2.46), which gives (2.48) and (2.49). Finally, (2.44) and (2.49) yield (2.50): $Q_{0,x_0}(f) = \mathscr{K}[1, f_{x_0}(0)] = f_{x_0}(0) = f(x_0)$. ☐

Corollary 2.18. *For any* $f \in \mathscr{C}^\infty_{\rm const}(X, \operatorname{End}(E))$*, we have*

$$T_{f,p}(x,x) = \sum_{r=0}^{\infty} \boldsymbol{b}_{r,f}(x) p^{n-r} + \mathscr{O}(p^{-\infty}), \quad \boldsymbol{b}_{r,f} \in \mathscr{C}^\infty(X, \operatorname{End}(E)). \tag{2.53}$$

uniformly on compact sets in the $\mathscr{C}^\infty$ *topology, analogous to the expansion* (2.36).

Proof. Take $Z = Z' = 0$ in (2.48). Then (2.53) holds with $\boldsymbol{b}_{r,f}(x) = (Q_{2r,x}(f)\,\mathcal{P}_x)(0,0)$. ☐

Since (2.49) gives $Q_{2r,x}(f)$ explicitly in terms of the $J_{r,x}$, one can compute the first coefficients $\boldsymbol{b}_{r,f}$. In particular, [51] computes $\boldsymbol{b}_{1,f}$ and $\boldsymbol{b}_{2,f}$; see also applications in Kähler geometry (e.g. [20, 21, 44]).

Lemma 2.16 and Theorem 2.17 give the local expansion of the kernel of $T_{f,p}$. The same method applies to compositions $T_{f,p}T_{g,p}$, yielding an expansion of the form (2.48). Conversely, the existence of such an expansion (with compactly supported kernel) provides a convenient criterion for a family to be Toeplitz.

Theorem 2.19 (Criterion for Toeplitz operators, compact support version)**.** *Let*

$$\mathcal{T} = \{T_p : L^2(X, L^p \otimes E) \longrightarrow L^2(X, L^p \otimes E) : p \geq 1\}$$

be a family of bounded linear operators with smooth kernels $T_p(\cdot,\cdot)$ *satisfying the following conditions:*

(i) For any $p \in \mathbb{N}$, $P_p\, T_p\, P_p = T_p$.

(ii) There exists a compact set $K \subset X$ *and a family* $\{S_p : L^2(X, L^p \otimes E) \longrightarrow L^2(X, L^p \otimes E) : p \geq 1\}$ *of operators with smooth kernels such that for all* $p \in \mathbb{N}$ *we have* $T_p = P_p S_p P_p$ *and* $\operatorname{supp} S_p(\cdot,\cdot) \subset K \times K$ *with the following property. For any* $\varepsilon_0 > 0$ *and any* $\ell \in \mathbb{N}$*, there exists* $C_{\ell,\varepsilon_0} > 0$ *such that for all* $p \geqslant 1$ *and all* $(x,x') \in K \times K$ *with* $d(x,x') > \varepsilon_0$*,*

$$|S_p(x,x')| \leqslant C_{\ell,\varepsilon_0} p^{-\ell}. \tag{2.54}$$

(iii) There exists a family of polynomials $\{\mathcal{Q}_{r,x_0}(\mathcal{T}) \in \operatorname{End}(E)_{x_0}[Z,Z']\}_{x_0\in X}$ *such that:*

- *(a) As a polynomial, each* $\mathcal{Q}_{r,x_0}(\mathcal{T})$ *possesses the same parity as* r.
- *(b) The family is smooth in* $x_0 \in X$*, the sections* $X \ni x_0 \longmapsto \mathcal{Q}_{r,x_0}(\mathcal{T})(0,0)$ *is supported in* K*,*
- *(c) There exists* $0 < \varepsilon' < a^K/4$ *such that for every* $x_0 \in K$*, every* $Z, Z' \in T_{x_0}X$ *with* $|Z|, |Z'| < \varepsilon'$*, and every* $k \in \mathbb{N}$ *we have*

$$p^{-n} T_{p,x_0}(Z,Z') \cong \sum_{r=0}^{k} (\mathcal{Q}_{r,x_0}(\mathcal{T})\mathscr{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z') p^{-\frac{r}{2}} + \mathcal{O}(p^{-\frac{k+1}{2}}), \tag{2.55}$$

in the sense of Notation 2.11.

Then $\mathcal{T} = \{T_p : p \geq 1\}$ *is a Toeplitz operator in the sense of Definition 2.4.*

Proof. We define inductively a sequence $(g_l)_{l\geqslant 0}$ with $g_l \in \mathscr{C}^\infty_0(X, \operatorname{End}(E))$ such that

$$T_p = \sum_{l=0}^{m} P_p\, g_l\, P_p\, p^{-l} + \mathcal{O}(p^{-m-1}), \quad \text{for every } m \geqslant 0. \tag{2.56}$$

Step 1: construction of g_0 and the case $m = 0$. Fix $x_0 \in X$ and set

$$g_0(x_0) = \mathcal{Q}_{0,x_0}(\mathcal{T})(0,0) \in \operatorname{End}(E_{x_0}). \tag{2.57}$$

By assumption (iii)(b), $g_0(x_0) = 0$ for $x_0 \notin K$. We claim that

$$p^{-n}\,(T_p - T_{g_0,p})_{x_0}(Z,Z') \cong \mathcal{O}(p^{-1})\,, \tag{2.58}$$

which implies

$$T_p = P_p\, g_0\, P_p + \mathcal{O}(p^{-1}). \tag{2.59}$$

The key analytic input is the spectral gap (2.19), which gives (as in the proof of Theorem 2.12) the identities/estimates

$$F_\varepsilon(D_p)s = P_p\, s, \qquad p \geq p_0,\ \ s \in H^0_{(2)}(X, L^p \otimes E), \tag{2.60}$$

$$\|F_\varepsilon(D_p) - P_p\| = \mathscr{O}(p^{-\infty}), \tag{2.61}$$

$$|F_\varepsilon(D_p)(x,x') - P_p(x,x')|_{\mathscr{C}^m(K\times K)} \leq C_{l,m,\varepsilon}\, p^{-l}, \qquad x,x' \in K, \tag{2.62}$$

together with finite propagation speed [65, §2.8], [47, Appendix D.2] (cf. also [16, Proposition 4.1]), implying that $F_\varepsilon(D_p)(x,\cdot)$ depends only on $D_p|_{B^X(x,\varepsilon)}$ and vanishes outside $B^X(x,\varepsilon)$.

A crucial point is the following (proved in [49, p. 596-597] by working with the compactly supported kernel of $F_\varepsilon(D_p)S_pF_\varepsilon(D_p)$ and then using (2.61)):

Proposition 2.20 ([49, Proposition 4.11]). *In the conditions of Theorem 2.19 we have* $\mathcal{Q}_{0,\,x_0}(\mathcal{T})(Z,Z') = \mathcal{Q}_{0,\,x_0}(\mathcal{T})(0,0)$ *for all* $x_0 \in X$ *and all* $Z, Z' \in T_{x_0}X$.

We now compare the expansions of T_p and $T_{g_0,p} = P_p\, g_0\, P_p$ near the diagonal. By (2.48) with $k = 1$,

$$p^{-n}T_{g_0,\,p,\,x_0}(Z,Z') \cong \big(g_0(x_0)\mathscr{P}_{x_0} + Q_{1,\,x_0}(g_0)\mathscr{P}_{x_0}\, p^{-1/2}\big)(\sqrt{p}Z, \sqrt{p}Z') + \mathcal{O}(p^{-1})\,, \tag{2.63}$$

since $Q_{0,x_0}(g_0) = g_0(x_0)$ by (2.50). On the other hand, the assumed expansion (2.55) with $k = 1$ gives

$$p^{-n}T_{p,\,x_0}(Z,Z') \cong \big(g_0(x_0)\mathscr{P}_{x_0} + \mathcal{Q}_{1,\,x_0}(\mathcal{T})\mathscr{P}_{x_0}\, p^{-1/2}\big)(\sqrt{p}Z, \sqrt{p}Z') + \mathcal{O}(p^{-1})\,, \tag{2.64}$$

where we used Proposition 2.20 and (2.57). Subtracting (2.63) from (2.64) yields

$$p^{-n}(T_p - T_{g_0,\,p})_{x_0}(Z,Z') \cong \big((\mathcal{Q}_{1,\,x_0}(\mathcal{T}) - Q_{1,\,x_0}(g_0))\mathscr{P}_{x_0}\big)(\sqrt{p}Z, \sqrt{p}Z')\, p^{-1/2} + \mathcal{O}(p^{-1})\,. \tag{2.65}$$

Thus it remains to show

$$F_{1,\,x} := \mathcal{Q}_{1,\,x}(\mathcal{T}) - Q_{1,\,x}(g_0) \equiv 0, \tag{2.66}$$

which is proved in [49, Lemma 4.18]. This establishes (2.58), hence (2.59), i.e. (2.56) for $m = 0$.

Step 2: induction. Assume (2.56) holds up to some $m \geq 0$. Set $\mathcal{T}^{(0)} := \mathcal{T}$ and define

$$\mathcal{T}^{(1)} = \{T^{(1)}_p = p(T_p - T_{g_0,\,p}) : p \geq 1\}. \tag{2.67}$$

Using (2.59) we can write $T^{(1)}_p = P_p\big(pS_p - F_\varepsilon(D_p)\, g_0\, F_\varepsilon(D_p)\big)P_p$, so (i)–(ii) are immediate. Condition (iii) follows by subtracting the asymptotic expansions of T_{p,x_0} (from (2.55)) and T_{g_0,p,x_0} (from (2.48)), and using Proposition 2.20 and (2.66) to see that the coefficients of p^0 and $p^{-1/2}$ vanish. Hence $\mathcal{T}^{(1)}$ again satisfies the hypotheses of Theorem 2.19, and admits an expansion

$$p^{-n}T^{(1)}_{p,\,x_0}(Z,Z') \cong \sum_{r=0}^{k}(\mathcal{Q}_{r,\,x_0}(\mathcal{T}^{(1)})\mathscr{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z')p^{-r/2} + \mathcal{O}(p^{-(k+1)/2}). \tag{2.68}$$

Define

$$g_1(x_0) = \mathcal{Q}_{0,\,x_0}(\mathcal{T}^{(1)})(0,0) \in \operatorname{End}(E_{x_0}). \tag{2.69}$$

Then (2.56) holds for $m = 1$. In general, for $l \in \mathbb{N}$ define

$$\mathcal{T}^{(l+1)} = \{T_p^{(l+1)} = p(T_p^{(l)} - T_{g_l,p}) : p \geq 1\}, \quad T_p^{(l+1)} = p^{l+1}T_p - \sum_{j=0}^{l} p^{l+1-j} T_{g_j,p}\,, \tag{2.70}$$

and repeat the above argument to verify that $\mathcal{T}^{(l+1)}$ satisfies (i)–(iii), hence admits an expansion

$$p^{-n} T_{p,x_0}^{(l+1)}(Z,Z') \cong \sum_{r=0}^{k} (\mathcal{Q}_{r,x_0}(\mathcal{T}^{(l+1)})\mathscr{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z') p^{-r/2} + \mathcal{O}(p^{-(k+1)/2}), \tag{2.71}$$

and define

$$g_{l+1}(x_0) = \mathcal{Q}_{0,x_0}(\mathcal{T}^{(l+1)})(0,0) \in \operatorname{End}(E_{x_0}). \tag{2.72}$$

This inductive construction yields (2.56) for all m, hence $\mathcal{T}$ is a Toeplitz family in the sense of Definition 2.4. □

2.5. **Proof of Theorem 2.7.**

Proof. (1) We follow [47, Theorems 4.1.1 and 6.1.1]. We first deduce a spectral gap for the Kodaira Laplacian $\square_p = \frac{1}{2}D_p^2$ acting on sections of $L^p \otimes E$. Let $f \in \operatorname{Dom}(\square_p) \cap L^2(X, L^p \otimes E)$ and set $s = \overline{\partial}_p^E f$. Then (2.19) gives

$$\|2\square_p f\|_{L^2}^2 = 2\langle \overline{\partial}_p^{E,*} s, \overline{\partial}_p^{E,*} s\rangle = \|D_p s\|_{L^2}^2 \geq (2C_0 p - C_L)\|s\|_{L^2}^2 = (2C_0p - C_L)\langle \square_p f, f\rangle. \tag{2.73}$$

Hence

$$\operatorname{Spec}(2\square_p) \subset \{0\} \cup [2C_0 p - C_L, \infty[\,. \tag{2.74}$$

By (2.74) we may localize as in Theorem 2.12 (cf. [47, §4.1.2]) and conclude from [47, Theorem 4.1.24] exactly as in the compact case [47, Theorem 4.1.1].

