Partial Bergman kernels and determinantal point processes on Kähler manifolds

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Abstract

We compute the full off-diagonal asymptotics of the equivariant and partial Bergman kernels associated with a circle action on a prequantized Kähler manifold with bounded geometry at infinity, then use these results to compute the asymptotics of the linear statistics of the associated determinantal point process as the number of points grows to infinity, showing that its distribution converges to a centered normal variable with variance given by the sum of an H^1-norm squared in the bulk and an $H^{\frac{1}{2}}$-norm squared on the boundary of the associated droplet.

1 Introduction

Given a measured space (X, dv_X) and a finite orthonormal family $\{\psi_j \in L^2(X, \mathbb{C})\}_{j=1}^N$ with $N \in \mathbb{N}$, the associated *determinantal point process* is the measure $d\nu_N$ on the N-fold product X^N defined for all $(x_1, \cdots, x_N) \in X^N$ by

$$d\nu_N(x_1, \ldots, x_N) := \frac{1}{N!} \left|\det(\psi_j(x_k))_{j,k=1}^N\right|^2 dv_X(x_1) \cdots dv_X(x_N)\,. \tag{1.1}$$

As explained for instance in [12, Lem. 4.5.1], this measure defines in fact a probability measure over X. Determinantal point processes were introduced by Macchi in [23] as general models of a probability distribution on configurations of N points over X exhibiting a repulsive behavior, since the determinant in (1.1) vanishes as soon as any two points of the configuration coincide, while also experiencing a confining potential, since square-integrable functions typically tend to 0 at infinity when $\mathrm{Vol}(X, dv_X) = +\infty$. In order to describe the distribution of typical configurations with respect to $d\nu_N$, it is natural to consider the associated *linear statistics* with respect to a test function $f \in L^\infty(X, \mathbb{R})$, which are defined as the random variable $\mathcal{N}[f] : X^N \to \mathbb{R}$ given for any $(x_1, \ldots, x_N) \in X^N$ by

$$\mathcal{N}[f](x_1, \ldots, x_N) := \sum_{j=1}^N f(x_j)\,, \tag{1.2}$$

and study its behavior in the *thermodynamic limit* as $N \to +\infty$. Specifically, a sequence of measures of the form (1.1) for each $N \in \mathbb{N}$ is said to admit an *equilibrium measure* ν

over X if the linear statistics (1.2) satisfy the following convergence in probability,

$$\frac{1}{N}\mathcal{N}[f] \xrightarrow{N\to\infty} \frac{1}{\mathrm{Vol}(X,d\nu)}\int_X f\,d\nu\,. \tag{1.3}$$

This property is called a *law of large numbers*. The support $D := \operatorname{Supp}\mu \subset X$ of the equilibrium measure is called the *droplet*, and the law of large numbers (1.3) shows in particular that typical configurations tend to accumulate in the droplet as $N \to \infty$. Their fluctuations, on the other hand, are usually described by a *central limit theorem* for the linear statistics (1.2), whose variance should then be described in terms of the variations of f over $D \subset X$.

In the context of this paper, the fundamental example of such a determinantal point process indexed by $N \in \mathbb{N}$ is the so-called *Ginibre ensemble*, which corresponds to $X = \mathbb{C}$ equipped with the Lebesgue measure $dv_{\mathbb{C}}$, together with the orthonormal family $\{\psi_j^{(N)} \in L^2(\mathbb{C},\mathbb{C})\}_{j=1}^N$ defined for all $1 \leqslant j \leqslant N$ and all $z \in \mathbb{C}$ by

$$\psi_j^{(N)}(z) := \sqrt{\frac{N^{j+1}}{\pi(j-1)!}}\; z^{j-1}\, e^{-\frac{N}{2}|z|^2}\,. \tag{1.4}$$

This determinantal point process has been introduced by Ginibre in [11, § 1], who showed that it describes the distribution of eigenvalues of a random matrix of size $N \times N$ with entries following independent complex centered Gaussians with variance $\frac{1}{N}$. As explained for instance in [27, § 1.2.2], it can also be interpreted as the Boltzman-Gibbs distribution for a *Coulomb gas* in the plane confined by a quadratic potential at inverse temperature $\beta = 2$. The main result of Ginibre in [11, § 1] then states that these determinantal point processes admit $d\nu := \mathbb{1}_{\mathbb{D}}\, dv_{\mathbb{C}}$ as an equilibrium measure as $N \to \infty$, where $\mathbb{D} \subset \mathbb{C}$ denotes the unit disk. This is the celebrated *circular law* for the Ginibre ensemble. Concerning the fluctuations, Rider and Virag established in [25] a central limit theorem for the Ginibre ensemble, showing that for any compactly supported $f \in \mathscr{C}_c^1(\mathbb{C},\mathbb{R})$, the centered random variable $\mathcal{N}[f] - \mathbb{E}[\mathcal{N}[f]]$ converges in distribution to a centered normal random variable with variance

$$\lim_{N\to\infty} \mathbb{V}[\mathcal{N}[f]] = \frac{1}{4\pi}\int_{\mathbb{D}} |df|^2\, dv_{\mathbb{C}} + \sum_{k\in\mathbb{Z}} |k|\,|\hat{f}_k|^2\,, \tag{1.5}$$

where for any $k \in \mathbb{Z}$, we write $\hat{f}_k \in \mathbb{C}$ for the k-th Fourier coefficient of $f \in \mathscr{C}_c^1(\mathbb{C},\mathbb{R})$ restricted to the unit circle $\partial\mathbb{D} \subset \mathbb{C}$. While the first term is an homogeneous H^1-norm squared of f restricted to the droplet $\mathbb{D} \subset \mathbb{C}$, the second term can be viewed as an homogoneous $H^{\frac{1}{2}}$-norm squared of f restricted to its boundary $\partial\mathbb{D} \subset \mathbb{C}$. This result has been extended to more general potentials over $\mathbb{C}$ with suitable growth at infinity by Ameur, Hedenmalm and Makarov in [1] and Leblé and Serfaty in [15]. As explained by Deleporte and Lambert in [10, Th. 1.2], the appearance of an $H^{\frac{1}{2}}$-norm over the boundary of the droplet in (1.5) can be understood as a manifestation of the universality of the *strong Szegö limit theorem* established by Szegö in [30].

In this paper, we extend these results to a much larger class of measured spaces, where X is a *Kähler manifold* equipped with its Riemannian volume form dv_X, in a general set-up first introduced by Berman in [4]. In this context, we consider a symplectic manifold (X, ω) without boundary together with a Hermitian line bundle (L, h^L) over X endowed with a Hermitian connection ∇^L satisfying the following *prequantization formula*,

$$\omega = \frac{\sqrt{-1}}{2\pi} R^L \,, \tag{1.6}$$

where $R^L \in \Omega^2(X, \mathbb{C})$ is the curvature of ∇^L. We let also X be equipped with an integrable complex structure $J \in \mathscr{C}^\infty(X, \mathrm{End}(TX))$ compatible with ω, making (X, ω, J) into a *Kähler manifold prequantized by* (L, h^L, ∇^L). We can then consider the associated *Kähler metric* g^{TX}, which is the Riemannian metric defined over X by the formula

$$g^{TX} := \omega(\cdot, J\cdot) \,, \tag{1.7}$$

and let dv_X be the associated Riemannian volume form. These data naturally endow (L, h^L) with a holomorphic structure for which ∇^L is the associated *Chern connection*. For any $p \in \mathbb{N}$, write $L^p := L^{\otimes p}$ for the p^{th}-tensor power of L equipped with the induced Hermitian metric h^{L^p}, and consider the space $H^0_{(2)}(X, L^p)$ of square-integrable holomorphic sections of L^p for the L^2-Hermitian product induced by h^{L^p} and dv_X. Given a finite orthonormal family $\{s_j \in H^0_{(2)}(X, L^p)\}_{j=1}^{N_p}$ with $N_p \in \mathbb{N}$, we consider the measure $d\nu_{N_p}$ over the N_p-fold product X^{N_p} defined for all $(x_1, \cdots, x_{N_p}) \in X^{N_p}$ by

$$d\nu_{N_p}(x_1, \ldots, x_{N_p}) := \frac{1}{N_p!} \left| \det(s_j(x_k))_{j,k=1}^{N_p} \right|_p^2 dv_X(x_1) \cdots dv_X(x_{N_p}) \,, \tag{1.8}$$

where $|\cdot|_p$ denotes the Hermitian norm induced by h^{L^p} on $\bigotimes_{1 \leqslant j \leqslant N_p} L^p_{x_j}$.