(2) Let $g \in \mathscr{C}_0^\infty(X, \operatorname{End}(E))$. We denote by $(F_\varepsilon(D_p)\, g\, F_\varepsilon(D_p))(x,x')$ the smooth kernel of $F_\varepsilon(D_p)\, g\, F_\varepsilon(D_p)$ with respect to $dv_X(x')$. For any relatively compact open $U \Subset X$ with $\operatorname{supp}(g) \subset U$ we have, using (2.60), (2.61), (2.62),

$$\begin{aligned} &T_{g,p} - F_\varepsilon(D_p)\, g\, F_\varepsilon(D_p) = \mathcal{O}(p^{-\infty}) \quad \text{in operator norm},\\ &T_{g,p}(x,x') - \big(F_\varepsilon(D_p)\, g\, F_\varepsilon(D_p)\big)(x,x') = \mathscr{O}(p^{-\infty}) \quad \text{on } U \times U. \end{aligned} \tag{2.75}$$

Fix $f, g \in \mathscr{C}_0^\infty(X, \operatorname{End}(E))$ and choose $U \Subset X$ such that $\operatorname{supp}(f) \cup \operatorname{supp}(g) \subset U$ and $d(x,y) > 2\varepsilon$ for all $x \in \operatorname{supp}(f) \cup \operatorname{supp}(g)$, $y \in X \setminus U$. Then (2.60) implies

$$T_{f,p}T_{g,p} = P_p\, F_\varepsilon(D_p)\, f\, P_p\, g\, F_\varepsilon(D_p)\, P_p. \tag{2.76}$$

The kernel of $F_\varepsilon(D_p) f P_p\, g F_\varepsilon(D_p)$ is supported in $U \times U$, and Lemmas 2.16, 2.17 and (2.62) show that it satisfies (2.54). Moreover, as in (2.51), for $x_0 \in U$ and $|Z|, |Z'| < \varepsilon/4$,

$$\begin{aligned} &(F_\varepsilon(D_p) f P_p\, g\, F_\varepsilon(D_p))_{x_0}(Z,Z') = (T_{f,p}\, T_{g,p})_{x_0}(Z,Z') + \mathscr{O}(p^{-\infty})\\ &= \int_{T_{x_0}X} T_{f,p,x_0}(Z,Z'')\rho\Big(\frac{4|Z''|}{\varepsilon}\Big) T_{g,p,x_0}(Z'',Z')\kappa_{x_0}(Z'')\, dv_{TX}(Z'') + \mathscr{O}(p^{-\infty}). \end{aligned} \tag{2.77}$$

Combining (2.77) with Theorem 2.17 gives, as in the proof of Theorem 2.17,

$$p^{-n}\big(F_\varepsilon(D_p) f P_p\, g F_\varepsilon(D_p)\big)_{x_0}(Z,Z') \cong \sum_{r=0}^{k} (Q_{r,x_0}(f,g)\mathscr{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z')\, p^{-r/2} + \mathcal{O}(p^{-(k+1)/2}), \tag{2.78}$$

with

$$Q_{r,x_0}(f,g)=\sum_{r_1+r_2=r}\mathscr{K}\big[Q_{r_1,x_0}(f),\,Q_{r_2,x_0}(g)\big],\tag{2.79}$$

where $Q_{r,x_0}(f)$ are given by (2.49). Hence, by Theorem 2.19 and (2.75), there exist $C_l(f,g)\in\mathscr{C}^\infty_0(X,\mathrm{End}(E))$ with $\mathrm{supp}(C_l(f,g))\subset\mathrm{supp}(f)\cap\mathrm{supp}(g)$ such that for any $k\geq 1$,

$$\Big\|F_\varepsilon(D_p)fP_p\,g\,F_\varepsilon(D_p)s-\sum_{l=0}^{k}F_\varepsilon(D_p)P_p\,C_l(f,g)\,p^{-l}\,P_pF_\varepsilon(D_p)s\Big\|\leqslant C_kp^{-k-1}\|s\|.\tag{2.80}$$

The estimates (2.76) and (2.80) imply that

$$\Big\|T_{f,p}T_{g,p}-\sum_{l=0}^{k}P_p\,C_l(f,g)\,p^{-l}P_p\Big\|\leq C_k\,p^{-k-1}.\tag{2.81}$$

This proves (i) in Definition 2.5 for sections f,g with compact support. In general, for $f,g\in\mathscr{C}^\infty_{\mathrm{const}}(X,\mathrm{End}(E))$ we write $f=f_0+c_f$, $g=g_0+c_g$, with f_0,g_0 with compact support and $c_f,c_g\in\mathbb{C}$. Then $T_{f,p}=T_{f_0,p}+c_fP_p$, $T_{g,p}=T_{g_0,p}+c_gP_p$, hence

$$T_{f,p}T_{g,p}=T_{f_0,p}T_{g_0,p}+c_gT_{f,p}+c_fT_{g,p}+c_fc_gP_p\,.$$

Using the expansion (2.81) for $T_{f_0,p}T_{g_0,p}$ we obtain the expansion for $T_{f,p}T_{g,p}$ taking into account that $C_0(f_0,g_0)+c_gf+c_fg+c_fc_g=C_0(f,g)$ and $C_r(f_0,g_0)=C_r(f,g)$ für $r\geq 1$. This completes the proof of preperty (i) from Definition 2.5.

We note that the coefficients $C_r(f,g)\in\mathscr{C}^\infty(X,\mathrm{End}(E))$ in (2.12), (2.81), are constructed inductively in Theorem 2.19. By setting $\mathcal{T}_{f,g}=\{T_{f,p}T_{g,p}\}$ we have $C_0(f,g)(x)=\mathcal{Q}_{0,x}(\mathcal{T}_{f,g})(0,0)=Q_{0,x}(f,g)$ by (2.57); for $l\geq 0$, we define inductively

$$\begin{aligned}\mathcal{T}^{(l+1)}_{f,g}&=\Big\{p^{l+1}T_{f,p}T_{g,p}-\sum_{j=0}^{l}p^{l+1-j}T_{C_j(f,g),p}\Big\},\\ C_{l+1}(f,g)(x)&=\mathcal{Q}_{0,x}(\mathcal{T}^{(l+1)}_{f,g})(0,0)\end{aligned}\tag{2.82}$$

as in (2.70) and (2.72).

Property (ii) from Definition 2.5 follows as in the compact case [47, Theorem 7.4.1], [49, Theorem 1.1]; in particular

$$C_0(f,g)(x)=Q_{0,x}(f,g)=\mathscr{K}[Q_{0,x}(f),Q_{0,x}(g)]=f(x)g(x),\tag{2.83}$$

and the commutator relation (2.13) follows from

$$C_1(f,g)(x)-C_1(g,f)(x)=\sqrt{-1}\{f,g\}\,\mathrm{Id}_E\,.\tag{2.84}$$

Finally, for (iii), we adapt [47, Theorem 7.4.2] and [49, Theorem 4.19]. Choose $x_0\in X$ and $u_0\in E_{x_0}$, $|u_0|=1$, such that $|f(x_0)(u_0)|=\|f\|_\infty$. In normal coordinates at x_0, trivialize L and E, and let e_L be the unit frame of L. Consider the peak sections

$$S^p_{x_0}(x)=p^{-n/2}P_p(x,x_0).(e^{\otimes p}_{L,x_0}\otimes u_0)\,.\tag{2.85}$$

Using (2.32) and (2.75), $F_\varepsilon(D_p)$ has the same local expansion as P_p, and thus by (2.35),

$$\big\|T_{f,p}S^p_{x_0}-f(x_0)S^p_{x_0}\big\|_{L^2}\leq\frac{C}{\sqrt{p}}\,\|S^p_{x_0}\|_{L^2}.\tag{2.86}$$

If f is real-valued, then $df(x_0)=0$ and the factor $C/\sqrt{p}$ improves to C/p. This is precisely (2.14), completing the proof. □

Let $f, g \in \mathscr{C}^\infty_{\mathrm{const}}(X, \mathrm{End}(E))$ and assume that $\omega = \Theta$. By [51, Theorem 0.3] we have:

$$\begin{aligned} C_1(f,g) &= -\frac{1}{2\pi}\big\langle \nabla^{1,0} f, \overline{\partial}^E g\big\rangle_\omega \in \mathscr{C}^\infty_0(X, \mathrm{End}(E)),\\ C_2(f,g) &= \boldsymbol{b}_{2,\,f,\,g} - \boldsymbol{b}_{2,\,fg} - \boldsymbol{b}_{1,\,C_1(f,\,g)}. \end{aligned} \tag{2.87}$$

If $f, g \in \mathscr{C}^\infty_{\mathrm{const}}(X)$, then

$$C_2(f,g) = \frac{1}{8\pi^2}\left\langle D^{1,0}\partial f, D^{0,1}\overline{\partial} g\right\rangle + \frac{\sqrt{-1}}{4\pi^2}\left\langle \mathrm{Ric}_\omega, \partial f \wedge \overline{\partial} g\right\rangle - \frac{1}{4\pi^2}\left\langle \partial f \wedge \overline{\partial} g, R^E\right\rangle_\omega . \tag{2.88}$$

Remark 2.21. There are two ways to prove (2.84), which also holds in the more general symplectic case. One is to compute directly the difference, as is done [49, p. 593-594], [47, p. 311], or one can explicitly compute each coefficient $C_1(f,g)$, which is more involved and was done in [31, Theorem 1.1] in the symplectic case, and then take the difference.

2.6. **The Berezin-Toeplitz star product.** Let (X,ω) be a Kähler manifold. The following notion was introduced by Bayen, Flato, Fronsdal, Lichnerowicz, and Sternheimer in [3] as a perspective on quantum mechanics that abstracts from Hilbert spaces to focus on observables, where quantum observables are represented by the space $\mathscr{C}^\infty(X,\mathbb{C})[[\hbar]]$ of formal series with coefficients in $\mathscr{C}^\infty(X,\mathbb{C})$, for which the quantum of action $\hbar$ plays the role of a formal variable.

Definition 2.22. Let (X,ω) be a Kähler manifold, and let $\{\cdot,\cdot\}$ be the Poisson bracket (2.11) on (X,ω). A *formal deformation* of a Poisson subalgebra $\mathscr{A} \subset (\mathscr{C}^\infty(X,\mathbb{C}), \{\cdot,\cdot\})$ is a linear associative product $\star$ on $\mathscr{A}[[\hbar]]$, called star product, admitting $1 \in \mathscr{C}^\infty(X,\mathbb{C})$ as unity and whose product $\star$ is given for all $f,\, g \in \mathscr{C}^\infty(X,\mathbb{C})$ by $f \star g = \sum_{r=0}^\infty \hbar^r\, C_r(f,g)$, where $\{C_r\}_{r\in\mathbb{N}}$ is a sequence of bidifferential operators acting on $\mathscr{C}^\infty(X,\mathbb{C})$, and satisfying $C_0(f,g) = fg$ and $C_1(f,g) - C_1(g,f) = \sqrt{-1}\{f,g\}$, where $\{f,g\}$ is defined in (2.11).

A formal deformation of the Poisson algebra $(\mathscr{C}^\infty(X,\mathbb{C}), \{\cdot,\cdot\})$ is also called a deformation quantization of the Kähler manifold (X,ω). Deformation quantization on a compact Kähler, or more generally on a compact symplectic manifold (X,ω), is subtle since associativity imposes infinitely many constraints on the bidifferential operators $\{C_r\}_{r\in\mathbb{N}}$. Existence was proved by De Wilde–Lecomte [17], and a geometric construction was given by Fedosov [19]. Kontsevich extended existence to general Poisson manifolds [40], though explicit computation of the operators C_r remains difficult. In this context, we obtain a corollary of Theorem 2.7 regarding the existence and computability of the so-called *Berezin-Toeplitz star product* over non-compact manifolds (cf. [37, 62] for the compact Kähler case).

Theorem 2.23. *Let (X,ω) be a Kähler manifold, let (L,h^L) be a holomorphic Hermitian line bundle on X such that $\omega = \frac{\sqrt{-1}}{2\pi}R^L$. Assume that the Kodaira-Laplacian possesses a spectral gap, as stated in (2.19). Then the sequence of bidifferential operators $\{C_r\}_{r\in\mathbb{N}}$ given by the asymptotic expansion (2.12) defines a $\star$-product on $\mathscr{C}^\infty_{\mathrm{const}}(X,\mathbb{C})$ on the Kähler manifold (X,ω) in the sense of Definition 2.22, and the calculus of Toeplitz kernels from Section 2.4 provides an algorithm to compute the sequence $\{C_r\}_{r\in\mathbb{N}}$ recursively in terms of local geometric data over X.*

Proof. Let $f, g \in \mathscr{C}^\infty_{\mathrm{const}}(X,\mathbb{C})$. By Theorem 2.7 there exists a unique sequence of bidifferential operators $C_r : \mathscr{C}^\infty_{\mathrm{const}}(X,\mathbb{C}) \times \mathscr{C}^\infty_{\mathrm{const}}(X,\mathbb{C}) \longrightarrow \mathscr{C}^\infty_{\mathrm{const}}(X,\mathbb{C})$, $r \in \mathbb{N}$, such that, as $p \to +\infty$, (2.12) holds, $T_{f,p}T_{g,p} = \sum_{r=0}^\infty p^{-r} T_{C_r(f,g),p} + \mathscr{O}(p^{-\infty})$. Moreover, C_r depends only on a finite jet of f, g and the geometric data at the point (locality). Set $\hbar = 1/p$ and define, for $f, g \in \mathscr{C}^\infty_{\rm const}(X, \mathbb{C})$,

$$f \star g := \sum_{r=0}^{\infty} \hbar^r (-\sqrt{-1})^r \, C_r(f,g) \in \mathscr{C}^\infty_{\rm const}(X,\mathbb{C})[[\hbar]]. \tag{2.89}$$

By (2.83), we have $C_0(f,g) = fg$, thus $\star$ deforms the pointwise product. Associativity of operator composition implies the associativity of $\star$. Indeed, for $f, g, h \in \mathscr{C}^\infty_{\rm const}(X, \operatorname{End}(E))$, we have $T_{f,p}\,(T_{g,p}T_{h,p}) = (T_{f,p}T_{g,p})\,T_{h,p}$, and expanding both sides using (2.12) yields, for each $k \geq 0$,

$$\sum_{r+s=k} C_r(f, C_s(g,h)) = \sum_{r+s=k} C_r(C_s(f,g),h). \tag{2.90}$$

Thus $\star$ defines an associative product on $\mathscr{C}^\infty_{\rm const}(X,\mathbb{C})[[\hbar]]$.

By (2.83), we have $C_0(f,1) = f$. Since $T_{1,p} = P_p$, we have $T_{f,p}T_{1,p} = T_{f,p}$; thus, $T_{f,p}T_{1,p} - T_{C_0(f,1),p} = T_{f,p} - T_{f,p} = 0$; so $\mathcal{T}^{(1)}_{f,1} = \{0\}$. Hence, $C_1(f,1) = 0$ by (2.82). By induction, it follows using (2.82) that $\mathcal{T}^{(r)}_{f,1} = \{0\}$; thus, $C_r(f,1) = 0$ for $r \geq 1$. In the same way, $C_0(1,f) = f$ and $C_r(1,f) = 0$ for $r \geq 1$. Therefore, 1 is the unit for $\star$.

Finally, because the C_r are local bidifferential operators and our symbols are constant outside a compact set, each $C_r(f,g)$ again lies in $\mathscr{C}^\infty_{\rm const}(X,\mathbb{C})$. □

Theorem 2.24. *Let (X,ω) be a compact Kähler manifold; let (L,h^L) be a holomorphic Hermitian line bundle on X such that $\omega = \frac{\sqrt{-1}}{2\pi}R^L$. Then the sequence of bidifferential operators $\{C_r\}_{r\in\mathbb{N}}$ given by the asymptotic expansion (2.12) defines a deformation quantization of the Kähler manifold (X,ω) in the sense of Definition 2.22, and the calculus of Toeplitz kernels developed in Section 2.4 provides an algorithm to compute the sequence $\{C_r\}_{r\in\mathbb{N}}$ recursively in terms of local geometric data over X.*

In the same way can define a formal deformation of the algebra $\mathscr{C}^\infty(X, \operatorname{End}(E))$, by setting for $f, g \in \mathscr{C}^\infty(X, \operatorname{End}(E))$, $f\star g := \sum_{k=0}^{\infty}(-\sqrt{-1})^k C_k(f,g)\hbar^k \in \mathscr{C}^\infty_{\rm const}(X, \operatorname{End}(E))[[\hbar]]$, where $C_k(f,g)$ is determined by (2.12). This is the *Berezin-Toeplitz star product* (cf. [47, 49] for the symplectic case and arbitrary twisting bundle E).

The coefficients $C_r(f,g)$, $r = 0,1,2$, are given by (2.87). Set

$$\{\{f,g\}\} := \frac{1}{2\pi\sqrt{-1}}\big(\langle\nabla^{1,0}g, \overline{\partial}^E f\rangle_\omega - \langle\nabla^{1,0}f, \overline{\partial}^E g\rangle_\omega\big)\,. \tag{2.91}$$

If $fg = gf$ on X we have

$$\big[T_{f,p}\,, T_{g,p}\big] = \tfrac{\sqrt{-1}}{p}\,T_{\{\{f,g\}\},p} + \mathcal{O}(p^{-2})\,, \quad p \to \infty. \tag{2.92}$$

Due to the fact that $\{\{f,g\}\} = \{f,g\}$ if E is trivial and comparing (2.13) to (2.92), one can regard $\{\{f,g\}\}$ defined in (2.91) as a non-commutative Poisson bracket.

2.7. Coherent states.

Let (X,ω) be a Hermitian manifold, and let (L,h^L) be a positive holomorphic Hermitian line bundle on X satisfying $\omega = \frac{\sqrt{-1}}{2\pi}R^L$. In this section, we assume that $E = \mathbb{C}$ is the trivial holomorphic line bundle, equipped with a possibly non-trivial Hermitian metric.

In quantum measurement theory, physical states are represented by positive rank-1 operators acting on the Hilbert spaces that correspond to the quantum physical system. In semiclassical analysis, one considers quantum states that best approximate the classical states associated with points on the symplectic manifold, the associated classical phase space. In Berezin-Toeplitz quantization, *coherent state* represent this notion.

Proposition 2.25. *For $p \in \mathbb{N}$ and $x \in X$, the coherent state projector $\Pi_p(x)$ is the orthogonal projector acting on $H^0_{(2)}(X,L^p)$ satisfying* $\operatorname{Ker}\Pi_p(x) = \{\, s \in H^0_{(2)}(X,L^p) \mid s(x) = 0\,\}$.

The link to Berezin-Toeplitz quantization is provided by the following result.

Proposition 2.26. *For any $x \in X$, any $p \in \mathbb{N}$ large enough, and any unit vector $e_x \in L^p_x$, the coherent state projector $\Pi_p(x)$ is the rank-1 orthogonal projector on the line spanned by the section $s_x \in H^0_{(2)}(X, L^p)$ defined for all $y \in X$ by*

$$s_x(y) = P_p(y,x).e_x\,, \tag{2.93}$$

where $e_x \in L^p_x$ is any given unit vector and satisfies

$$\|s_x\|^2_{L^2} = P_p(x,x)\,. \tag{2.94}$$

Proof. Equation (2.94) is a straightforward consequence of the definition (2.93) of the section $s_x \in H^0_{(2)}(X, L^p)$, together with the standard properties of Schwartz kernels under composition and the fact that $P_pP_p = P_p$. Now, for any $x \in X$, Theorem 2.7 shows that there exists $p_0 \in \mathbb{N}$ such that, for any $p \geq p_0$, we have $P_p(x,x) \neq 0$, so that $s_x \in H^0_{(2)}(X, L^p)$ does not vanish identically. This implies that $s_x(x) \neq 0$, so that in particular $\Pi_p(x)$ is, in fact, a rank one orthogonal projector. We are thus reduced to showing that for any $s \in H^0_{(2)}(X, L^p)$ satisfying $s(x) = 0$, we have $\langle s, s_x\rangle = 0$. But, using basic properties of the Bergman kernel, for any $s \in H^0_{(2)}(X, L^p)$ satisfying $s(x) = 0$, we have

$$\langle s, s_x\rangle = \int_X \langle s(y), P_p(y,x).e_x\rangle_{L^p} = \int_X \langle P_p(x,y)\, s(y), e_x\rangle_{L^p} = s(x).e_x = 0\,. \tag{2.95}$$

This establishes the result. □

The following result shows that Berezin-Toeplitz quantization coincides with the *coherent state quantization* associated with Definition 2.25.

Proposition 2.27. *For any $f \in \mathscr{C}^\infty_0(X, \mathbb{R})$ and any $p \in \mathbb{N}$ large enough, the Berezin-Toeplitz quantization of f satisfies*

$$T_{f,p} = \int_X f(x)\, \Pi_p(x)\, P_p(x,x)\, dv_X(x)\,. \tag{2.96}$$

Proof. By Definition 2.3 of the Berezin-Toeplitz quantization of $f \in \mathscr{C}^\infty_0(X, \mathbb{R})$, the reproductive property of the Bergman kernel and using Proposition 2.26, for any $s \in H^0_{(2)}(X, L^p)$ and any $y \in X$, we have

$$\begin{aligned} T_{f,p}s(y) &= \int_X f(x)\, P_p(y,x).\left(\int_X P_p(x,w).s(w)\, dv_X(w)\right) dv_X(x) \\ &= \int_X f(x)\, \frac{\langle s, s_x\rangle\, s_x(y)}{\|s_x\|^2_{L^2}}\, P_p(x,x)\, dv_X(x) \\ &= \int_X f(x)\, \left(\Pi_p(x)s\right)(y)\, P_p(x,x)\, dv_X(x)\,. \end{aligned} \tag{2.97}$$

This establishes the result. □

Let us introduce the *Berezin symbol*, which associates a classical observable with a quantum observable by its expectation value at coherent states. Semiclassically, it represents the classical observable that best approximates the quantum observable.

Definition 2.28. The *Berezin symbol* of a linear bounded operator A on $H^0_{(2)}(X, L^p)$ is the function $\sigma(A) \in \mathscr{C}^\infty(X, \mathbb{R})$ defined for any $x \in X$ by $\sigma(A)(x) = \mathrm{Tr}[\Pi_p(x)\, A]$.

2.8. **The Berezin transform.** Berezin introduced his transform in the context of quantization on bounded symmetric domains [4]. Already in this setting, the construction was tied to the Kähler geometry induced by the Bergman metric. Beginning with a function f defined on the base manifold, assigning to it its Toeplitz operator $T_{f,p}$, and subsequently computing the covariant symbol of the Toeplitz operator will result in a function known as the Berezin transform B_pf of f. In [37], it is shown that its asymptotic expansion gives a formal Berezin transform in the sense of Karabegov, associated with a star product equivalent to the Berezin-Toeplitz star product.

Let (X,ω) be a Hermitian manifold, and let (L,h^L) be a positive holomorphic Hermitian line bundle on X satisfying $\omega=\frac{\sqrt{-1}}{2\pi}R^L$. In this section, we assume that $E=\mathbb{C}$ is the trivial holomorphic line bundle equipped with a possibly non-trivial Hermitian metric. Following [18], we introduce the following basic tool in Berezin-Toeplitz quantization.

Definition 2.29. The *Berezin transform* of $f\in\mathscr{C}^\infty_0(X,\mathbb{R})$, is defined by

$$B_pf=\sigma(T_{f,p}),\quad \text{for } p\in\mathbb{N}\,. \tag{2.98}$$

We summarize the properties of the Berezin transform seen as a *Markov operator* [33].

Proposition 2.30. *For any $x\in X$ and any $p\in\mathbb{N}$ such that $P_p(x,x)\neq 0$, the Berezin transform of $f\in\mathscr{C}^\infty_b(X,\mathbb{R})$ satisfies*

$$B_pf(x)=\int_X\frac{|P_p(x,y)|_p^2}{P_p(x,x)}\,f(y)\,dv_X(y)\,. \tag{2.99}$$

where $|\cdot|_p$ is the norm induced by h^L on $L^p_x\otimes(L^p_y)^*$ for all x, $y\in X$.

Furthermore, for any $p\in\mathbb{N}$, the Berezin transform sends measurable bounded positive functions to measurable bounded positive functions and extends to a self-adjoint bounded positive operator acting on $L^2(X,\mathbb{R})$.

Proof. By Definition 2.3 of the Berezin-Toeplitz quantization of $f\in\mathscr{C}^\infty_b(X,\mathbb{R})$, Definition 2.28 of the Berezin symbol and using Proposition 2.26, for any $x\in X$, we have

$$\begin{aligned}B_pf(x)&=\frac{\langle T_{f,p}s_x,s_x\rangle}{\|s_x\|^2_{L^2}}=\int_X\frac{\langle s_x(y),s_x(y)\rangle_{L^p}}{P_p(x,x)}\,f(y)\,dv_X(y)\\&=\int_X\frac{|P_p(x,y)|^2_p}{P_p(x,x)}\,f(y)\,dv_X(y)\,.\end{aligned} \tag{2.100}$$

This shows (2.99). Now for any positive $f\in\mathscr{C}^\infty_b(X,\mathbb{R})$ and any $x\in X$ such that $P_p(x,x)\neq 0$, using the elementary fact that $\|T_{f,p}\|_{op}\leq\|f\|_\infty$ for $f\in\mathscr{C}^\infty_b(X,\mathbb{R})$, and since $\Pi_p(x)$ is a rank-1 projection by Proposition 2.26, we get

$$B_pf(x)=\operatorname{Tr}[\Pi_p(x)T_{f,p}]\leq\operatorname{Tr}[\Pi_p(x)]\,\|T_{f,p}\|_{op}\leq\|f\|_\infty\,. \tag{2.101}$$

Since the Toeplitz operator $T_{f,p}$ associated to a positive symbol f is a positive operator, the Berezin transform maps the set of bounded measurable positive functions to itself.

To show that it extends as a bounded self-adjoint positive operator acting on $L^2(X,\mathbb{R})$, let us set $K:=\{x\in X\mid P_p(x,x)\neq 0\}\subset X$, and note from (2.5) that for any $x\in X$ such that $P_p(x,x)=0$, Definition 2.25 shows that $\Pi_p(x)=0$. Then for all $x\in K$ and $y\in X$, writing $B_p(x,y)\geq 0$ for the Schwartz kernel of the Berezin transform as given by formula (2.99), we get via Cauchy-Schwarz inequality for all $f\in\mathscr{C}^\infty_0(X,\mathbb{R})$,

$$\begin{aligned}\|B_pf\|^2_{L_2}&\leq\int_K\left(\int_X B_p(x,y)\,dv_X(y)\right)\left(\int_X B_p(x,y)\,|f(y)|^2\,dv_X(y)\right)dv_X(x)\\&\leq\sup_{x\in X}\left(\int_X B_p(x,y)\,dv_X(y)\right)\sup_{y\in K}\left(\int_X B_p(x,y)\,dv_X(x)\right)\|f\|^2_{L_2}\,.\end{aligned} \tag{2.102}$$

Together with (2.101) applied to $f \equiv 1$, this implies that $\|B_p f\|^2_{L_2} \leq \|f\|^2_{L_2}$, so that B_p defines in fact a bounded self-adjoint operator on $L^2(X, \mathbb{R})$ by density. The fact that it is positive and self-adjoint follows directly from Definition 2.29. ☐

The following result gives the asymptotic expansion of the Berezin transform, extending a result of Englis in [18] in the case of pseudoconvex domains considered in Section 3.4, and of Karabegov-Schlichenmaier in [37] and Ioos-Kaminker-Polterovich-Shmoish in [33, Proposition 3.8] in the case of compact manifolds considered in Theorem 2.8. As explained in [33, Remark 3.12], the weighted case considered in [18, (1)] corresponds to the case $E = \mathbb{C}$ equipped with a non-trivial Hermitian metric.