In case X is compact, the space of holomorphic sections $H^0_{(2)}(X, L^p) =: H^0(X, L^p)$ is finite-dimensional for all $p \in \mathbb{N}$, and the determinantal point processes (1.8) associated with orthonormal bases of $H^0(X, L^p)$ have first been studied by Berman in [4]. When X is not necessarily compact, a natural construction of finite orthonormal families in $H^0_{(2)}(X, L^p)$ consists in considering a compatible action of the circle S^1 on (L, h^L, ∇^L) over (X, ω, J), inducing a unitary representation of S^1 on $H^0_{(2)}(X, L^p)$ for each $p \in \mathbb{N}$. One can then consider the *weight decomposition*

$$H^0_{(2)}(X, L^p) = \widehat{\bigoplus_{m \in \mathbb{Z}}} H^0_{(2)}(X, L^p)_m \,, \tag{1.9}$$

given by the L^2-orthogonal Hilbert direct sum of the *weight spaces* $H^0_{(2)}(X, L^p)_m$ defined for all $m \in \mathbb{Z}$ by

$$H^0_{(2)}(X, L^p)_m := \{ s \in H^0_{(2)}(X, L^p) \mid \varphi_t^* s = e^{2\pi\sqrt{-1}tm} s \,, \text{ for all } t \in \mathbb{R} \} \,, \tag{1.10}$$

where φ_t^* denotes the pullback by the action of $S^1 \simeq \mathbb{R}/\mathbb{Z}$ on L over X for all $t \in \mathbb{R}$. As we explain in Section 2.3, a compatible action of S^1 on (L, h^L, ∇^L) over (X, ω, J) actually determines a *moment map* $\mu \in \mathscr{C}^\infty(X, \mathbb{R})$ for the Hamiltonian action of S^1 on the symplectic manifold (X, ω), called *Kostant moment map*. Under the assumption described in Definition 4.1 that $\mu \in \mathscr{C}^\infty(X,\mathbb{R})$ has *polynomial growth*, so that in particular it is proper and bounded from below, we show in Proposition 4.2 that for any $p \in \mathbb{N}$ large enough, the subspace

$$\mathscr{H}_p := \widehat{\bigoplus_{m \leqslant 0}} H^0_{(2)}(X,L^p)_m \subset H^0_{(2)}(X,L^p)\,, \tag{1.11}$$

given by the Hilbert direct sum of weight spaces (1.10) with negative weights, has finite dimension $N_p \in \mathbb{N}$, so that one can consider the determinantal point process (1.8) associated with an orthonormal basis $\{s_j \in \mathscr{H}_p\}_{j=1}^{N_p}$. As we explain in Example 4.5, in the special case of $X = \mathbb{C}$ endowed with the action of $S^1 \subset \mathbb{C}$ by multiplication, we recover in this way the Ginibre ensemble (1.4).

Assume also that S^1 acts freely on the compact submanifold $\mu^{-1}(0) \subset X$, and consider the associated *symplectic reduction* $X_0 := \mu^{-1}(0)/S^1$, with induced Riemannian metric g^{TX_0} and Riemanian volume form dv_{X_0}. For any $f \in \mathscr{C}^\infty(X,\mathbb{C})$ and any $k \in \mathbb{N}$, consider the function $|\hat{f}_k|^2 : X_0 \to \mathbb{R}$ induced by its *k-th Fourier coefficient*, defined for any $x \in \mu^{-1}(0)$ by

$$\hat{f}_k(x) := \int_0^1 e^{2\pi\sqrt{-1}tk}\, f(\varphi_t(x))\, dt\,. \tag{1.12}$$

Under the additional assumption described in Section 2.1 that the Kähler manifold (X,ω,J) prequantized by (L,h^l,∇^L) has *bounded geometry at infinity*, the main result of this paper is the following.

Theorem 1.1. *Let (X,ω,J) be a Kähler manifold prequantized by (L,h^L,∇^L) with bounded geometry at infinity endowed with a compatible S^1-action such that its Kostant moment map $\mu \in \mathscr{C}^\infty(X,\mathbb{R})$ has polynomial growth, and assume that S^1 acts freely on $\mu^{-1}(0) \subset X$. Then for any $f \in L^\infty(X,\mathbb{R})$, the linear statistics $\mathcal{N}_p[f] : X^{N_p} \to \mathbb{R}$ defined as in (1.2) satisfy the following convergence in probability as $p \to \infty$,*

$$\frac{1}{N_p}\mathcal{N}_p[f] \xrightarrow{p\to\infty} \frac{1}{\mathrm{Vol}\,(\{\mu<0\})}\int_{\{\mu<0\}} f\, dv_X\,. \tag{1.13}$$

Furthermore, for any smooth $f \in \mathscr{C}^\infty_c(X,\mathbb{R})$ with compact support, the variance of the associated linear statistics satisfies

$$\lim_{p\to\infty} p^{-n+1}\mathbb{V}\left[\mathcal{N}_p[f]\right] = \frac{1}{4\pi}\int_{\{\mu<0\}} |df|^2\, dv_X(x) + \frac{1}{2}\int_{X_0}\sum_{k\in\mathbb{Z}} |k|\, |\hat{f}_k(x)|^2\, dv_{X_0}(x)\,, \tag{1.14}$$

where $n := \frac{\dim X}{2}$, and the random variable $N_p^\alpha(\mathcal{N}_p[f] - \mathbb{E}[\mathcal{N}_p[f]])$ with $\alpha = \frac{1}{2n} - \frac{1}{2}$ converges in distribution to a centered normal random variable with variance (1.14) as $p \to \infty$.

Theorem 1.1 thus states that as $p \to \infty$, the determinantal point processes (1.8) associated with the spaces (1.11) admit the equilibrium measure $d\nu := \mathbb{1}_D\, dv_X$ over X with droplet $D := \{\mu < 0\}$, and satisfy a central limit theorem with fluctuations given by the sum of an homogeneous H^1-norm squared over $D \subset X$ and an homogeneous $H^{\frac{1}{2}}$-norm squared over its boundary $\mu^{-1}(0) \subset X$ in the directions of the circle action, as in the strong Szegö limit theorem [30]. The proof of Theorem 1.1 is described in Section 4. In particular, the law of large numbers (1.13) is a consequence of Theorem 4.6 while the asymptotics (1.14) of the variance is established in Theorem 4.7, and those asymptotics imply the central limit theorem by a general argument from the theory of determinantal point processes due to Soshnikov in [29, Th. 1].

In the special case of $X = \mathbb{C}$ endowed with the action of $S^1 \subset \mathbb{C}$ by multiplication, the law of large numbers (1.13) in Theorem 1.1 recovers the classical circular law for the Ginibre ensemble established by Ginibre in [11, § 1], while the asymptotics (1.14) for the variance in Theorem 1.1 recover the asymptotics (1.5) established by Rider and Virag in [25] together with the corresponding central limit theorem. On the other hand, in the case of the trivial S^1-action on a compact prequantized Kähler manifold, one can arrange for the Kostant moment map $\mu \in \mathscr{C}^\infty(X, \mathbb{R})$ to be constant and strictly positive, so that $\mu^{-1}(0) = \emptyset$ and the hypotheses of Theorem 1.1 are trivially satisfied. Then while the spaces $\mathscr{H}_p = H^0(X, L^p)$ coincide with the space of holomorphic sections for all $p \in \mathbb{N}$, and Theorem 1.1 recovers results of Berman in [4, Th. 1.4, 1.5]. Note that the $H^{\frac{1}{2}}$-term in formula (1.14) for the variance vanishes in that case.

The hypotheses of Theorem 1.1 are satisfied more generally in the case of a compact prequantized with a compatible S^1-action acting freely on $\mu^{-1}(0) \subset X$, extending the results of Berman in [4] to include the case of smooth functions whose support are not necessarily included in the droplet $D \subset X$, displaying the extra $H^{\frac{1}{2}}$-term in (1.5) in the case $\mu^{-1}(0) \neq \emptyset$. Up to a standard shift of weights in (1.11), the assumptions of Theorem 1.1 are also satisfied in the important case when X is a *toric manifold*, with circle action induced by the choice of $S^1 \subset \mathbb{T}^n$ inside the associated torus, which includes in particular the case $X = \mathbb{C}^n$ for general $n \in \mathbb{N}$. The law of large numbers (1.13) then recovers a result of Berman in [3, Th. 3.4], while Theorem 1.1 establishes a central limit theorem extending the result of Rider and Virag in [25] in all these cases. This also includes the case of the real and complex hyperbolic spaces, the relevance of the associated determinantal point processes being studied for instance by Bufetov, Fan and Qiu in [5] in case $X = \mathbb{H}^n$ is the real hyperbolic space, and by Bufetov and Qiu in [6] in case $X = \mathbb{B}^n$ is the complex hyperbolic space.