Theorem 2.31. *Let (X, ω) be a Hermitian manifold and let (L, h^L) be a positive holomorphic Hermitian line bundle on X satisfying $\omega = \frac{\sqrt{-1}}{2\pi} R^L$. Assume $E = \mathbb{C}$ is the trivial holomorphic line bundle, equipped with a possibly non-trivial Hermitian metric, and that the Kodaira-Laplacian has a spectral gap in the sense of Definition 2.6. Then there exists a sequence of differential operators $\{D_j\}_{j \in \mathbb{N}^*}$ acting on $\mathscr{C}^\infty(X, \mathbb{R})$ such that for any compact set $K \subset X$ and any $m, r \in \mathbb{N}$, there exist $l \in \mathbb{N}$ and a constant $C_m > 0$, uniform in the $\mathscr{C}^m$-norm of the derivatives of h^L and h^E up to order l, such that for any $f \in \mathscr{C}^\infty_0(X, \mathbb{R})$ and all $p \in \mathbb{N}$ big enough, we have*

$$\left| B_p f - f - \sum_{j=1}^{r-1} p^{-j} D_j f \right|_{\mathscr{C}^m(K)} \leq C_m p^{-r} |f|_{\mathscr{C}^{m+2r}(K)} \,. \tag{2.103}$$

Furthermore, we have $D_1 = -\frac{\Delta}{4\pi}$, where Δ is the Laplace-Beltrami operator of (X, g^{TX}).

Proof. First, recall that for any compact set $K \subset X$, Theorem 2.7 implies that there exists $p_0 \in \mathbb{N}$ such that for all $x \in K$, we have $P_p(x, x) \neq 0$ for all $p \geq p_0$, so that for any $f \in \mathscr{C}^\infty_0(X, \mathbb{R})$, formula (2.99) holds for its Berezin transform $B_p f$. Now, using (2.7), we readily get from formula (2.99) that

$$B_p f(x) = \frac{T_{f,p}(x, x)}{P_p(x, x)} \,. \tag{2.104}$$

Thus, following the analogous computations in the proof of [36, Proposition 3.4], this is a straightforward consequence of the explicit formulas for the coefficients of the asymptotic expansion of the Bergman kernel and of the Toeplitz operators along the diagonal given by (2.37) and [51, Theorem 0.1] in the case $E = \mathbb{C}$. ☐

3. NON-COMPACT MANIFOLDS

3.1. **General framework.** In this section, we give geometric conditions ensuring the spectral gap (2.19) for the Kodaira Laplacian on $(0,1)$-forms with values in $L^p \otimes E$. Combined with Theorem 2.7, these hypotheses yield the Berezin–Toeplitz package for $\mathscr{C}^\infty_{\rm const}(X, \mathrm{End}(E))$. We first collect the analytic input on complete Hermitian manifolds used below; see [47, Lemma 3.3.1, Corollaries 3.3.3–3.3.4].

Theorem 3.1. *Let (X, Θ) be a complete Hermitian manifold and let (F, h^F) be a holomorphic Hermitian vector bundle.*

(i) Let $\overline{\partial}^F_{max}$ and $\overline{\partial}^{F,*}_{max}$ be the maximal extensions of $\overline{\partial}^F$ and $\overline{\partial}^{F,*}$, respectively. Then $\Omega^{0,\bullet}_0(X, F)$ is dense in

$$\mathrm{Dom}(\overline{\partial}^F_{max}), \;\; \mathrm{Dom}(\overline{\partial}^{F,*}_{max}), \;\; \mathrm{Dom}(\overline{\partial}^F_{max}) \cap \mathrm{Dom}(\overline{\partial}^{F,*}_{max}),$$

in the graph norms of $\overline{\partial}^F_{max}$, $\overline{\partial}^{F,*}_{max}$, and $\overline{\partial}^F_{max} + \overline{\partial}^{F,*}_{max}$, respectively.

(ii) The Hilbert space adjoint of the maximal extension and the maximal extension of the formal adjoint of $\overline{\partial}^F$ coincide. that is, $\overline{\partial}^{F,*}_H = \overline{\partial}^{F,*}_{max}$.

(iii) The Kodaira-Laplacian $\square^F : \Omega_0^{0,\bullet}(X,F) \to \Omega_{(2)}^{0,\bullet}(X,F)$ *is essentially self-adjoint. In particular, its Gaffney and Friedrichs extensions coincide, and their associated quadratic form is the form* Q *given by* (2.18).

We denote by $R^{\det}$ the curvature of the holomorphic Hermitian connection $\nabla^{\det}$ on $K_X^* = \det(T^{(1,0)}X)$. We have the following spectral gap result.

Condition 3.2. Let (X,J,Θ) be a Hermitian manifold and let (L,h^L) and (E,h^E) be holomorphic Hermitian vector bundles of rank one and r, respectively. We assume that the Riemannian metric g^{TX} induced from Θ is complete and we suppose that there exist $C,\varepsilon>0$ such that

$$\sqrt{-1}R^L > \varepsilon\Theta\,, \qquad \sqrt{-1}(R^{\det}+R^E) > -C\Theta\,\mathrm{Id}_E\,, \qquad |\partial\Theta|_{g^{TX}} < C\,. \tag{3.1}$$

If $L = K_X := \det(T^{*(1,0)}X)$ is the canonical line bundle on X, the first two conditions in (3.1) are to be replaced by

$$h^L \text{ is induced by } \Theta \text{ and } \sqrt{-1}R^{\det} < -\varepsilon\Theta,\ \sqrt{-1}R^E > -C\Theta\,\mathrm{Id}_E\,.$$

Theorem 3.3 ([47, Theorem 6.1.1], [48, Theorem 3.11])**.** *Assume that Condition 3.2 holds. Then there exist* $C_1>0$ *and* $p_0\in\mathbb{N}$ *such that for* $p\geqslant p_0$ *the quadratic form* Q_p *associated to the Kodaira Laplacian* $\square_p$ *acting on* $\Omega_{(2)}^{0,q}(X,L^p\otimes E)$ *satisfies*

$$Q_p(s,s) \geqslant C_1 p\,\|s\|_{L^2}^2\,, \quad \text{for } s\in \mathrm{Dom}(Q_p)\cap\Omega_{(2)}^{0,q}(X,L^p\otimes E),\ q>0\,. \tag{3.2}$$

Especially, there exists $p_0\in\mathbb{N}$ *such that for* $p\geqslant p_0$*,*

$$H_{(2)}^{0,q}(X,L^p\otimes E) = 0\,, \quad \text{for } q>0 \tag{3.3}$$

and the spectrum $\mathrm{Spec}(\square_p)$ *of* $\square_p$ *acting on* $L^2(X,L^p\otimes E)$ *is contained in* $\{0\}\cup[\,pC_1,\infty[$.

Proof. The proof is based on the Bochner-Kodaira-Nakano formula [47, Theorem 1.4.12] and its consequence Nakano's inequality [47, Corollary 1.4.17]. Let (F,h^F) be a holomorphic Hermitian bundle on X. Set $\widetilde{F} = F\otimes K_X^*$ where $K_X^* = \Lambda^n(T^{(1,0)}X) = \det(T^{(1,0)}X)$. Since $K_X\otimes K_X^*\cong\mathbb{C}$, there exists a natural isometry

$$\begin{aligned} &\Psi = \sim : \Lambda^{0,q}(T^*X)\otimes F \longrightarrow \Lambda^{n,q}(T^*X)\otimes\widetilde{F},\\ &\Psi\, s = \widetilde{s} = (w^1\wedge\ldots\wedge w^n\wedge s)\otimes(w_1\wedge\ldots\wedge w_n), \end{aligned} \tag{3.4}$$

where $\{w_j\}_{j=1}^n$ is a local orthonormal frame of $T^{(1,0)}X$ and $\{w^j\}_{j=1}^n$ is its dual frame. Let us denote by $\mathcal{T} = [i(\Theta),\partial\Theta]$ the Hermitian torsion of the metric Θ. The Bochner-Kodaira-Nakano formula [47, Corollary 1.4.17] shows that for any $s\in\Omega_0^{0,q}(X,F)$,

$$\frac{3}{2}\big(\|\overline{\partial}^F s\|^2+\|\overline{\partial}^{F,*}s\|^2\big) \geqslant \big\langle R^{F\otimes K_X^*}(w_j,\overline{w}_k)\overline{w}^k\wedge i_{\overline{w}_j}s,s\big\rangle - \frac{1}{2}\big(\|\mathcal{T}^*\widetilde{s}\|^2+\|\overline{\mathcal{T}}\widetilde{s}\|^2+\|\overline{\mathcal{T}}^*\widetilde{s}\|^2\big). \tag{3.5}$$

By applying (3.5) for $F = L^p\otimes E$ and taking into account that $R^{L^p} = pR^L$ and (3.1) we immediately obtain that for $Q_p(s,s)\geqslant C_1 p\,\|s\|_{L^2}^2$ for any $s\in\Omega_0^{0,q}(X,L^p\otimes E)$ and $q>0$. By the density result of Theorem 3.1 we obtain (3.2). Then as in the proof of Theorem 2.7 we obtain the spectral gap of $\square_p$ on $L_{0,0}^2(X,L^p\otimes E)$. ☐

Theorems 2.7 and 3.3 imply the following result.

Theorem 3.4. *Let* (X,Θ) *be a complete Hermitian manifold of dimension* n *and* L *and* E *be two holomorphic vector bundles on* X*, where* $\mathrm{rk}\,L=1$*, such that condition* (3.1) *is fulfilled. Set* $\omega = \frac{\sqrt{-1}}{2\pi}R^L$*. Then:*

(1) The Bergman kernel asymptotics for $H_{(2)}^0(X,L^p\otimes E,\Theta^n/n!)$ *holds on compact sets of* X*.*

(2) The Berezin-Toeplitz quantization package holds for the Kähler manifold (X,ω)*, the algebra* $\mathscr{C}_{\mathrm{const}}^\infty(X,\mathrm{End}(E))$ *and quantum spaces* $H_{(2)}^0(X,L^p\otimes E,\Theta^n/n!)$.

Next, we consider an interesting analogue of Theorem 3.3 for $(n,0)$-forms with values in $L^p\otimes E$, where $n=\dim X$. Let us first note that for such forms there exists a canonical L^2 condition. Indeed, let (F,h^F) be a holomorphic Hermitian vector bundle over the manifold X and let Θ be any Hermitian metric on X. Let $\Omega^{0,0}_{(2)}(X,F\otimes K_X)$ be the space of L^2 sections in $F\otimes K_X$, where the L^2 norm is calculated with respect to the metrics h^L, h^{K_X} induced by Θ and the volume form of Θ. Of course, $\Omega^{0,0}_{(2)}(X,F\otimes K_X)$ equals the space $\Omega^{n,0}_{(2)}(X,F)$ of square integrable $(n,0)$-forms with values in F. For any $(n,0)$-form s with values in F, and any metrics g^{TX}, g^{TX}_1 on X, with Riemannian volume forms dv_X, $dv_{X,1}$, respectively, we have $|s|^2_{g^{TX}}dv_X=|s|^2_{g^{TX}_1}dv_{X,1}$ pointwise. This shows that the L^2 condition for $(n,0)$-forms is independent of the choice of Hermitian metric Θ on X. Secondly, if we work with $(n,0)$-forms, it is not necessary to impose any condition regarding $R^{\det}$.

Condition 3.5. Let (X,J,Θ) be a Hermitian manifold and let (L,h^L) and (E,h^E) be holomorphic Hermitian vector bundles of rank one and r, respectively. We assume that the Riemannian metric g^{TX} induced from Θ is complete and we suppose that there exist $C,\varepsilon>0$ such that

$$\sqrt{-1}R^L>\varepsilon\Theta\,,\quad \sqrt{-1}R^E>-C\Theta\,\mathrm{Id}_E\,,\quad |\partial\Theta|_{g^{TX}}<C\,. \tag{3.6}$$

Theorem 3.6. *Assume that Condition 3.5 holds. Then there exist $C_1>0$ and $p_0\in\mathbb{N}$ such that for $p\geqslant p_0$ the quadratic form Q_p associated with the Kodaira Laplacian $\square_p$ acting on $\Omega^{n,q}_{(2)}(X,F)$ satisfies*

$$Q_p(s,s)\geqslant C_1p\,\|s\|^2\,,\quad \text{for } s\in\mathrm{Dom}(Q_p)\cap\Omega^{n,q}_{(2)}(X,L^p\otimes E),\ q>0\,. \tag{3.7}$$

Especially, $H^{n,q}_{(2)}(X,L^p\otimes E)=0$ for all $p\geqslant p_0$, $q>0$, and the spectrum $\mathrm{Spec}(\square_p)$ of $\square_p$ acting on $L^2(X,L^p\otimes E)$ is contained in $\{0\}\cup[\,p\,C_1,\infty[$.

Proof. In this case, we can use the following form of Nakano's inequality (cf. [47, Theorem 1.4.14]), we have for any $s\in\Omega^{\bullet,\bullet}_0(X,F)$,

$$\frac{3}{2}\langle\square^F s,s\rangle\geqslant\big\langle[\sqrt{-1}R^F,\Lambda]s,s\big\rangle-\frac{1}{2}(\|\mathcal{T}s\|^2+\|\mathcal{T}^*s\|^2+\|\overline{\mathcal{T}}s\|^2+\|\overline{\mathcal{T}}^*s\|^2)\,. \tag{3.8}$$

For an (n,q)-form $s\in\Omega^{n,q}_0(X,F)$ we have $\big\langle[\sqrt{-1}R^F,\Lambda]s,s\big\rangle\geqslant qa_1(x)|s|^2$ at every point $x\in X$, where $a_1(x)$ is the smallest eigenvalue of R^F_x with respect to Θ. Setting $F=L^p\otimes E$, (3.7) follows immediately from (3.6) and Theorem 3.1. □

By Theorems 2.7 and 3.6 we have:

Theorem 3.7. *Let (X,ω) be a complete Kähler manifold and (L,h^L) be a prequantum line bundle, that is, $\omega=\frac{\sqrt{-1}}{2\pi}R^L$. Let (E,h^E) be a holomorphic Hermitian vector bundle. Assume that there exists $C>0$ such that $\sqrt{-1}R^E>-C\omega\,\mathrm{Id}_E$. Then:*
(1) The Bergman kernel asymptotics for $H^{n,0}_{(2)}(X,L^p\otimes E)$ holds on compact sets of X.
(2) The Berezin-Toeplitz quantization package holds for the Kähler manifold (X,ω), the algebra $\mathscr{C}^\infty_{\mathrm{const}}(X,\mathrm{End}(E))$, and quantum spaces $H^{n,0}_{(2)}(X,L^p\otimes E)$.

The advantage of this result is that we do not need any condition on $R^{\det}$, and the Hilbert quantum spaces do not depend on the chosen Hermitian metric on X.

Corollary 3.8. *Let (X,ω) be a complete Kähler-Einstein manifold with $\mathrm{Ric}\,\omega=-\omega$. Then:*
(1) The Bergman kernel asymptotics for $H^0_{(2)}(X,K^p_X)$ is valid on compact subsets of X.
(2) The Berezin-Toeplitz quantization package holds for the Kähler manifold (X,ω), the algebra $\mathscr{C}^\infty_{\mathrm{const}}(X)$, and quantum spaces $H^0_{(2)}(X,K^p_X)$.

Example 3.9. Let us recall some classes of complete Kähler-Einstein manifolds.
(1) The Bergman metric ω_B on an irreducible bounded symmetric domain in $D \subset \mathbb{C}^n$ is a complete Kähler-Einstein metric with negative Ricci curvature; see e.g., [55, Proposition 3, p. 59]. The canonical line bundle K_D endowed with the metric induced by ω_D is positive.
(2) Cheng-Yau [13] showed that every bounded strictly pseudoconvex domain in $\mathbb{C}^n$ admits a unique complete Kähler-Einstein metric with negative Ricci curvature, which is a biholomorphic invariant. Mok-Yau [56] extended this result to bounded pseudoconvex domains in $\mathbb{C}^n$. For general strictly pseudoconvex domains, the Bergman metric is not equal to the Cheng-Yau metric.
(3) if X is a compact projective manifold, Σ is an effective divisor of X, such that $L = K_X \otimes \mathscr{O}(\Sigma)$ is ample. Then by [39] there exists a unique complete Kähler-Einstein metric ω with $\operatorname{Ric}\omega = -\omega$ on $X \setminus \Sigma$.

In this section, we consider very important classes of complex manifolds for the theory of several complex variables. These classes satisfy certain complex convexity conditions.

Stein manifolds are the natural framework for the complex function theory of several variables [11, 63]. The classical results of complex analysis in one variable, such as the Mittag-Leffler and Weierstrass theorems, generalize to Stein manifolds. Stein manifolds are characterized by the existence of enough holomorphic functions that provide an embedding in Euclidean space. They are the non-compact analogs of projective manifolds. On a Stein manifold, any holomorphic line bundle becomes a prequantum line bundle since we can endow it with a positively curved Hermitian metric.

Definition 3.10. Let X be a complex manifold and let $\mathscr{O}_X(X)$ be the algebra of holomorphic functions on X. We say that:
(a) X is *holomorphically separable* if for any $x, y \in X$, $x \neq y$, there exists $f \in \mathscr{O}_X(X)$ with $f(x) \neq f(y)$.
(b) X is *holomorphically convex* if for any compact set $K \subset X$, the holomorphic hull $\widehat{K} := \{x \in X : |f(x)| \leqslant \sup_K |f| \text{ for all } f \in \mathscr{O}_X(X)\}$ is compact.
(c) X is a *Stein manifold* if X is holomorphically separable and convex.

Examples. (i) Every non-compact Riemann surface is a Stein manifold, by a theorem of Behnke-Stein.
(ii) $\mathbb{C}^n$ is Stein. A domain in $\mathbb{C}^n$ is Stein if and only if it is a domain of holomorphy.
(iii) A product of Stein manifolds is Stein.
(iv) Any closed complex submanifold of $\mathbb{C}^n$ is a Stein manifold. Conversely, by a theorem of Remmert-Bishop-Narasimhan [6, 58, 61], any Stein manifold of dimension n can be properly embedded in $\mathbb{C}^{2n+1}$ and is thus biholomorphic to a closed submanifold of $\mathbb{C}^{2n+1}$.

An important characterization of Stein manifolds is provided through the use of plurisubharmonic functions. Let us introduce some convexity concepts for complex manifolds.

Definition 3.11. Let X be a complex manifold. A smooth function $\varphi : X \to \mathbb{R}$ is called *(strictly) plurisubharmonic*, for short *(strictly) psh*, if for any local holomorphic chart (U, z) on X the matrix $(\partial^2\varphi/\partial z_j \partial \overline{z}_k)_{j,k}$ is positive semidefinite (definite), that is, if the $(1,1)$-form $\sqrt{-1}\partial\overline{\partial}\varphi$ is semipositive (positive). The manifold X is called:
(a) 1-*complete* if it admits a strictly plurisubharmonic exhaustion function,
(b) *weakly* 1-*complete* if it admits a plurisubharmonic exhaustion function,
(c) 1-*convex* if it admits an exhaustion function that is strictly plurisubharmonic outside a compact set of X.

We note that a smooth function $\varphi : X \to \mathbb{R}$ is strictly plurisubharmonic if and only if $\sqrt{-1}\partial\overline{\partial}\varphi$ defines a Kähler metric on X. The notions of 1-complete and 1-convex manifolds were introduced in the seminal paper by Andreotti-Grauert [1] and the notion of weakly 1-complete manifold was introduced by Nakano [57].

A complex manifold is Stein if and only if it is 1-complete, as shown by Grauert's solution to the Levi problem [24]; cf. also [28, Theorem 5.2.10]. Any Hermitian metric of a holomorphic line bundle (L, h^L) can be modified to a positively curved metric $h^L_\chi = h^L e^{-\chi(\varphi)}$, with φ a strictly psh exhaustion function and χ a rapidly increasing convex function. Therefore, on a Stein manifold, any line bundle is a prequantum line bundle.

On a Stein manifold, we can construct complete metrics as follows. Let $\lambda : \mathbb{R} \to \mathbb{R}$ be a smooth, convex, increasing function such that

$$\int_0^\infty \sqrt{\lambda''(t)}\, dt = \infty\,. \tag{3.9}$$

Then for any Hermitian metric Θ on the X, the metric

$$\Theta_\lambda = \Theta + \sqrt{-1}\,\partial\overline{\partial}\lambda(\varphi) = \Theta + \sqrt{-1}\,\lambda'(\varphi)\,\partial\overline{\partial}\varphi + \sqrt{-1}\,\lambda''(\varphi)\,\partial\varphi\wedge\overline{\partial}\varphi\,. \tag{3.10}$$

is a complete Hermitian metric. We will denote in the following by $\mathscr{C}$ the cone of smooth convex increasing functions on $\mathbb{R}$.

Theorem 3.12. *Let X be a Stein manifold and let φ be a strictly plurisubharmonic exhaustion function. Let (L, h^L), (E, h^E) be holomorphic Hermitian vector bundles, with L of rank one. Set $\omega = \frac{\sqrt{-1}}{\pi}\partial\overline{\partial}\varphi$.*
(a) Then there exists $\lambda \in \mathscr{C}$ such that ω_λ is a complete Kähler metric and

$$\sqrt{-1}R^E \geq -\omega_\lambda \otimes \mathrm{Id}_E\,. \tag{3.11}$$

(b) There exists $\chi_0 \in \mathscr{C}$ such that for any $\chi \in \chi_0 + \mathscr{C}$, we have with $h^L_\chi := h^L e^{-\chi(\varphi)}$:
(1) The Bergman kernel asymptotics for $H^{n,0}_{(2)}(X, L^p \otimes E, (h^L_\chi)^p \otimes h^E)$ holds on compact sets of X.
(2) The Berezin-Toeplitz quantization package holds for the Kähler manifold (X, ω_χ), the algebra $\mathscr{C}^\infty_{\mathrm{const}}(X, \mathrm{End}(E))$, and quantum spaces $H^{n,0}_{(2)}(X, L^p \otimes E, (h^L_\chi)^p \otimes h^E)$.

Proof. (a) We first choose $\lambda_1 \in \mathscr{C}$ such that (3.9) is satisfied for $\lambda = \lambda_1$. Given that $\sqrt{-1}\,\partial\overline{\partial}\varphi$ is a Kähler form and φ is an exhaustion function, we use (3.10) to determine $\lambda_2 \in \mathscr{C}$ with λ_2' so rapidly increasing that (3.11) holds for $\lambda = \lambda_2$ (as done in e.g., [28, Theorem 4.2.2]). Then $\lambda = \lambda_1 + \lambda_2$ satisfies both conditions (3.9) and (3.11).
(b) As in (a), we can find χ_0 such that

$$\sqrt{-1}R^{(L,h^L_{\chi_0})} = \sqrt{-1}R^{(L,h^L)} + \sqrt{-1}\partial\overline{\partial}\chi_0(\varphi) \geq \omega_\lambda. \tag{3.12}$$

We check next that Condition 3.5 is satisfied in the present context for the Kähler manifold (X, ω_λ), and the bundles (L, h^L_χ) and (E, h^E). Since $\chi \in \chi_0 + \mathscr{C}$, we have $\sqrt{-1}\partial\overline{\partial}\chi(\varphi) \geq \sqrt{-1}\partial\overline{\partial}\chi_0(\varphi)$; hence

$$\sqrt{-1}R^{(L,h^L_\chi)} = \sqrt{-1}R^{(L,h^L)} + \sqrt{-1}\partial\overline{\partial}\chi(\varphi) \geq \omega_\lambda. \tag{3.13}$$

By (3.11), (3.13) and the fact that ω_λ is Kähler, we deduce that Condition 3.5 is satisfied, thereby ensuring that (1) and (2) follow from Theorem 3.6. □

Let us consider now the case of 1-convex manifolds. Since the exhaustion function of a 1-convex manifold is strictly psh only outside a compact set, one cannot use it to construct a Hermitian metric of positive curvature on any line bundle. We will thus assume the existence of a positive line bundle.

On a 1-convex manifold, we can construct a very natural exhaustion function. For this purpose, we recall the following analytic characterization of 1-convex manifold X (see

e. g. [1]): There exists a Stein space Y, a proper holomorphic surjective map $\rho : X \to Y$ satisfying $\rho_* \mathscr{O}_X = \mathscr{O}_Y$, and a finite set $A \subset Y$ such that the induced map $X \setminus \rho^{-1}(A) \to Y \setminus A$ is biholomorphic. The Stein space Y is called the Remmert reduction of X and $\Sigma := \rho^{-1}(A)$ the exceptional analytic set of X. Consider a strictly psh smooth exhaustion function φ_Y of Y, such that $\varphi_Y \geq 0$ and $\{\varphi_Y = 0\} = A$ (see [15, p. 563]). This is constructed by embedding Y into a Euclidean space $\mathbb{C}^N$ and constructing such a strictly psh exhaustion function on $\mathbb{C}^N$. Then $\varphi = \varphi_Y \circ \rho$ is a smooth psh exhaustion function of X, such that $\varphi \geq 0$, $\{\varphi = 0\} = \Sigma$ and φ is strictly psh on $X \setminus \Sigma$.

Theorem 3.13. *Let X be a 1-convex manifold and let φ be an exhaustion function as above. Let (L, h^L) be a positive line bundle on X, and (E, h^E) be a holomorphic Hermitian vector bundle. Let ω be a Kähler form on X. Then the following holds:*
(a) There exist $\lambda \in \mathscr{C}$ and $C > 0$ such that ω_λ is a complete Kähler metric and

$$\sqrt{-1}R^E \geq -C\omega_\lambda \otimes \mathrm{Id}_E\,. \tag{3.14}$$

(b) There exists $\chi_0 \in \mathscr{C}$ such that for any $\chi \in \chi_0 + \mathscr{C}$, we have for $h^L_\chi := h^L e^{-\chi(\varphi)}$:
(1) The Bergman kernel asymptotics for $H^{n,0}_{(2)}(X, L^p \otimes E, (h^L_\chi)^p \otimes h^E)$ holds on compact sets of X.
(2) The Berezin-Toeplitz quantization package holds for the Kähler manifold (X, ω_χ), the algebra $\mathscr{C}^\infty_{\mathrm{const}}(X, \mathrm{End}(E))$, and quantum spaces $H^{n,0}_{(2)}(X, L^p \otimes E, (h^L_\chi)^p \otimes h^E)$.