The proof of Theorem 1.1 is based on the full asymptotic expansion of the *partial Bergman kernel* as $p \to \infty$ that we establish in Section 3.3, recovering in particular the results of Ross and Singer in [26, Th. 1.2], Zelditch and Zhou in [32, (8), Th. 4] and Shabtai in [28, Th. 1.7], who establish the asymptotic expansion of the partial Bergman kernel in neighborhoods of size of order $\frac{1}{\sqrt{p}}$ around the boundary $\mu^{-1}(0) \subset X$ and outside neighborhoods of size of order 1 as $p \to \infty$. The full off-diagonal expansion over arbitrary neighborhoods of $\mu^{-1}(0)$ established in Theorems 3.5 and 3.6 plays a crucial role in the proof of Theorem 1.1. These results are in turn based on the full off-diagonal asymptotic expansion of the *equivariant Bergman kernel* associated with the weight space (1.10) for each $m \in \mathbb{Z}$, extending the analogous results in the case $m = 0$ of Ma and Zhang in [21, Th. 0.2]. The tools used in this paper are based on the full off-diagonal asymptotic expansion of the Bergman kernel established by Dai, Liu and Ma in [9, Th. 4.18'] and its extension to non-compact manifolds established by Ma and

Marinescu in [18, § 3.5], which we recall in Section 2.2. A comprehensive introduction of this theory can be found in their book [17]. Near-diagonal asymptotic expansions for a larger class of equivariant Bergman kernels in the case $m = 0$ have also been established by Paoletti in [24, Th. 1.2], and of a larger class of partial Bergman kernels by Coman and Marinescu in [8] and Zelditch and Zhou in [31].

In the setting of the trivial S^1-action on a compact prequantized Kähler manifold originally considered by Berman in [4], Charles and Estienne computed in [7, Cor. 1.7] the asymptotics of the linear statistics with respect to the characteristic function $f = \mathbb{1}_U$ of an open subset $U \subset X$ with smooth boundary. In particular, they establish a central limit theorem in this case. On the other hand, Berman also considers in [4, Th. 1.4, 1.5] the case of a compact prequantized Kähler manifold with singular Hermitian metric h^L, in which case the droplet $D \subset X$ does not necessarily coincide with X, and computes the asymptotics of the linear statistics with respect to $f \in L^\infty(X, \mathbb{R})$ sufficiently regular and with support strictly included in D, so that the second term in formula (1.14) still vanishes. We hope that the methods of this paper can be used to extend both of these results to the case of a non-trivial S^1-action on a not necessarily compact prequantized Kähler manifold.

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2 Bergman kernels and circle actions

After describing the general setting of the paper in Section 2.1, we recall in Section 2.2 the results of Ma and Marinescu in [17, Chap. 6] on the full asymptotic expansion of the Bergman kernel of prequantized Kähler manifolds with bounded geometry which will constitute the fundamental tool of this paper, then introduce in Section 2.3 the Kostant moment map of a compatible S^1-action on a prequantized Kähler manifold.

2.1 Setting

Let (X, ω) be a symplectic manifold of dimension $2n \in \mathbb{N}$ without boundary, together with a Hermitian line bundle (L, h^L) over X endowed with a Hermitian connection ∇^L satisfying the *prequantization formula* (1.6). We also assume that X is equipped with an integrable complex structure $J \in \mathscr{C}^\infty(X, \mathrm{End}(TX))$ compatible with ω, making (X, ω, J) into a *Kähler manifold* and (L, h^L) into a holomorphic Hermitian line bundle of which ∇^L is the Chern connection. We write g^{TX} for the associated *Kähler metric* (1.7) and dv_X for the induced Riemannian volume form. In this paper, we will always make the assumption that these data have *bounded geometry at infinity*, meaning that (X, g^{TX}) is complete with positive injectivity radius and that the derivatives at any order of R^L, J and g^{TX} are uniformly bounded in the norms induced by h^L and g^{TX}.

Note that this assumption is automatically satisfied in the important case when X is compact.

For any $p \in \mathbb{N}$, write h^{L^p} and ∇^{L^p} for the Hermitian metric and connection on the p^{th}-tensor power $L^p := L^{\otimes p}$ respectively induced by h^L and ∇^L on L. We denote by $\mathscr{C}^\infty_c(X, L^p)$ the space of compactly supported smooth sections of L^p, endowed with the L^2-*Hermitian product* $\langle \cdot, \cdot \rangle_p$ given for any $s_1, s_2 \in \mathscr{C}^\infty_c(X, L^p)$ by the formula

$$\langle s_1, s_2 \rangle_p := \int_X h^{L^p}(s_1(x), s_2(x))\, dv_X(x)\,. \tag{2.1}$$

Let $L^2(X, L^p)$ denote the completion of $\mathscr{C}^\infty_c(X, L^p)$ with respect to the associated L^2-norm, and write $H^0_{(2)}(X, L^p) \subset L^2(X, L^p)$ for the space of L^2-holomorphic sections of L^p. The following result is a consequence of standard elliptic theory, and introduces the basic fundamental tool of this paper.

Proposition 2.1. *[17, Rmk. 1.4.3] For any $p \in \mathbb{N}$, the orthogonal projection onto $H^0_{(2)}(X, L^p) \subset L^2(X, L^p)$ with respect to the L^2-product (2.1), denoted by*

$$P_p : L^2(X, L^p) \longrightarrow H^0_{(2)}(X, L^p) \tag{2.2}$$

admits a smooth Schwartz kernel $P_p(\cdot, \cdot) \in \mathscr{C}^\infty(X \times X, L^p \boxtimes (L^p)^*)$ with respect to dv_X, called the Bergman kernel*, characterized for any $s \in \mathscr{C}^\infty(X, L^p)$ and $x \in X$ by the formula*

$$(P_p s)(x) = \int_X P_p(x, y).s(y)\, dv_X(y). \tag{2.3}$$

Finally, for any Riemannian manifold (Y, g^{TY}), we will write $d^Y(\cdot, \cdot)$ for the associated distance over Y, and for any $x \in Y$ and $\varepsilon > 0$, we will write $B^Y(x, \varepsilon) \subset Y$ for the geodesic ball of center $x \in Y$ and radius $\varepsilon > 0$.

2.2 Asymptotic expansion of the Bergman kernel

Let us consider the setting described in Section 2.1, and for any $r, p \in \mathbb{N}$, let $|\cdot|_{\mathscr{C}^r}$ denote the local $\mathscr{C}^r$-norm on $L^p \boxtimes (L^p)^*$ induced by h^{L^p} and ∇^{L^p}. The following result describes the off-diagonal decay of the Bergman kernel introduced in Proposition 2.1.

Theorem 2.2. *[20, Th. 1] There exists $c > 0$ such that for any $r \in \mathbb{N}$, there is $C_r > 0$ such that for all $p \in \mathbb{N}$ and $x, y \in X$, the following estimate holds,*

$$|P_p(x, y)|_{\mathscr{C}^r} \leqslant C_r p^{n+\frac{r}{2}}\, e^{-c\sqrt{p}\, d^X(x,y)}\,. \tag{2.4}$$

For any $x \in X$, let $|\cdot|$ denote the Euclidean norm on T_xX and on T^*_xX induced by g^{T_xX}, and for any subspace $E_x \subset T_xX$, let $B^{E_x}(0, \varepsilon_0) \subset E_x$ denote the open ball in E_x of radius $\varepsilon_0 > 0$ with respect to the induced norm. We write $B^{TX}(0, \varepsilon_0) \subset TX$ for the ball bundle over X whose fibre over any $x \in X$ is given by $B^{T_xX}(0, \varepsilon_0) \subset T_xX$. Recall that since (X, g^{TX}) has bounded geometry at infinity, its injectivity radius is bounded from below.

To describe asymptotic estimates for the Bergman kernel in a neighborhood of the diagonal, we will need the following definition.

Definition 2.3. Given $\varepsilon_0 > 0$ smaller than the injectivity radius of (X, g^{TX}), we say that a smooth map $\psi : B^{TX}(0, \varepsilon_0) \to X$ is a *bounded family of charts* if for any $x \in X$, its restriction $\psi_x : B^{T_xX}(0, \varepsilon_0) \to X$ to $B^{T_xX}(0, \varepsilon_0) \subset B^{TX}(0, \varepsilon_0)$ is a diffeomorphism on its image $U_x \subset X$ satisfying $\psi_x(0) = x$ and $d\psi_{x,0} = \mathrm{Id}_{T_xX}$, and if for any $r \in \mathbb{N}$, there exists $C_r > 0$ such that for all $x \in X$, we have

$$\left|\exp_x^X \circ \psi_x^{-1}\right|_{\mathscr{C}^r} \leqslant C_r\,, \tag{2.5}$$

where $\exp^X : B^{TX}(0, \varepsilon_0) \to X$ is the *exponential map* of (X, g^{TX}).

Note that the the exponential map $\exp^X : B^{TX}(0, \varepsilon_0) \to X$ is itself a bounded family of charts by definition. We fix one of them for the rest of the section.