Proof. (a) We first observe that the $(1,1)$-form ω_λ in (3.10) is a Kähler form since φ is psh on X. Moreover, the same argument as above shows that ω_λ is complete provided λ satisfies (3.9). For $a \in \mathbb{R}$, we denote by $X_a = \{x \in X : \varphi(x) < a\} \Subset X$. Let us consider $c < d$ such that $\Sigma \subset X_c \subset X_d$. There exists $C > 0$ such that $\sqrt{-1}R^E \geq -C\omega \otimes \mathrm{Id}_E$ on X_d, and thus also $\sqrt{-1}R^E \geq -C\omega_\lambda \otimes \mathrm{Id}_E$. Given that φ is strictly psh outside Σ, we can select λ to be rapidly increasing such that $\sqrt{-1}R^E \geq -\omega_\lambda \otimes \mathrm{Id}_E$ on $X \setminus X_c$, and ω_λ is complete.
(b) We fix $\lambda \in \mathscr{C}$ as in (a). There exists $\varepsilon > 0$ such that $\sqrt{-1}R^{(L,h^L)} \geq \varepsilon\omega_\lambda$ on X_d. Since$\sqrt{-1}\partial\overline{\partial}\chi(\varphi)$ is semi-positive on X for any $\chi \in \mathscr{C}$, we have $\sqrt{-1}R^{(L,h^L_\chi)} \geq \varepsilon\omega_\lambda$ on X_d. Given that φ is strictly psh outside Σ, there exists $\chi_0 \in \mathscr{C}$ such that $\sqrt{-1}\partial\overline{\partial}\chi_0(\varphi) \geq \varepsilon\omega_\lambda$ on $X \setminus X_c$. Since $\sqrt{-1}R^{(L,h^L)}$ is positive, we have $\sqrt{-1}R^{(L,h^L_{\chi_0})} \geq \varepsilon\omega_\lambda$ on $X \setminus X_c$. Hence,

$$\sqrt{-1}R^{(L,h^L_{\chi_0})} \geq \varepsilon\omega_\lambda \quad \text{on } X. \tag{3.15}$$

By (3.14), (3.15), and the fact that ω_λ is Kähler, we deduce that Condition 3.5 is satisfied, thereby ensuring that (1) and (2) follow from Theorem 3.6. □

We consider now weakly 1-complete manifolds. To give examples, note that:
(i) Any 1-convex manifold is weakly 1-complete.
(ii) If X and Y are complex manifolds and there exists a proper holomorphic map $\pi : X \to Y$ and Y is weakly 1-complete, then X is weakly 1-complete, too. Indeed, if φ is a smooth psh exhaustion function on Y, Then $\varphi \circ \pi$ is a smooth psh exhaustion function on X.

Theorem 3.14. *Let X be a weakly 1-complete manifold of dimension n and let $\varphi : X \to \mathbb{R}$ be a smooth psh exhaustion function. Let (L, h^L) be a positive line bundle. We consider the Kähler metric $\omega = \sqrt{-1}R^{(L,h^L)}$ on X. Then the following holds:*
(a) There exists $\lambda \in \mathscr{C}$ such that ω_λ is a complete Kähler.
(b) For any $\chi \in \lambda + \mathscr{C}$, we have with $h^L_\chi := h^L e^{-\chi(\varphi)}$:
(1) The Bergman kernel asymptotics for $H^{n,0}_{(2)}(X, L^p, (h^L_\chi)^p)$ holds on compact sets of X.
(2) The Berezin-Toeplitz quantization package holds for the Kähler manifold (X, ω_χ), the algebra $\mathscr{C}^\infty_{\mathrm{const}}(X)$ and quantum spaces $H^{n,0}_{(2)}(X, L^p, (h^L_\chi)^p)$.

Proof. (a) If $\lambda \in \mathscr{C}$ satisfies (3.9) then ω_λ is a complete Kähler metric.
(b) For $\chi \in \lambda + \mathscr{C}$ we have $\sqrt{-1}R^{(L,h^L_\chi)} \geq \omega_\lambda$ so Condition 3.5 is satisfied. □

3.2. **Big line bundles and quasiprojective manifolds.** Let X be a compact complex manifold X of dimension n. A holomorphic line bundle L on X is called big if its Kodaira-Iitaka dimension equals the dimension of X, equivalently if

$$\limsup_{p\to\infty} p^{-n} \dim H^0(X, L^p) > 0.$$

If a compact manifold X admits a big line bundle then X is Moishezon and L admits a singular metric h^L, smooth outside a proper analytic subset Σ of X, and with strictly positive curvature current $\sqrt{-1}R^{h^L}$ (see e. g. [47, Lemma 2.3.6]). The main result of this section is the Berezin-Toeplitz quantization of this Zariski open set endowed with the generalized Poincaré metric.

Let X be a compact connected complex manifold of dimension n. Let Σ be a closed analytic subset of X. Let $\pi : \widetilde{X} \longrightarrow X$ be a resolution of singularities such that $\pi : \widetilde{X} \setminus \pi^{-1}(\Sigma) \longrightarrow X \setminus \Sigma$ is biholomorphic and $\pi^{-1}(\Sigma)$ is a divisor with normal crossings. More precisely, there exists a finite sequence of blow-ups

$$\widetilde{X} = X_m \xrightarrow{\tau_m} X_{m-1} \xrightarrow{\tau_{m-1}} \cdots \xrightarrow{\tau_2} X_1 \xrightarrow{\tau_1} X_0 = X \tag{3.16}$$

such that

(a): τ_i is the blow-up along a non-singular center Y_{i-1} contained in the strict transform of Σ in X_{i-1}, $i \geqslant 1$,

(b): the strict transform of Σ in $\widetilde{X} = X_m$ through $\pi = \tau_1 \circ \tau_2 \circ \cdots \circ \tau_m$ is smooth and $\pi^{-1}(\Sigma)$ is a divisor with normal crossings.

Let $g_0^{T\widetilde{X}}$ be an arbitrary smooth J-invariant metric on $\widetilde{X}$ and $\Theta'(\cdot,\cdot) = g_0^{T\widetilde{X}}(J\cdot,\cdot)$ the corresponding $(1,1)$-from. The *generalized Poincaré metric* on $X \setminus \Sigma = \widetilde{X} \setminus \pi^{-1}(\Sigma)$ is defined by the Hermitian form

$$\Theta_{\varepsilon_0} = \Theta' + \varepsilon_0\sqrt{-1}{\textstyle\sum_i}\overline{\partial}\partial \log\left((-\log(\|\sigma_i\|_i^2))^2\right), \quad 0 < \varepsilon_0 \ll 1 \text{ fixed}, \tag{3.17}$$

where $\pi^{-1}(\Sigma) = \cup_i \Sigma_i$ is the decomposition into irreducible components Σ_i of $\pi^{-1}(\Sigma)$ and each Σ_i is non-singular; σ_i are holomorphic sections of the associated holomorphic line bundle $\mathscr{O}_{\widetilde{X}}(\Sigma_i)$ which vanish to first order on Σ_i, and $\|\sigma_i\|_i$ is the norm for a smooth Hermitian metric $\|\cdot\|_i$ on $\mathscr{O}_{\widetilde{X}}(\Sigma_i)$ such that $\|\sigma_i\|_i < 1$. Let $R^{\mathscr{O}_{\widetilde{X}}(\Sigma_i)}$ be the curvature of $(\mathscr{O}_{\widetilde{X}}(\Sigma_i), \|\cdot\|_i)$.

Lemma 3.15 ([47, Lemma 6.2.1]). (i) *The generalized Poincaré metric* (3.17) *is a complete Hermitian metric of finite volume. Its Hermitian torsion* $\mathcal{T}_{\varepsilon_0} = [i(\Theta_{\varepsilon_0}), \partial\Theta_{\varepsilon_0}]$ *and the curvature* $R^{\det} = R^{K_X^*}$ *is also bounded.*
(ii) *If* (E, h^E) *is a holomorphic Hermitian vector bundle over* X*, set*

$$H^0_{(2)}(X \setminus \Sigma, E) = \left\{u \in L^2_{0,0}(X \setminus \Sigma, E\,,\, \Theta_{\varepsilon_0}\,, h^E) : \overline{\partial}^E u = 0\right\}, \tag{3.18}$$

then

$$H^0_{(2)}(X \setminus \Sigma, E) = H^0(X, E). \tag{3.19}$$

Lemma 3.16 ([47, Lemma 6.2.2]). *There exists a singular Hermitian line bundle* $(\widetilde{L}, h^{\widetilde{L}})$ *on* $\widetilde{X}$ *which is strictly positive and* $\widetilde{L}|_{\widetilde{X}\setminus\pi^{-1}(\Sigma)} \cong \pi^*(L^{k_0})$*, for some* $k_0 \in \mathbb{N}$.

We introduce on $L|_{X\setminus\Sigma}$ the metric $(h^{\widetilde{L}})^{1/k_0}$ whose curvature extends to a strictly positive $(1,1)$–current on $\widetilde{X}$. Set

$$h^L_\varepsilon := (h^{\widetilde{L}})^{1/k_0} \prod_i (-\log(\|\sigma_i\|_i^2))^\varepsilon, \quad 0 < \varepsilon \ll 1\,, \tag{3.20a}$$

$$H^0_{(2)}(X\setminus\Sigma,L^p):=\big\{u\in L^2_{0,0}(X\setminus\Sigma,L^p\,,\,\Theta_{\varepsilon_0}\,,h^L_\varepsilon):\overline{\partial}^{L^p}u=0\big\}. \tag{3.20b}$$

The space $H^0_{(2)}(X\setminus\Sigma,L^p)$ is the space of L^2-holomorphic sections relative to the metrics Θ_{ε_0} on $X\setminus\Sigma$ and h^L_ε on $L|_{X\setminus\Sigma}$. Since $(h^{\widetilde{L}})^{1/k_0}$ is bounded away from zero (having plurisubharmonic weights), and its curvature extends a strictly positive $(1,1)$-current on $\widetilde{X}$, the elements of this space are L^2 integrable with respect to the Poincaré metric and a smooth metric h^L_* of L over the whole X. By the proof of Lemma 3.15 given in [47, Proof of Lemma 6.2.1.(ii)], we have $H^0_{(2)}(X\setminus\Sigma,L^p)\subset H^0(X,L^p)$. The space $H^0_{(2)}(X\setminus\Sigma,L^p)$ is our space of polarized sections.

Theorem 3.17. *Let X be a compact complex manifold with an integral Kähler current ω. Let (L,h^L) be a singular polarization of $[\omega]$ with strictly positive curvature current having singular support contained in a proper analytic set Σ. Then the following statements hold:*
(1) The Bergman kernel of the space of polarized sections (3.20b) has the asymptotic expansion on compact sets of $X\setminus\Sigma$.
(2) The Berezin-Toeplitz quantization package holds for the Kähler manifold $(X\setminus\Sigma,\omega_\varepsilon)$, with $\omega_\varepsilon=\sqrt{-1}R^{h^L_\varepsilon}$.

Proof. It was shown in [47, Theorem 6.2.3], by using Lemmas 3.15 and 3.16, that the spectral gap (2.19) holds in the situation at hand. Thus, both statements follow from Theorem 2.7. □

3.3. Manifolds of bounded geometry.

The purpose of this section is to establish the Berezin-Toeplitz quantization of manifolds with bounded geometry. The result itself already appears in [22, Lemma 4.6] (derived there from our method [50, 52]); see also [41] for a related statement. We provide here a short, self-contained proof. Let us first introduce the notion of bounded geometry in the form we will need it.

Definition 3.18. Let (X,J,Θ) be a Hermitian manifold and let $g^{TX}=\Theta(\cdot,J\cdot)$ be the associated Riemannian metric. Let (F,h^F) be a holomorphic Hermitian vector bundle. We say that (X,J,Θ) and (F,h^F) have bounded geometry if the derivatives of any order of R^F, J, g^{TX} are uniformly bounded on X in the norm induced by g^{TX}, h^F, and the injectivity radius of (X,g^{TX}) is positive.

Let us denote by

$$\mathscr{C}^\infty_b(X,F):=\left\{f\in\mathscr{C}^\infty(X,F):\sup_{x\in X}|(\nabla^F)^k s|_{g^{TX},h^F}<\infty\text{ for any }k\in\mathbb{N}\right\}, \tag{3.21}$$

where ∇^F is the connection induced by the Chern connection ∇^F and the Levi-Civita connection ∇^{TX} on the tensor algebra of T^*X, and the norm $|\cdot|_{g^{TX},h^F}$ is induced by g^{TX} and h^F.

Assumption 3.19. Let (X,J,Θ) be a Hermitian manifold of dimension n with associated Riemannian metric $g^{TX}=\Theta(\cdot,J\cdot)$. Let (L,h^L), (E,h^E) be holomorphic Hermitian vector bundles, with L of rank one. Suppose that (X,g^{TX}) is complete and (X,J,Θ), (L,h^L), (E,h^E) have bounded geometry.

We recall the following result about the exponential decay of the Bergman kernel.

Theorem 3.20 ([52, Theorem 1])**.** *In the situation of Assumption 3.19 assume that there exists $\varepsilon>0$ such that on X, $\sqrt{-1}R^L>\varepsilon\Theta$. Then there exist $\boldsymbol{c}>0$, $\boldsymbol{p}_0>0$, which can be determined explicitly from the geometric data such that for any $k\in\mathbb{N}$, there exists $C_k>0$ such that for any $p\geqslant\boldsymbol{p}_0$, $x,x'\in X$, we have*

$$|P_p(x,x')|_{\mathscr{C}^k}\leqslant C_k\,p^{n+\frac{k}{2}}\exp(-\boldsymbol{c}\sqrt{p}\,d(x,x'))\,. \tag{3.22}$$

In the context of bounded geometry, we have the following:

Theorem 3.21 ([22, Lemma 4.6]). *Under the hypotheses of Theorem 3.20, the Berezin-Toeplitz package holds for the algebra* $\mathscr{C}^\infty_b(X, \mathrm{End}(E))$.