For any $x \in X$, identify L over the image $U_x \subset X$ of $\psi_x : B^{T_xX}(0, \varepsilon_0) \to X$ with L_x through parallel transport with respect to ∇^L along radial lines of $B^{T_xX}(0, \varepsilon_0)$ and pick a unit vector $e_x \in L_x$ to identify L_x with $\mathbb{C}$. Under the natural isomorphism $\mathrm{End}(L^p) \simeq \mathbb{C}$, the formulas below will not depend on this identification. For any $p \in \mathbb{N}$ and any kernel $K_p(\cdot, \cdot) \in \mathscr{C}^\infty(X \times X, L^p \boxtimes (L^p)^*)$, we write

$$K_{p,x}(Z, Z') \in \mathbb{C} \tag{2.6}$$

for its image in this trivialization evaluated at $Z, Z' \in B^{T_xX}(0, \varepsilon_0)$, and for any smooth $f \in \mathscr{C}^\infty(X, \mathbb{C})$, we write $f_x := \psi_x^* f$ for its pullback in these coordinates. We then get the following immediate consequence of Theorem 2.2.

Corollary 2.4. *For any r, $k \in \mathbb{N}$ and $\varepsilon > 0$, there exists $C_k > 0$ such that for all $p \in \mathbb{N}$, $x \in X$, and $Z, Z' \in B^{T_xX}(0, \varepsilon_0)$ satisfying $|Z - Z'| > \varepsilon p^{\frac{-1+\varepsilon}{2}}$, the following estimate holds,*

$$|P_{p,x}(Z, Z')|_{\mathscr{C}^r} \leqslant C_k\, p^{-k}. \tag{2.7}$$

We use the following explicit local model for the Bergman kernel from [19, (3.25)], for any $x \in M$ and $Z, Z' \in T_xX$,

$$\mathscr{P}_x(Z, Z') := \exp\left(-\frac{\pi}{2}|Z - Z'|^2 - \pi\sqrt{-1}\omega_x(Z, Z')\right)\,. \tag{2.8}$$

Let $|\cdot|_{\mathscr{C}^r(X)}$ denote the $\mathscr{C}^r$-norm over the fibred product $B^{TX}(0, \varepsilon_0) \times_X B^{TX}(0, \varepsilon_0)$ in the direction of X via the Levi-Civita connection. We can now state the following fundamental result on the near diagonal expansion of the Bergman kernel, which was first established by Ma and Marinescu in [18, § 3.5] following Dai, Liu and Ma in [9, Th. 4.18']. We present this result in a simplified form following the approach of Lu, Ma and Marinescu in [16, Th. 2.1], and which can be directly deduced from [17, Problem 6.1].

Theorem 2.5. *There is a family $\{J_{r,x}(Z, Z') \in \mathbb{C}[Z, Z']\}_{r \in \mathbb{N}}$ of polynomials in Z, $Z' \in T_xX$ depending smoothly on $x \in X$, of the same parity as $r \in \mathbb{N}$ and of total degree less than $3r \in \mathbb{N}$, such that for any k, j, $j' \in \mathbb{N}$ and $\delta > 0$, there is $\varepsilon_0 > 0$ and $C > 0$ such* *that for all* $\varepsilon\in]0,\varepsilon_0[$, $x\in X$, $p\in\mathbb{N}$ *and all* $Z,Z'\in T_xX$ *satisfying* $|Z|,|Z'|<\varepsilon p^{\frac{-1+\varepsilon}{2}}$, *we have*

$$\sup_{|\alpha|+|\alpha'|\leqslant j}\Big|\frac{\partial^\alpha}{\partial Z^\alpha}\frac{\partial^{\alpha'}}{\partial Z'^{\alpha'}}\Big(p^{-n}P_{p,x}(Z,Z') - \sum_{r=0}^{k-1}J_{r,x}(\sqrt{p}Z,\sqrt{p}Z')\mathscr{P}_x(\sqrt{p}Z,\sqrt{p}Z')\,p^{-\frac{r}{2}}\Big)\Big|_{\mathscr{C}^{j'}(X)}\leqslant Cp^{-\frac{k-j}{2}+\delta}. \tag{2.9}$$

Furthermore, for all $x\in X$, *we have* $J_{0,x}\equiv 1$.

The following result is a direct consequence of Theorems 2.2 and 2.5.

Corollary 2.6. *For any* $r\in\mathbb{N}$, *there exists* $C_r>0$ *such that for any* $p\in\mathbb{N}$ *and any* $x,\,y\in X$, *we have*

$$\Big|p^{-n}\,P_p(x,y)\Big|_{\mathscr{C}^r}\leqslant C_r\quad\text{and}\quad\Big|p^{-n}\,P_p(x,x)-1\Big|_{\mathscr{C}^r}\leqslant C_r\,p^{-1}\,. \tag{2.10}$$

2.3 Circle actions on prequantized Kähler manifolds

Let (X,ω) be a symplectic manifold together with a Hermitian line bundle (L,h^L) over X with Hermitian connection ∇^L satisfying the prequantization formula (1.6), and assume that L is endowed with a compatible action of the circle S^1 lifting an action on X, so that it preserves h^L and ∇^L. Write $\varphi_t:X\to X$ for the flow of diffeomorphisms of X induced by the action of $S^1\simeq\mathbb{R}/\mathbb{Z}$ for all $t\in\mathbb{R}$, and for any $p\in\mathbb{N}$, write $\varphi_{t,p}:L^p\to L^p$ for its lift to L^p. We consider the induced action by pullback on a smooth sections $s\in\mathscr{C}^\infty(X,L^p)$ defined for all $x\in X$ by

$$(\varphi_t^*s)(x):=\varphi_{t,p}^{-1}\,s(\varphi_t(x))\,. \tag{2.11}$$

Let $\xi\in\mathscr{C}^\infty(X,TX)$ denote the associated fundamental vector field over X, defined for all $x\in X$ by

$$\xi_x:=\frac{d}{dt}\Big|_{t=0}\varphi_t(x)\,. \tag{2.12}$$

The following definition is due to Kostant in [14, Th. 4.5.1].

Definition 2.7. The *moment map* $\mu\in\mathscr{C}^\infty(X,\mathbb{R})$ of the action of S^1 on (L,h^L,∇^L) over X is defined by the following formula, for all $s\in\mathscr{C}^\infty(X,L)$,

$$\mu\,s=\frac{\sqrt{-1}}{2\pi}\left(\nabla^L_\xi s-\frac{d}{dt}\Big|_{t=0}\varphi_t^*s\right)\,. \tag{2.13}$$

Formula (2.13) is called the *Kostant formula*. The right-hand side of formula (2.13) is $\mathscr{C}^\infty(X,\mathbb{R})$-linear, so that the left-hand side defines in fact a function $\mu\in\mathscr{C}^\infty(X,\mathbb{R})$, invariant by the action of S^1. Furthermore, Definition 2.7 together with the prequantization formula (1.6) implies that for all $\eta\in\mathscr{C}^\infty(X,TX)$, we have

$$\omega(\eta,\xi)=d\mu.\eta\,, \tag{2.14}$$

where $\xi \in \mathscr{C}^\infty(X, TX)$ is the fundamental vector field (2.12), making the action of S^1 into a *Hamiltonian action* on the symplectic manifold (X, ω), with Hamiltonian $\mu \in \mathscr{C}^\infty(X, \mathbb{R})$.

Now for any $p \in \mathbb{N}$, the Kostant formula (2.13) applied to any $s \in \mathscr{C}^\infty(X, L^p)$ gives

$$\frac{d}{dt}\Big|_{t=0} \varphi_t^* s = \left(\nabla_\xi^{L^p} + 2\pi\sqrt{-1}p\mu\right)s\,. \tag{2.15}$$

For all $t \in \mathbb{R}$ and $x \in X$, write $\tau_{t,p} : L^p_{\varphi_t(x)} \to L^p_x$ for the parallel transport with respect to ∇^{L^p} along the path $u \mapsto \varphi_u(x)$ for $u \in [0, t]$ from $u = t$ to $u = 0$, defined for any $s \in \mathscr{C}^\infty(X, L^p)$ by

$$\left(\nabla_\xi^{L^p} s\right)(x) = \frac{d}{dt}\Big|_{t=0} \tau_{t,p}\, s(\varphi_t(x))\,. \tag{2.16}$$

Then formulas (2.11) and (2.15) imply that for any $p \in \mathbb{N}$ and $t \in \mathbb{R}$, we have

$$\varphi_{t,p}^{-1} = e^{2\pi\sqrt{-1}tp\mu}\,\tau_{t,p}\,. \tag{2.17}$$

In the sequel, we will always assume that the action of S^1 on X preserves an integrable complex structure $J \in \mathscr{C}^\infty(X, \mathrm{End}(TX))$ compatible with ω as in Section 2.1, so that we have an induced action of S^1 on the space $H^0_{(2)}(X, L^p)$ of L^2-holomorphic sections preserving the L^2-product (2.1), and for each $m \in \mathbb{Z}$, we define the weight space $H^0_{(2)}(X, L^p)_m \subset H^0_{(2)}(X, L^p)$ of weight $m \in \mathbb{Z}$ as in (1.10) and the Hilbert direct sum $\mathscr{H}_p \subset H^0_{(2)}(X, L^p)$ of the negative weight spaces as in (1.11).

3 Partial and equivariant Bergman kernels

In this section, we consider a Kähler manifold (X, ω, J) prequantized by (L, h^L, ∇^L) in the sense of (1.6) with bounded geometry at infinity in the sense of Section 2.1 and endowed with a compatible S^1-action as in Section 2.3.

In Section 3.1, we introduce the equivariant Bergman kernels and establish their off-diagonal properties, while in Section 3.2, we establish their full off-diagonal expansion as $p \to \infty$. In Section 3.3, we introduce the partial Bergman kernel and use the results of Sections 3.1 and 3.2 to establish its full off-diagonal expansion as $p \to \infty$.

3.1 Equivariant Bergman kernels

Recall the Bergman kernel defined in Proposition 2.1 for any $p \in \mathbb{N}$ as the smooth Schwartz kernel of the orthogonal projection $P_p : L^2(X, L^p) \to H^0_{(2)}(X, L^p)$. We start this section by the following Proposition, which serves as a definition of the equivariant Bergman kernels.

Proposition 3.1. *For any $m \in \mathbb{Z}$, $p \in \mathbb{N}$, the section $P_p^{(m)}(\cdot, \cdot) \in \mathscr{C}^\infty(X \times X, L^p \boxtimes (L^p)^*)$ defined for all x, $y \in X$ by the formula*

$$P_p^{(m)}(x, y) := \int_0^1 e^{-2\pi\sqrt{-1}tm} \varphi_{t,p}^{-1} P_p(\varphi_t(x), y)\, dt\,, \tag{3.1}$$

coincides with the smooth Schwartz kernel with respect to dv_X of the orthogonal projection $P_p^{(m)} : L^2(X, L^p) \to H^0_{(2)}(X, L^p)_m$ with respect to the L^2-product (2.1) onto the weight space (1.10) of weight $m \in \mathbb{Z}$.

Furthermore, for all $t \in \mathbb{R}$, $p \in \mathbb{N}$ and x, $y \in X$, it satisfies

$$\varphi_{t,p}^{-1} P_p^{(m)}(\varphi_t(x), y) = P_p^{(m)}(x, \varphi_t^{-1}(y))\varphi_{t,p}^{-1} = e^{2\pi\sqrt{-1}tm} P_p^{(m)}(x, y). \tag{3.2}$$

Proof. For any $p \in \mathbb{N}$, $s \in H^0_{(2)}(X, L^p)$, $m \in \mathbb{Z}$ and $u \in \mathbb{R}$, using $\varphi_u \varphi_t = \varphi_{t+u}$ and the change of variable $t \mapsto t - u$ for all $t \in S^1 \simeq \mathbb{R}/\mathbb{Z}$, we compute

$$\begin{aligned} \varphi_u^* \left(\int_0^1 e^{-2\pi\sqrt{-1}tm} \varphi_t^* s \, dt \right) &= \int_0^1 e^{-2\pi\sqrt{-1}tm} \varphi_{t+u}^* s \, dt \\ &= e^{2\pi\sqrt{-1}um} \int_0^1 e^{-2\pi\sqrt{-1}tm} \varphi_t^* s \, dt \, . \end{aligned} \tag{3.3}$$

Together with the definition (1.10) of the weight space of weight $m \in \mathbb{Z}$, since the action of S^1 on $L^2(X, L^p)$ is unitary and recalling that $P_p : L^2(X, L^p) \to H^0_{(2)}(X, L^p)$ denotes the orthogonal projection, this shows that the map

$$\begin{aligned} P_p^{(m)} : L^2(X, L^p) &\longrightarrow H^0_{(2)}(X, L^p)_m \\ s &\longmapsto \int_0^1 e^{-2\pi\sqrt{-1}tm} \varphi_t^* P_p s \, dt \end{aligned} \tag{3.4}$$

is the orthogonal projection. This implies formula (3.1) by Proposition 2.1 introducing the Bergman kernel and definition (2.11) of the action of S^1 on sections, while formula (3.2) for the equivariance follows from the computation (3.3) in the same way, together with the usual formula $P_p^{(m)}(x, y) = P_p^{(m)}(y, x)^*$ for all x, $y \in X$, holding for Schwartz kernels of smoothing Hermitian operators. □

The Schwartz kernel defined in Proposition 3.1 for all $p \in \mathbb{N}$ is called the *equivariant Bergman kernel* of weight $m \in \mathbb{Z}$. Recalling the fundamental vector field (2.12) of the S^1-action, the following result is adapted from [13, Prop. 4.1].

Proposition 3.2. *For any $k \in \mathbb{N}$ and $r \in \mathbb{N}$, there exists $C_k > 0$ such that for any $m \in \mathbb{N}$, $p \in \mathbb{N}$ and x, $y \in X$ with $\mu(x) \neq m/p$, we have*

$$\left| P_p^{(m)}(x, y) \right|_{\mathscr{C}^r} \leqslant \frac{C_k \, |\xi_x|^k}{\left| \mu(x) - \frac{m}{p} \right|^k} \, p^{n+r-\frac{k}{2}} \, . \tag{3.5}$$

Proof. For any $x \in X$ and $t_0 \in \mathbb{R}$, consider a chart around $x_0 := \varphi_{t_0}(x) \in X$ as in Definition 2.3 such that the radial line in $B^{T_{x_0}X}(0, \varepsilon_0)$ generated by the fundamental vector field $\xi_x \in T_x X$ defined by (2.12) is sent to the path $u \mapsto \varphi_u(x_0)$ in the image $U_x \subset X$ of $\psi_x : B^{T_x X}(x, \varepsilon_0) \to X$. Then L^p is identified with $L^p_{x_0}$ along this path by the parallel transport $\tau_{t,p}$ introduced in (2.16), for all $p \in \mathbb{N}$ and $|t| < \varepsilon_0$. Thus for any $Z \in B^{T_{x_0}X}(0, \varepsilon)$ and any $t \in \mathbb{R}$ small enough, in the notation (2.6) we have

$$\tau_{t,p} P_p(\varphi_{t_0+t}(x), \psi_{x_0}(Z)) = P_{p,x_0}(t\xi_{x_0}, Z) \, . \tag{3.6}$$

Using also that $\tau_{t_0+t,p} = \tau_{t,p}\tau_{t_0,p}$, we can thus apply Theorems 2.2 and 2.5 for all $t_0 \in S^1 \simeq \mathbb{R}/\mathbb{Z}$, so that for any r, $k \in \mathbb{N}$, we get $C_k > 0$ such that for any x, $y \in X$, $t \in [0,1]$ and $p \in \mathbb{N}$, we have

$$\left| \frac{\partial^k}{\partial t^k} \tau_{t,p} P_p(\varphi_t(x), y) \right|_{\mathscr{C}^r} \leqslant C_k \, p^{n+r+\frac{k}{2}} \, |\xi_x|^k \,. \tag{3.7}$$

Then using the exponentiated form (2.17) of the Kostant formula, we can integrate by parts to get from Proposition 3.1 that for any $x, y \in X$ and $k \in \mathbb{N}$,

$$\begin{aligned} P_p^{(m)}(x,y) &= \int_0^1 \tau_{t,p} P_p(\varphi_t(x), y) e^{2\pi\sqrt{-1}t(p\mu(x)-m)} dt \\ &= \frac{1}{(2\pi\sqrt{-1}(p\mu(x)-m))^k} \int_0^1 \tau_{t,p} P_p(\varphi_t(x), y) \frac{\partial^k}{\partial t^k} e^{2\pi\sqrt{-1}t(p\mu(x)-m)} dt \\ &= \left(\frac{\sqrt{-1}}{2\pi}\right)^k \frac{p^{-k}}{(\mu(x)-\frac{m}{p})^k} \int_0^1 \frac{\partial^k}{\partial t^k} \left(\tau_{t,p} P_p(\varphi_t(x), y)\right) e^{2\pi\sqrt{-1}t(p\mu(x)-m)} dt \,. \end{aligned} \tag{3.8}$$

Together with (3.7), this establishes the result. □

3.2 Asymptotic expansion of equivariant Bergman kernels

Recall the setting described at the beginning of this section, and let us assume in addition that S^1 acts freely on $\mu^{-1}(0) \subset X$. This means in particular that the fundamental vector field $\xi \in \mathscr{C}^\infty(X, TX)$ defined by (2.12) nowhere vanishes over $\mu^{-1}(0) \subset X$, and the definition (1.7) of the Kähler metric together with the moment map equation (2.14) then imply that for all $x \in \mu^{-1}(0)$, we have

$$d\mu.J\xi_x = -|\xi_x|^2 \neq 0 \,. \tag{3.9}$$

This shows in particular that 0 is a regular value of $\mu \in \mathscr{C}^\infty(X, \mathbb{R})$, so that $\mu^{-1}(0) \subset X$ is a smooth submanifold. For any $x \in \mu^{-1}(0)$, we set

$$N_x := \{\eta \in T_xX \mid g^{TX}(\eta, \xi_x) = g^{TX}(\eta, J\xi_x) = 0\} \,. \tag{3.10}$$