Proof. Let $0 < 4\varepsilon < a^X$, where $a^X > 0$ is the injectivity radius of X. At first, for any $0 < c_1 < \boldsymbol{c}$, $k \in \mathbb{N}$, $f \in \mathscr{C}^\infty_b(X, \mathrm{End}(E))$, there exists $C_k > 0$ such that for any $p \geqslant \boldsymbol{p}_0$, $x, x' \in X$, we have

$$|T_{f,p}(x,x')|_{\mathscr{C}^k} \leqslant C_k\, p^{n+\frac{k}{2}} \exp(-c_1 \sqrt{p}\, d(x,x')) \,. \tag{3.23}$$

In fact, by (3.22), we have

$$\begin{aligned} |P_p(x,y)f(y)P_p(y,x')|_{\mathscr{C}^k \text{on } x,x'} &\leqslant Cp^{2n+\frac{k}{2}} e^{-\boldsymbol{c}\sqrt{p}(d(x,y)+d(y,x'))} \\ &\leqslant Cp^{2n+\frac{k}{2}} e^{-c_1\sqrt{p}d(x,x')-(\boldsymbol{c}-c_1)\sqrt{p}d(x,y)}. \end{aligned} \tag{3.24}$$

Now, under our assumption of bounded geometry, there exists $K > 0$ such that the Ricci curvature of (X, g^{TX}) is bounded below by $-(2n-1)K^2 g^{TX}$. By Bishop's inequality, this implies that the volume of $B^X(x,r) \subset X$ is smaller than or equal to the volume of a geodesic ball of radius $r > 0$ in the space of constant curvature $-K$ (see, for example, [59, Lemma 7.1.3]). Then, by a classical estimate for the volume of large balls in the space of constant curvature $-K$, which can be found for example in [54, p. 3], there exists a universal constant $C_{n,K} > 0$, depending only on K and on the dimension n of X, such that for any $x \in X$ and $r > 0$,

$$\mathrm{Vol}\, B^X(x,r) \leq C_{n,K}\, e^{(2n-1)Kr} \,. \tag{3.25}$$

Then by (2.96), (3.24), (3.25), we get

$$\begin{aligned} |T_{f,p}(x,x')|_{\mathscr{C}^k} &\leqslant \sum_{k=0}^{\infty} \int_{B^X(x,(k+1)\varepsilon)\setminus B^X(x,k\varepsilon)} |P_p(x,y)f(y)P_p(y,x')|_{\mathscr{C}^k \text{on } x,x'} dv_X(y) \\ &\leqslant \sum_{k=0}^{\infty} Ce^{(2n-1)K(k+1)\varepsilon} p^{2n+\frac{k}{2}} e^{-(\boldsymbol{c}-c_1)\sqrt{p}k\varepsilon} e^{-c_1\sqrt{p}kd(x,x')}. \end{aligned} \tag{3.26}$$

From (3.26), for any $p > \left(\frac{(2n-1)K}{\boldsymbol{c}-c_1}\right)^2$, the above series is bounded by

$$Cp^{2n+\frac{k}{2}} e^{-c_1\sqrt{p}kd(x,x')},$$

and by slightly increasing c_1, we obtain (3.23) if $d(x,x') > \varepsilon$. If $d(x,x') \leqslant \varepsilon$, then the summation $\sum_{k=2}^{\infty}$ in (3.26) can be estimated by

$$\sum_{k=2}^{\infty} Ce^{(2n-1)K(k+1)\varepsilon} p^{2n+\frac{k}{2}} e^{-\boldsymbol{c}\sqrt{p}(2k-1)\varepsilon} \leqslant Ce^{-\boldsymbol{c}\sqrt{p}\varepsilon}. \tag{3.27}$$

The term $\sum_{k=0}^{1}$ can be estimated on the normal coordinate centered at x using (3.24):

$$\sum_{k=0}^{1} \cdots \leqslant C \int_{B^X(x,2\varepsilon)} p^{2n+\frac{k}{2}} e^{-(\boldsymbol{c}-c_1)\sqrt{p}d(x,y)} e^{-c_1\sqrt{p}d(x,x')} dv_X(y) \leqslant Cp^{n+\frac{k}{2}} e^{-c_1\sqrt{p}d(x,x')}. \tag{3.28}$$

From (3.26)-(3.28), we get (3.23). Subsequently, we establish the following criterion for Toeplitz operators, which serves as an analog of Theorem 2.19.

Lemma 3.22 ([22, Theorem 3.18]). *For a family of operators in Theorem 2.19, we replace the conditions ii) and iii) with ii)' : For any* $\varepsilon_0 > 0$, *there exist* $p_0 > 0$, $C > 0$, *and* $c_1 > 0$ *such that for* $p > p_0$ *and* $x, x' \in X$ *with* $d(x,x') \geq \varepsilon_0$.

$$|T_p(x,x')| \leqslant Cp^n \exp(-c_1 \sqrt{p}\, d(x,x')) \,. \tag{3.29}$$

We assume in iv) that for each r, the polynomial $\mathcal{Q}_{r,x_0}(\mathcal{T})$ as a section of $\mathrm{End}(E)$ twisted with tensor algebras of T^*X is uniformly bounded with derivatives and its degree on $x_0 \in X$. Then $\{T_p\}_p$ is a Toeplitz operator.

Proof. We follow step by step the proof of Theorem 2.19. At first, from our assumption iv) on polynomial $\mathcal{Q}_{r,x_0}(\mathcal{T})$, we know that g_0 in (2.57) is in $\mathscr{C}^\infty_b(X, \mathrm{End}(E))$. To obtain Proposition 2.20, we need to modify the argument in the proof of [49, Lemma 4.13] as follows by using assumption ii)': We use the notation in [49, Lemma 4.13]; by (3.29), note that $R_{r,p}(x, y) = 0$ for $d(x, y) \geq \varepsilon'$, and for any $k > 0$, there exist $c_1 > 0$ and $C > 0$ such that

$$\Big|\big(\mathscr{T}_p - \textstyle\sum_{r=1}^k p^{-r/2}R_{r,p}\big)(x,y)\Big| \begin{aligned}[t] &\leqslant C\exp(-c_1\sqrt{p}d(x,y)) \text{ if } d(x,y) \geq \varepsilon', \\ &Cp^{n-(k+1)/2}\exp(-c_1\sqrt{p}d(x,y)) + \mathscr{O}(p^{-\infty}) \text{ if } d(x,y) < \varepsilon'. \end{aligned} \tag{3.30}$$

Now, by the argument in the proof of (3.23), i.e., we decompose the integral $\int_X$ by the sum of the integrals on $B^X(x, (k+1)\varepsilon) \setminus B^X(x, k\varepsilon)$, and using (3.30), we obtain

$$\begin{aligned} &\int_X \Big|\big(\mathscr{T}_p - \sum_{r=1}^k p^{-r/2}R_{r,p}\big)(y,x)\Big| dv_X(y) = \mathscr{O}(p^{-1}), \\ &\int_X \Big|\big(\mathscr{T}_p - \sum_{r=1}^k p^{-r/2}R_{r,p}\big)(x,y)\Big| dv_X(y) = \mathscr{O}(p^{-1}). \end{aligned} \tag{3.31}$$

From (3.31), we obtain [49, (4.48)]

$$\begin{aligned} \|\big(\mathscr{T}_p - \sum_{r=1}^k p^{-r/2}R_{r,p}\big)\, s\|^2_{L^2} \leqslant &\int_X \Big(\int_X \Big|\big(\mathscr{T}_p - \sum_{r=1}^k p^{-r/2}R_{r,p}\big)(x,y)\Big| dv_X(y)\Big) \\ &\times \Big(\int_X \Big|\big(\mathscr{T}_p - \sum_{r=1}^k p^{-r/2}R_{r,p}\big)(x,y)\Big| |s(y)|^2 dv_X(y)\Big) dv_X(x) \leq Cp^{-2}\|s\|^2_{L^2}. \end{aligned} \tag{3.32}$$

Thus [49, Lemma 4.13] holds in our situation. □

Now we fix $f, g \in \mathscr{C}^\infty_b(X, \mathrm{End}(E))$. From (3.23), and by using the same trick as in (3.26)-(3.28), we get that $(T_{f,p}T_{g,p})(x, x')$ satisfies conditions ii)' and iv) in Lemma 3.22. this implies that $T_{f,p}T_{g,p}$ is a Toeplitz operator by Lemma 3.22.

We adapt the proof of (2.14) presented in Theorem 2.7. Suppose the supremum of f is attained at a specific point x_0. In that case, this proof remains unchanged, as the expansion of the peak section (2.85), derived from the asymptotic expansion of the Bergman kernel, continues to hold on manifolds characterized by bounded geometry, as shown by Theorem 3.20. If the supremum is not attained, then the same proof gives us that for any $\varepsilon > 0$, there exists $p_0 > 0$, such that for every $p \geq p_0$, we have $\|f\|_\infty - \varepsilon \leqslant \|T_{f,p}\|$, and this entails (2.14) in the present case. □

Example 3.23. (1) Let (X, J, Θ) be a compact Hermitian manifold and with associated Riemannian metric $g = \Theta(\cdot, J\cdot)$. Let (L, h^L), (E, h^E) be holomorphic Hermitian vector bundles, with L of rank one. Let $\rho : \widetilde{X} \to X$ be a Galois covering. Let us decorate by $\sim$ pullbacks of objects on X by ρ. Then $(\widetilde{X}, \widetilde{g})$ is complete and $(X, \widetilde{J}, \widetilde{\Theta})$, $(\widetilde{L}, \widetilde{h^L})$, $(\widetilde{E}, \widetilde{h^E})$ have bounded geometry. Moreover there exists $\varepsilon > 0$ such that $\sqrt{-1}R^{\widetilde{L}} > \varepsilon\,\widetilde{\Theta}$.

(2) Let D be a smoothly bounded strictly pseudovonvex domain in $\mathbb{C}^n$ or in a Stein manifold. Then, for each fixed $\lambda < 0$, there exists a unique complete Kähler metric ω on D satisfying $\mathrm{Ric}(\omega) = \lambda\omega$. For the unique complete Kähler-Einstein metric ω_{CY} with $\mathrm{Ric}(\omega_{\mathrm{CY}}) = -\omega_{\mathrm{CY}}$ (the Cheng-Yau metric), the canonical bundle is positive and polarizes the metric. It is known to have bounded geometry, as a result of its asymptotically

complex hyperbolic nature, proven through the work of Cheng-Yau [13] and the boundary regularity analysis of Lee-Melrose [42]. Hence the results in this section apply to $(D, \omega_{\rm CY})$ and the canonical bundle K_D endowed with the metric induced by $\omega_{\rm CY}$.

3.4. **Pseudoconvex domains.** In this section, we consider relatively compact pseudoconvex domains with smooth boundary in a complex manifold. They are endowed with an incomplete Hermitian metric, and we will thus work with the Kodaira Laplacian with $\overline{\partial}$-Neumann boundary conditions. This generalizes the results of Englis for strictly pseudoconvex domains in $\mathbb{C}^n$.

Let M be a complex manifold and let X be a smooth, relatively compact domain in M. We set $X = \{x \in M \,:\, \varrho(x) < 0\}$, where $\varrho \in \mathscr{C}^\infty(M)$ is a defining function that satisfies $|d\varrho| = 1$ on ∂X. The *Levi form* of ∂X is the restriction of $\partial\overline{\partial}\varrho$ to the holomorphic tangent bundle of ∂X. The domain X is called *strictly pseudoconvex (pseudoconvex)* if the Levi form is positive definite (semi-definite) at each point of ∂X. Let us consider a holomorphic Hermitian vector bundle (F, h^F) on M. The complex manifold M is endowed with a Hermitian metric with $(1,1)$-form Θ, and we consider its restriction to X with volume form $dv_X = \Theta^n/n!$. We construct, as in (2.1), (2.2), the spaces $L^2(X,F)$ and $H^0_{(2)}(X,F)$. Let $\overline{\partial}^F : \Omega^{0,\bullet}(M,F) \to \Omega^{0,\bullet+1}(M,F)$ be the Dolbeault operator; we denote by $\overline{\partial}^{F,*}$ its formal adjoint. Let $\overline{\partial}^F : \mathrm{Dom}(\overline{\partial}^F) \subset L^2_{0,\bullet}(X,F) \to L^2_{0,\bullet+1}(X,F)$ be its maximal extension on $L^2_{0,\bullet}(X,F)$. Let $\overline{\partial}^{F,*}_H$ be the Hilbert space adjoint of $\overline{\partial}^F$ on X. In order to describe the domain of $\overline{\partial}^{F,*}_H$ we now present the following concepts. Let $-e_{\mathfrak{n}} \in TM$ be the metric dual of $d\varrho$. Then $e_{\mathfrak{n}} \in TM$ is the inward pointing unit normal at ∂X. We decompose $e_{\mathfrak{n}}$ as $e_{\mathfrak{n}} = e^{(1,0)}_{\mathfrak{n}} + e^{(0,1)}_{\mathfrak{n}} \in T^{(1,0)}M \oplus T^{(0,1)}M$. We introduce the space

$$B^{0,q}(X,F) = \left\{ s \in \Omega^{0,q}(\overline{X},F) : i_{e^{(0,1)}_{\mathfrak{n}}} s = 0 \text{ on } \partial X \right\}. \tag{3.33}$$

It is then well-known that $B^{0,q}(X,F) = \mathrm{Dom}(\overline{\partial}^{F,*}_H) \cap \Omega^{0,q}(\overline{X},F)$ and $\overline{\partial}^{F,*}_H = \overline{\partial}^{F,*}$ on $B^{0,q}(X,F)$ (cf. [23, 27], [47, Proposition 1.4.19]). Thus

$$\big\langle \overline{\partial}^F s_1, s_2 \big\rangle = \big\langle s_1, \overline{\partial}^{F,*} s_2 \big\rangle \,, \qquad \text{for } s_1 \in \Omega^{0,q}(\overline{X},F)\,, s_2 \in B^{0,q+1}(X,F). \tag{3.34}$$

We consider the operator

$$\begin{aligned} \mathrm{Dom}(\square^F) &:= \big\{ s \in B^{0,q}(X,F) \,:\, \overline{\partial}^F s \in B^{0,q+1}(X,F) \big\}, \\ \square^F s &= \overline{\partial}^F \overline{\partial}^{F,*} s + \overline{\partial}^{F,*} \overline{\partial}^F s\,, \quad \text{for } s \in \mathrm{Dom}(\square^F)\,, \end{aligned} \tag{3.35}$$

which by (3.34) is positive. Then the boundary conditions of $\mathrm{Dom}(\square^E)$ in (3.35) are called *$\overline{\partial}$-Neumann boundary conditions* [23, 27] is given by :

$$\mathrm{Dom}(\square^F) = \big\{ s \in \Omega^{0,\bullet}(\overline{X},F);\ i_{e^{(0,1)}_{\mathfrak{n}}} s = i_{e^{(0,1)}_{\mathfrak{n}}} \overline{\partial}^F s = 0 \quad \text{on } \partial X \big\}. \tag{3.36}$$

An extension of the associated quadratic form Q is

$$\mathrm{Dom}(Q) := B^{0,q}(X,F), \ \ Q(s_1,s_2) := \langle \overline{\partial}^F s_1, \overline{\partial}^F s_2 \rangle + \langle \overline{\partial}^{F,*} s_1, \overline{\partial}^{F,*} s_2 \rangle\,. \tag{3.37}$$

It is easy to see that Q is closable, so there exists a self-adjoint operator associated with the closure $\overline{Q}$ (the Friedrichs extension of $\square^F$) called, in this context , *Kodaira Laplacian with $\overline{\partial}$-Neumann boundary conditions*. We still denote this operator by $\square^F$. We have an analog of the Andreotti-Vesentini density result (Theorem 3.1).

Lemma 3.24 ([23, 27]). *$\Omega^{0,\bullet}(\overline{X},F)$ is dense in $\mathrm{Dom}(\overline{\partial}^F)$ in the graph-norm of $\overline{\partial}^F$, and $B^{0,q}(M,F)$ is dense in $\mathrm{Dom}(\overline{\partial}^{F,*}_H)$ and in $\mathrm{Dom}(\overline{\partial}^F) \cap \mathrm{Dom}(\overline{\partial}^{F,*}_H)$ in the graph-norms of $\overline{\partial}^{F,*}$ and $\overline{\partial}^E + \overline{\partial}^{F,*}$, respectively.*

From this we deduce immediately the following (see e. g. [47, Proposition 3.5.2]).

Proposition 3.25. *The Kodaira Laplacian with $\overline{\partial}$-Neumann conditions on X coincides with the Gaffney extension* (2.17) *of the Kodaira Laplacian.*

We recall now the Bochner-Kodaira-Nakano formula with boundary term [47, Theorem 1.4.21]. For $s \in \Omega^{0,q}(\overline{X}, F)$ and $y \in \partial X$, set

$$\mathscr{L}_\varrho(s,s) = (\partial\overline{\partial}\varrho)(w_k, \overline{w}_j)\langle \overline{w}^j \wedge i_{\overline{w}_k} s, s\rangle_{\Lambda^{\bullet,\bullet}\otimes E,y}. \tag{3.38}$$

By [47, Theorem 1.4.21] we have for any $s \in B^{0,\bullet}(X, F)$,

$$\begin{aligned} \|\overline{\partial}^F s\|^2_{L^2} + \|\overline{\partial}^{F,*} s\|^2_{L^2} = \|(\nabla^{\widetilde{F}})^{1,0*}\widetilde{s}\|^2_{L^2} + \big\langle R^{F\otimes K^*_X}(w_j, \overline{w}_k)\overline{w}^k \wedge i_{\overline{w}_j} s, s\big\rangle \\ -\big\langle \overline{\partial}^F s, \Psi^{-1}\overline{\mathcal{T}}\,\widetilde{s}\big\rangle - \big\langle \Psi^{-1}\overline{\mathcal{T}}^*\widetilde{s}, \overline{\partial}^{F,*} s\big\rangle + \big\langle \mathcal{T}^*\widetilde{s}, (\nabla^{\widetilde{F}})^{1,0*}\widetilde{s}\big\rangle \\ + \int_{\partial X} \mathscr{L}_\varrho(s,s)\, dv_{\partial X}\,. \end{aligned} \tag{3.39}$$

Especially, we obtain the following Bochner-Kodaira-Nakano inequalitiy [47, Corollary 1..4.22]. For any $s \in B^{0,q}(X, F)$,

$$\begin{aligned} \frac{3}{2}\big(\|\overline{\partial}^F s\|^2_{L^2} + \|\overline{\partial}^{F,*} s\|^2_{L^2}\big) \geqslant \frac{1}{2}\,\|(\nabla^{\widetilde{F}})^{1,0*}\widetilde{s}\|^2_{L^2} + \big\langle R^{F\otimes K^*_X}(w_j, \overline{w}_k)\overline{w}^k \wedge i_{\overline{w}_j} s, s\big\rangle \\ + \int_{\partial X} \mathscr{L}_\varrho(s,s)\, dv_{\partial X} - \frac{1}{2}\big(\|\mathcal{T}^*\widetilde{s}\|^2_{L^2} + \|\overline{\mathcal{T}}\widetilde{s}\|^2_{L^2} + \|\overline{\mathcal{T}}^*\widetilde{s}\|^2_{L^2}\big). \end{aligned} \tag{3.40}$$

Theorem 3.26. *Let X be a relatively compact pseudoconvex domain with smooth boundary in a complex manifold M. Let L and E be two holomorphic vector bundles on M, where* $\operatorname{rk} L = 1$. *Assume that (L, h^L) is positive on a neighbourhood of $\overline{X}$. Then we have*

(1) The Bergman kernel asymptotics for $H^0_{(2)}(X, L^p \otimes E)$ holds on compact sets of X.

(2) The Berezin-Toeplitz quantization package holds for the Kähler manifold (X, ω), where $\omega = \frac{\sqrt{-1}}{2\pi} R^{(L,h^L)}$, the algebra $\mathscr{C}^\infty_{\mathrm{const}}(X, \operatorname{End}(E))$ and quantum spaces $H^0_{(2)}(X, L^p \otimes E)$.

Proof. Since X is pseudoconvex we have $\mathscr{L}_\varrho(s,s) \geq 0$ pointwise on ∂X. Moreover the torsion of the the metric Θ and $R^{K^*_X}$ are bounded on $\overline{X}$. Hence (3.40) yields immediately the spectral gap (2.19) and we can apply Theorem 2.7. □

4. SZEGŐ-TYPE LIMIT FORMULAS

Boutet de Monvel and Guillemin [10, 25] obtained complex variable analogues of the classical Szegő theorem [64]. The analogous result for projective manifolds endowed with the restriction of the hyperplane bundle was originally proved in [10, Theorem 13.13], [25, Theorem 1, p. 248] and for arbitrary positive line bundles in [5], see also [43]. In [30, Theorem 1.6] the asymptotics are proved for a semi-classical spectral function of the Kodaira Laplacian on an arbitrary manifold.

Lemma 4.1. *Let (X, Θ) be a Hermitian manifold, (L, h^L) and (E, h^E) be holomorphic Hermitian vector bundles on X of rank one and r, respectively. Let $f \in \boldsymbol{L}^\infty(X)$ be non-negative and have compact support. Then the Toeplitz operator $T_{f,p}$ is a compact operator on $H^0_{(2)}(X, L^p \otimes E)$. If the set where f does not vanish has a non-empty interior, then $T_{f,p}$ is injective.*

Proof. $T_{f,p}$ is a positive operator and

$$\operatorname{Tr}\big[T_{f,p}\big] = \int_X f(x) \operatorname{Tr}\big[P_p(x,x)\big]\, dv_X(x) < +\infty. \tag{4.1}$$

By (4.1), $T_{f,p}$ is of trace class and is therefore compact (cf. [60, Theorem VI. 21]). On the other hand, if the set where f does not vanish has a non-empty interior, then for any

$s \in H^0_{(2)}(X, L^p \otimes E)$, we have by definition

$$\langle T_{f,p}s, s\rangle = \int_X f\,|s|^2_{L^p\otimes E}\, dv_X > 0\,, \tag{4.2}$$

by holomorphicity of s and since f is non-negative by assumption. This implies that $T_{f,p}$ is injective, hence concluding the proof of the Lemma. ☐

We denote by $\operatorname{Spec}(T)$ the spectrum of an operator T. Then $\operatorname{Spec}(T_{f,p}) \subset \big[0, \|f\|_\infty\big]$ and $\operatorname{Spec}(T_{f,p}) \cap \big]0, \|f\|_\infty\big]$ consists of at most a countable set of eigenvalues of finite multiplicity that can cluster only at 0. If $H^0_{(2)}(X, L^p \otimes E)$ is infinite dimensional we have $0 \in \operatorname{Spec}(T_{f,p})$. We denote the positive eigenvalues of $T_{f,p}$ counted with multiplicity by

$$\lambda_{p,1} \geq \lambda_{p,2} \geq \ldots \geq \lambda_{p,j} \geq \ldots, \tag{4.3}$$

so $\operatorname{Spec}(T_{f,p}) \cap \big]0, \|f\|_\infty\big] = \{\lambda_{p,j} : j \in J_p\}$. The spectral density measure of $T_{f,p}$ on the interval $[0, \infty[$ is $\mu_{f,p} = \sum_{j\in J_p} \delta_{\lambda_{p,j}}$. We define the spectral counting function of $T_{f,p}$ as follows:

$$N_p(u) = \#\{j; \lambda_{p,j} > u\} = \int_{]u,\|f\|_\infty]} d\mu_{f,p}\,. \tag{4.4}$$

Theorem 4.2. *Under the hypotheses of Theorem 2.7, let f be a continuous nonnegative function with compact support. Then the sequence of normalized spectral measures $p^{-n}\mu_{f,p}$ converges weakly on $]0, \|f\|_\infty]$ to the pushforward of the Liouville measure $\operatorname{rk}(E)c_1(L, h^L)^n/n!$ by f.*

$$p^{-n}\mu_{f,p} \to \operatorname{rk}(E) f_*\left(\frac{1}{n!} c_1(L, h^L)^n\right)\,, \text{ on }]0, \|f\|_\infty] \text{ as } p \to \infty. \tag{4.5}$$

The counting function of the spectrum of $T_{f,p}$ has the following asymptotics: for any $\lambda > 0$ as $p \to +\infty$.

$$N_p(\lambda) = \operatorname{rk}(E)\frac{p^n}{n!} \int\limits_{\{f>\lambda\}} \left(\frac{\sqrt{-1}}{2\pi} R^L\right)^n + o(p^n)\,. \tag{4.6}$$

If X is compact, the asymptotics in (4.5) hold on $[0, \|f\|_\infty]$ for the full spectral measures $\widetilde{\mu}_{f,p} = \sum_{\lambda\in \operatorname{Spec} T_{f,p}} \delta_\lambda$ (multiplicities counted), and in (4.6) also for $\lambda = 0$.

Proof. We follow the proof of [48, Theorem 3.1] (cf. also [53, Theorem 32]). We denote the normalized spectral density measure on $]0, \|f\|_\infty]$ by $v_p = p^{-n}\mu_{f,p}$. Then

$$v_p = -p^{-n}\frac{d}{du}N_p(u), \quad u \in]0, \|f\|_\infty]. \tag{4.7}$$

Clearly, v_p is a sum of Dirac measures supported on $\operatorname{Spec}(T_{f,p}) \cap]0, \|f\|_\infty]$.

We claim that the weak limit of the sequence $\{v_p\}_{p\geqslant 1}$ is the direct image measure $\operatorname{rk}(E)f_*(\frac{\omega^n}{n!})$ with $\omega = \frac{\sqrt{-1}}{2\pi}R^L$, that is, for every continuous function $\varphi \in \mathscr{C}_0\big(]0, \|f\|_\infty]\big)$, we have

$$\lim_{p\to+\infty} \int_0^{\|f\|_\infty} \varphi\, dv_p = \int_X (\varphi \circ f)\, \frac{\omega^n}{n!}. \tag{4.8}$$

By the argument of the proof of [2, Theorem 3.5], which is local, and our assumption that $f \in \mathscr{C}^0(X, [0, \infty[)$, we have for any $m \geqslant 1$,

$$\int_0^{\|f\|_\infty} x^m dv_p = p^{-n}\operatorname{Tr}\big[T^m_{f,p}\big] = p^{-n}\int_X \operatorname{Tr}\big[\underbrace{T_{f,p}\cdots T_{f,p}}_{m \text{ times}}(x, x)\big]\, dv_X(x)$$

$$\begin{aligned}(4.9)\qquad &= p^{-n}\int_X f(x)\operatorname{Tr}\big[P_p\underbrace{T_{f,p}\cdots T_{f,p}}_{m-1\ \text{times}}(x,x)\big]dv_X(x).\\ &= \operatorname{rk}(E)\int_X f(x)^m\frac{\omega^n}{n!}+o(1).\end{aligned}$$

Now we apply the Weierstrass approximation theorem for $\frac{1}{x}\varphi\in\mathscr{C}_0\big(]0,\|f\|_\infty]\big)$, we know that any $\varphi\in\mathscr{C}_0\big(]0,\|f\|_\infty]\big)$ can be approximated uniformly by polynomials without a constant term. Now from (4.9), we get (4.8). By approximating the characteristic function $1_{]\lambda,\|f\|_\infty]}$ for $\lambda>0$ by continuous functions f_k, we obtain

$$\lim_{p\to+\infty}\int_0^{\|f\|_\infty} f_k dv_p=\int_X(f_k\circ f)\,\frac{\omega^n}{n!}. \tag{4.10}$$

Letting $k\to+\infty$ yields

$$\lim_{p\to+\infty}\int_0^{\|f\|_\infty} 1_{]\lambda,\|f\|_\infty]}dv_p=\int_X(1_{]\lambda,\|f\|_\infty]}\circ f)\,\frac{\omega^n}{n!}=\int_{\{f>\lambda\}}\frac{\omega^n}{n!}. \tag{4.11}$$

By (4.7), we find

$$\int_0^{\|f\|_\infty} 1_{]\lambda,\|f\|_\infty]}dv_p=p^{-n}N_p(\lambda). \tag{4.12}$$

Then (4.6) follows from (4.11) and (4.12). The proof of Theorem 4.2 is completed. ☐

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