Note from (1.7) and (2.14) that $J\xi_x \in T_xX$ is orthogonal to $T_x\mu^{-1}(0) \subset T_xX$, and consider the unit orthogonal vectors in T_xX defined by

$$e_0 := \frac{\xi_x}{|\xi_x|} \in T_x\mu^{-1}(0) \quad \text{and} \quad e_1 := \frac{J\xi_x}{|\xi_x|} \in T_x\mu^{-1}(0)^\perp \,, \tag{3.11}$$

so that we have an orthogonal decomposition of (T_xX, g_x^{TX}) given by

$$T_xX = \mathbb{R}\, e_0 \oplus \mathbb{R}\, e_1 \oplus N_x \,. \tag{3.12}$$

Fix now $x_0 \in \mu^{-1}(0)$. In order to build a chart around $x_0 \in X$ adapted to this decomposition, let $\varepsilon_0 > 0$ be such that the exponential map $\exp_{x_0}^{\mu^{-1}(0)} : B^{T_{x_0}\mu^{-1}(0)}(0, \varepsilon_0) \to \mu^{-1}(0)$ of $(\mu^{-1}(0), g^{TX}|_{\mu^{-1}(0)})$ is injective, and write

$$V_0 := \exp_{x_0}^{\mu^{-1}(0)}(B^{N_{x_0}}(0, \varepsilon_0)) \subset \mu^{-1}(0) \,, \tag{3.13}$$

where $N_{x_0} \subset T_{x_0}\mu^{-1}(0)$ by construction. This defines a local section of the quotient map $\pi : \mu^{-1}(0) \to X_0 := \mu^{-1}(0)/S^1$ considered as an S^1-principal bundle. Let now $\varepsilon_0 > 0$ be so small that for all $u \in I_0 :=]-\varepsilon_0, \varepsilon_0[$ and $x \in V_0$, the ODE

$$\frac{d}{du}\phi_u(x) = -\frac{J\xi_{\phi_u(x)}}{|\xi_{\phi_u(x)}|^2} \quad \text{with} \quad \phi_0(x) = x \tag{3.14}$$

defines an embedding $\phi : I_0 \times V_0 \xrightarrow{\sim} V \subset X$, which by (3.9) satisfies $\mu(\phi_u(x)) = u$ for all $u \in I_0$ and $x \in V_0$. Since formula (3.14) is S^1-invariant in the sense that for all $t \in S_1 \simeq \mathbb{R}/Z$, $u \in\in I_0$ and $x \in V_0$, we have $\phi_u(\varphi_t(x)) = \varphi_t(\phi_u(x))$, we get a natural S^1-equivariant open embedding

$$\begin{aligned} \Phi : S^1 \times I_0 \times V_0 &\xrightarrow{\sim} U \subset X \\ (t\,,\,u\,,\,x) &\longmapsto \varphi_t(\phi_u(x))\,. \end{aligned} \tag{3.15}$$

Using a partition of unity, we can then consider a bounded family of charts $\psi : B^{TX}(0, \varepsilon_0) \to X$ in the sense of Definition 2.3 such that for any $x_0 \in \mu^{-1}(0)$, its restriction $\psi_{x_0} : B^{T_{x_0}X}(0, \varepsilon_0) \xrightarrow{\sim} U_{x_0} \subset X$ is defined for any t, $u \in \mathbb{R}$ and $\eta \in N_{x_0}$ satisfying $|te_1 + ue_1 + \eta| < \varepsilon_0$ in the decomposition (3.12) by

$$\psi_{x_0}(te_0 + ue_1 + \eta) = \Phi\left(\frac{t}{|\xi_{x_0}|}\,,\, -|\xi_{x_0}|\,u\,,\, \exp_{x_0}^{\mu^{-1}(0)}(\eta)\right). \tag{3.16}$$

Note that we have in fact $d\psi_{x_0} = \mathrm{Id}_{T_{x_0}X}$ by (3.11) and (3.14). For any $x \in \mu^{-1}(0)$, we consider the *horizontal tangent bundle*

$$\begin{aligned} T^H_x X &:= \{\eta \in T_xX \mid g^{TX}(\eta, \xi_x) = 0\} \\ &= \mathbb{R}e_1 \oplus N_x \subset T_xX \end{aligned} \tag{3.17}$$

and for any $Z \in T^H_x X$, we write

$$Z = u\,e_1 + Z^\perp \in \mathbb{R}\,e_1 \oplus N_x\,, \tag{3.18}$$

for its decomposition induced by (3.12), with $u \in \mathbb{R}$ and $Z^\perp \in N_x$. In the same way, for any $Z' \in T^H_x X$, we write $Z' = u'\,e_1 + Z'^\perp$ with $u' \in \mathbb{R}$ and $Z'^\perp \in N_x$. The following result establishes the asymptotic expansion for the equivariant Bergman kernels introduced in Proposition 3.1.

Theorem 3.3. *There is a family $\{Q_{r,x} \in \mathbb{C}[Y, Z, Z']\}_{r\in\mathbb{N}}$ of polynomials in $Z, Z' \in T^H_x X$ and $Y \in \mathbb{R}$, depending smoothly on $x \in \mu^{-1}(0)$, of the same total parity as $r \in \mathbb{N}$ and of total degree less than $3r \in \mathbb{N}$, such that for any compact set $K \subset \mu^{-1}(0)$, any k, $j \in \mathbb{N}$ and $\delta > 0$, there is $\varepsilon_0 > 0$ and $C > 0$ such that for all $\varepsilon \in]0, \varepsilon_0[$, $x \in K$, $p \in \mathbb{N}$, $m \in \mathbb{Z}$ and Z, $Z' \in T^H_x X$ satisfying $|Z|$, $|Z'| < \varepsilon\, p^{\frac{-1+\varepsilon}{2}}$, we have*

$$\left| p^{-n+\frac{1}{2}} P^{(m)}_{p,x}(Z, Z') - \sum_{r=0}^{k-1} Q_{r,x}\left(\frac{m}{\sqrt{p}}, \sqrt{p}Z, \sqrt{p}Z'\right) \mathscr{P}^{(m)}_x(\sqrt{p}Z, \sqrt{p}Z')p^{-\frac{r}{2}} \right|_{\mathscr{C}^j(K)} \leqslant Cp^{-\frac{k}{2}+\delta}, \tag{3.19}$$

where for any $Z, Z' \in T_x^H X$ as in (3.18), we have

$$\mathscr{P}_x^{(m)}(Z,Z') = \frac{\sqrt{2}}{|\xi_x|}\exp\left(-2\pi\left(\frac{u+u'}{2}+\frac{m}{|\xi_x|\sqrt{p}}\right)^2\right) \exp\left(-\frac{\pi}{2}(u-u')^2\right)\mathscr{P}_x(Z^\perp, Z'^\perp)\,. \quad (3.20)$$

Furthermore, for all $x \in M$, we have $Q_{0,x} \equiv 1$.

Proof. Let $x \in \mu^{-1}(0)$, and consider the trivialization of L by parallel transport along radial lines in the charts defined by (3.16). Then for any $Z \in B^{T_x^H X}(0,\varepsilon_0)$ as in (3.18), by a standard computation which can be found for example in [17, (1.2.31)] and which holds in any trivialization of L along radial lines, the connection ∇^L at $\psi_x(Z) \in X$ has the form

$$\begin{aligned}\nabla_\xi^L &= d + \frac{1}{2}R^L(Z,\xi_x) + O(|Z|^2)\\ &= d - \sqrt{-1}\pi\omega_x(Z,\xi_x) + O(|Z|^2)\\ &= d + \sqrt{-1}\pi|\xi_x|\,u + O(|Z|^2)\,,\end{aligned} \quad (3.21)$$

where we used formulas (1.6) and (1.7) for ω and g^{TX} to get the first line, and formulas (3.11) and (3.18) for the second. On the other hand, by (3.9) and (3.14), we have $\mu(\psi_x(Z)) = -|\xi_x|\,u$ in the charts defined by (3.16). Hence by definition (2.16) of the parallel transport with respect to $\nabla_\xi^{L^p}$ and by the form (3.21) of ∇^L at $Z \in B^{T_x^H X}(0,\varepsilon_0)$, formula (2.17) reads as

$$\begin{aligned}\varphi_{t,p}^{-1} &= \exp(2\pi\sqrt{-1}tp\mu(\psi_x(Z)))\exp(\sqrt{-1}\pi tp(|\xi_x|\,u + O(|Z|^2)))\\ &= \exp(-\sqrt{-1}\pi tp(|\xi_x|\,u + O(|Z|^2)))\,.\end{aligned} \quad (3.22)$$

Let us now choose $\varepsilon > 0$ so small that for any $Z \in B^{T_x^H X}(0,\varepsilon)$ and $t \in \,]-\varepsilon,\varepsilon[$, we have $|te_0 + Z| < \varepsilon_0$. Then in the charts defined by (3.16), we get

$$\varphi_t(Z) = |\xi_x|\,t\,e_0 + Z\,. \quad (3.23)$$

Hence by definition of the local model (2.8), we have the following asymptotic expansion as $p \to \infty$, uniform in $t \in \,]-\varepsilon p^{\frac{\varepsilon}{2}}, \varepsilon p^{\frac{\varepsilon}{2}}[$ and in $Z, Z' \in B^{T_x^H X}(0,\varepsilon p^{\frac{\varepsilon}{2}})$ as in (3.18),

$$\begin{aligned}&\mathscr{P}_x(\sqrt{p}\,\varphi_{t/\sqrt{p}}(Z/\sqrt{p}), Z')\\ &= \exp\left(-\frac{\pi}{2}|\xi_x|^2t^2 - \frac{\pi}{2}|Z-Z'|^2 - \pi\sqrt{-1}t\omega_x(\xi_x,Z') - \pi\sqrt{-1}t\omega_x(Z,Z')\right)\exp\left(O(p^{\varepsilon-\frac{1}{2}})\right)\\ &= \exp\left(-\frac{\pi}{2}|\xi_x|^2t^2 - \pi\sqrt{-1}|\xi_x|tu'\right)\exp(-\frac{\pi}{2}(u-u')^2)\mathscr{P}_x(Z^\perp,Z'^\perp) + O(p^{\varepsilon-\frac{1}{2}})\,.\end{aligned} \quad (3.24)$$

Then by Theorems 2.2 and 2.5, Corollary 2.4, and Proposition 3.1, using the classical formula for the Fourier transform of the Gaussian and the change of variables

$(t,Z,Z') \mapsto (t/\sqrt{p}, Z/\sqrt{p}, Z'/\sqrt{p})$, for any $\delta>0$, we get $\varepsilon>0$ such that we have the following asymptotic expansion as $p\to\infty$, uniform in $m\in\mathbb{Z}$, in $\varepsilon\in]0,\varepsilon_0[$ and in $Z,\,Z'\in B^{T^H_xX}(0,\varepsilon p^{\frac{-1+\varepsilon}{2}})$ as in (3.18),

$$\begin{aligned}
&p^{-n+\frac{1}{2}}\,P^{(m)}_{p,x}(Z/\sqrt{p},Z'/\sqrt{p})\\
&= p^{-n+\frac{1}{2}}\int_{-\varepsilon p^{\frac{-1+\varepsilon}{2}}}^{\varepsilon p^{\frac{-1+\varepsilon}{2}}} e^{-2\pi\sqrt{-1}tm}\varphi_{t,p}^{-1}P_{p,x}(\varphi_t(Z/\sqrt{p}),Z'/\sqrt{p})\,dt + O(p^{-\infty})\\
&= \int_{-\varepsilon p^{\frac{\varepsilon}{2}}}^{\varepsilon p^{\frac{\varepsilon}{2}}} e^{-2\pi\sqrt{-1}t\frac{m}{\sqrt{p}}}\exp(-\sqrt{-1}\pi t|\xi_x|\,u)\mathscr{P}_x(\sqrt{p}\,\varphi_{t/\sqrt{p}}(Z/\sqrt{p}),Z')\,dt + O(p^{\delta-\frac{1}{2}})\\
&= \left(\int_{-\infty}^{\infty} e^{-\sqrt{-1}\pi t|\xi_x|\,(u+u')-2\sqrt{-1}\pi t\frac{m}{\sqrt{p}}-\frac{\pi}{2}|\xi_x|^2t^2}\,dt\right)\exp(-\frac{\pi}{2}(u-u')^2)\mathscr{P}_x(Z^\perp,Z'^\perp) + O(p^{\delta-\frac{1}{2}})\\
&= \frac{\sqrt{2}}{|\xi_x|}\exp\left(-2\pi\left(\frac{(u+u')}{2}+\frac{m}{|\xi_x|\sqrt{p}}\right)^2\right)\exp(-\frac{\pi}{2}(u-u')^2)\mathscr{P}_x(Z^\perp,Z'^\perp) + O(p^{\delta-\frac{1}{2}})\,.
\end{aligned} \tag{3.25}$$

This establishes the first order of the asymptotics expansion (3.19). To get the next order, one simply needs to consider the next order of the Taylor expansions performed in formulas (3.21) to (3.24), then realize that all integrations performed in the computation (3.25) extend to the case when the exponential terms are multiplied by polynomials in $(\sqrt{p}t,\sqrt{p}Z,\sqrt{p}Z')$, leading to an asymptotic expansion of the form (3.19). This proves the result. □

Remark 3.4. Theorem 3.3 gives the asymptotic expansion as $p\to\infty$ of the equivariant Bergman kernel with respect to each weight $m\in\mathbb{Z}$ in a neighborhood of $\mu^{-1}(0)$ and in the directions orthogonal to the S^1-action, since it only holds for $Z,\,Z'\in T^HX$ as in (3.17). Together with Proposition 3.1 prescribing its behavior in the direction of the S^1-action and Proposition 3.2 giving its asymptotic decay outside the same neighborhood of $\mu^{-1}(0)$, Theorem 3.3 thus computes the full asymptotic expansion of the equivariant Bergman kernels as $p\to\infty$, uniformly in the weight $m\in\mathbb{Z}$. In the special case $m=0$, Theorem 3.3 is a consequence of the result of Ma and Zhang in [21, Th. 0.2].

3.3 Asymptotic expansion of the partial Bergman kernel

Recall the setting described at the beginning of the section. The following Proposition serves as a definition of the partial Bergman kernel, and gives its first fundamental asymptotic properties.

Theorem 3.5. *For any $p\in\mathbb{N}$, the orthogonal projection to the space $\mathscr{H}_p\subset L^2(X,L^p)$ defined by (1.11) admits a smooth Schwartz kernel $P_p^{(-)}(\cdot,\cdot)\in\mathscr{C}^\infty(X\times X,L^p\boxtimes(L^p)^*)$, and for any $r,\,k\in\mathbb{N}$ and $\alpha>0$, there is $C_k>0$ such that for all $\varepsilon\in]0,\alpha/2[$, $p\in\mathbb{N}$*

and $x,\, y \in X$, the following estimates holds

$$\begin{aligned} \Big| P_p^{(-)}(x,y) - \sum_{m \geqslant -\alpha p^{\frac{1+\alpha}{2}}}^{m=0} P_p^{(m)}(x,y) \Big|_{\mathscr{C}^r} &\leqslant C_k\, p^{-k}\, |\xi_x|^k \quad \textit{if} \quad |\mu(x)| \leqslant \varepsilon p^{\frac{-1+\varepsilon}{2}}\,, \\ |P_p^{(-)}(x,y)|_{\mathscr{C}^r} &\leqslant C_k\, p^{-k}\, |\xi_x|^k \quad \textit{if} \quad \mu(x) > \alpha p^{\frac{-1+\alpha}{2}}\,, \\ |P_p^{(-)}(x,y) - P_p(x,y)|_{\mathscr{C}^r} &\leqslant C_k\, p^{-k}\, |\xi_x|^k \quad \textit{if} \quad \mu(x) < -\alpha p^{\frac{-1+\alpha}{2}}\,, \end{aligned} \tag{3.26}$$

and $|P_p^{(-)}(x,y)|_{\mathscr{C}^r} \leqslant C_k\, p^{-k}\, |\xi_x|^k$ if $d^X(x,y) > \alpha p^{\frac{-1+\alpha}{2}}$.

Proof. For any $p \in \mathbb{N}$ and $m \in \mathbb{Z}$, recall the orthogonal projection (3.4) on the weight space (1.10) of weight $m \in \mathbb{Z}$, and note from (1.11) that the map

$$P_p^{(-)} := \sum_{m \leqslant 0} P_p^{(m)} : L^2(X, L^p) \longrightarrow \mathscr{H}_p \tag{3.27}$$

converges to the orthogonal projection onto the space (1.11) in the space of bounded operators acting on $L^2(X, L^p)$. In the same way, by the weight decomposition (1.9), the orthogonal projection $P_p : L^2(X, L^p) \to H^0_{(2)}(X, L^p)$ satisfies $P_p = \sum_{m \in \mathbb{Z}} P_p^{(m)}$, where the sum converges not only in the space of bounded operators acting on $L^2(X, L^p)$, but also in local $\mathscr{C}^r$-norms of their Schwartz kernels for each $r \in \mathbb{N}$ by Proposition 3.1 and standard Fourier theory. This shows in particular that the sum (3.27) converges in local $\mathscr{C}^r$-norms for each $r \in \mathbb{N}$, so that (3.27) admits a smooth Schwartz kernel $P_p^{(-)}(\cdot,\cdot) \in \mathscr{C}^\infty(X \times X, L^p \boxtimes (L^p)^*)$, and we have the following smooth convergence for all $x,\, y \in X$,

$$P_p^{(-)}(x,y) = \sum_{m \in \mathbb{Z}} P_p^{(m)}(x,y)\,. \tag{3.28}$$

Using Proposition 3.2, for any $k,\, r \in \mathbb{N}$ and $\alpha > 0$, we get a constant $C_k > 0$ such that for any $\varepsilon \in]0, \alpha/2[$, $p \in \mathbb{N}$ and $x,\, y \in X$ such that $|\mu(x)| \leqslant \varepsilon p^{\frac{-1+\varepsilon}{2}}$, we have

$$\begin{aligned} \sum_{m < -\alpha p^{\frac{1+\alpha}{2}}} \Big| P_p^{(m)}(x,y) \Big|_{\mathscr{C}^r} &\leqslant C_k\, p^{n+r-\frac{k}{2}}\, |\xi_x|^k \sum_{m < -\alpha p^{\frac{1+\alpha}{2}}} \Big| \mu(x) - \frac{m}{p} \Big|^{-k} \\ &\leqslant C_k p^{n+r-\frac{k}{2}}\, |\xi_x|^k \sum_{m=1}^{\infty} \Big| \Big(\mu(x) - 2\alpha p^{\frac{-1+\alpha}{2}}\Big) + \frac{m}{p} \Big|^{-k} \\ &\leqslant C_k \alpha^{-k} p^{-\alpha\frac{k}{2}} p^{n+r}\, |\xi_x|^k \sum_{m=1}^{\infty} \left(\frac{m}{\varepsilon p^{\frac{1+\alpha}{2}}} + 1 \right)^{-k} \\ &\leqslant C_k \alpha^{1-k} p^{n+r+\frac{1}{2}} p^{-\alpha\frac{k-1}{2}}\, |\xi_x|^k \int_0^\infty (t+1)^{-k}\, dt\,. \end{aligned} \tag{3.29}$$

Since the integral in the last line of (3.29) converges, this shows that the sum in the left-hand side converges as well and proves the first estimate of (3.26) by formula (3.28) and taking $k \in \mathbb{N}$ large enough.

Letting now $x, y \in X$ satisfy $\mu(x) > \alpha p^{\frac{-1+\alpha}{2}}$, a strictly analogous computation allows to directly estimate the sum $\sum_{m\leqslant 0} |P_p^{(m)}(x,y)|_{\mathscr{C}^r}$ for all $r \in \mathbb{N}$ in the same way, establishing the second estimate of (3.26) by formula (3.28).

Finally, if $x, y \in X$ satisfy $\mu(x) < -\alpha p^{\frac{-1+\alpha}{2}}$, a computation stricly analogous to (3.29) allows to estimate the sum $\sum_{m=1}^{\infty} |P_p^{(m)}(x,y)|_{\mathscr{C}^r}$ for all $r \in \mathbb{N}$ in the same way, giving the third estimate of (3.26) thanks to the identity $P_p - P_p^{(-)} = \sum_{m=1}^{\infty} P_p^{(m)}$ following from (3.27) as above. This concludes the proof.

□

The Schwartz kernel defined in Theorem 3.5 for all $p \in \mathbb{N}$ is called the *partial Bergman kernel*. Recalling the bounded family of charts constructed in Section 3.2, the main goal of this section is to establish the following result on the near-diagonal expansion of the partial Bergman kernel.

Theorem 3.6. *There are two families $\{Q_{r,x}^{(-)} \in \mathbb{C}[Z,Z']\}_{r\in\mathbb{N}}$ and $\{Q_{r,x}^{(0)} \in \mathbb{C}[Z,Z']\}_{r\in\mathbb{N}}$ of polynomials in $Z, Z' \in T_x^H X$ depending smoothly on $x \in \mu^{-1}(0)$, of the same total parity as $r \in \mathbb{N}$ and of degree less that $3r \in \mathbb{N}$, such that for any compact set $K \subset \mu^{-1}(0)$, any $k, j \in \mathbb{N}$ and $\delta > 0$, there is $\varepsilon_0 > 0$ and $C > 0$ such that for any $\varepsilon \in]0,\varepsilon_0[$, $x \in K$, $p \in \mathbb{N}$ and $Z, Z' \in T_x^H X$ satisfying $|Z|, |Z'| < \varepsilon p^{\frac{-1+\varepsilon}{2}}$, we have*

$$\Big| p^{-n} P_{p,x}^{(-)}(Z,Z') - \sum_{r=0}^{k-1} Q_{r,x}^{(-)}(\sqrt{p}Z,\sqrt{p}Z')\mathscr{P}_x^{(-)}(\sqrt{p}Z,\sqrt{p}Z')p^{-\frac{r}{2}} \\ -p^{-\frac{1}{2}} \sum_{r=0}^{k-2} Q_{r,x}^{(0)}(\sqrt{p}Z,\sqrt{p}Z')\mathscr{P}_x^{(0)}(\sqrt{p}Z,\sqrt{p}Z')p^{-\frac{r}{2}} \Big|_{\mathscr{C}^j(K)} \leqslant Cp^{-\frac{k}{2}+\delta}. \quad (3.30)$$

where $\mathscr{P}_x^{(0)}$ is the local model (3.20) for $m = 0$ and where for any $Z, Z' \in T_x^H X$ as in (3.18), we have

$$\mathscr{P}_x^{(-)}(Z,Z') = \sqrt{2}\left(\int_{-\infty}^{\frac{u+u'}{2}} e^{-2\pi t^2}\,dt\right)\exp\left(-\frac{\pi}{2}(u-u')^2\right)\,\mathscr{P}_x(Z^\perp, Z'^\perp)\,. \quad (3.31)$$

Furthermore, for all $x \in \mu^{-1}(0)$ and $Z, Z' \in T_x^H X$, we have $Q_{0,x}^{(-)} \equiv 1$.

Proof. First recall the following Euler-MacLaurin formula, which can be found for instance in [26, Th. 9.1], and which states the existence of a universal sequence $\{a_j \in \mathbb{R}\}_{j\in\mathbb{N}}$ such that for any $f \in \mathscr{C}^\infty(\mathbb{R},\mathbb{R})$ whose derivatives of all order tend to 0 at infinity, and for any $r \in \mathbb{N}$, we have

$$\sum_{m\in\mathbb{N}} f(m) = \int_0^\infty f(t)\,dt + \sum_{j=0}^{r-1} a_j\, f^{(j)}(0) + \int_0^\infty a_r\,(t - \lceil t\rceil)\, f^{(r)}(t)\,dt\,. \quad (3.32)$$

In particular, this implies the existence of polynomials $\{P_{a,j} \in \mathbb{R}[v]\}_{j\in\mathbb{N}}$ of degree at most $j \in \mathbb{N}$ and depending smoothly on $a > 0$ such that for any compact interval $I \subset]0,+\infty[$, we have the following estimate as $p \to \infty$, uniform in $v \in \mathbb{R}$ and $a \in I$,

$$\sum_{m\in\mathbb{N}} e^{-2\pi\left(v-\frac{m}{a\sqrt{p}}\right)^2} = a\sqrt{p}\int_{-\infty}^{v} e^{-2\pi t^2}\,dt + \sum_{j=0}^{r-1} p^{-\frac{j}{2}}\, P_{a,j}(v)\, e^{-2\pi v^2} + O(p^{-\frac{r}{2}})\,. \quad (3.33)$$

More generally, for any $k \in \mathbb{N}$, we get sequences of polynomials $\{P_{a,k,j},\, \widetilde{P}_{a,k,j} \in \mathbb{R}[v]\}_{j\in\mathbb{N}}$ of degree at most $j+k \in \mathbb{N}$ and depending smoothly on $a > 0$ such that for any compact interval $I \subset]0, +\infty[$, we have the following estimate as $p \to \infty$, uniform in $v \in \mathbb{R}$ and $a \in I$,

$$\begin{aligned}
&\sum_{m\in\mathbb{N}} \left(\frac{m}{\sqrt{p}}\right)^k e^{-2\pi\left(v-\frac{m}{a\sqrt{p}}\right)^2} \\
&= a^{k+1}\sqrt{p}\int_{-\infty}^{v} (v-t)^k\, e^{-2\pi t^2}\, dt + \sum_{j=0}^{r-1} p^{-\frac{j}{2}}\, P_{a,k,j}(v)\, e^{-av^2} + O(p^{-\frac{r}{2}})\,. \\
&= a^{k+1}\sqrt{p}\left(v^k \int_{-\infty}^{v} e^{-2\pi t^2}\, dt + \widetilde{P}_{a,k,0}(v)\, e^{-av^2}\right) + \sum_{j=0}^{r-1} p^{-\frac{j}{2}}\, \widetilde{P}_{a,k,j+1}(v)\, e^{-2\pi v^2} + O(p^{-\frac{r}{2}})\,.
\end{aligned} \tag{3.34}$$

Then by Theorems 3.3 and 3.5, for any compact subset $K \subset \mu^{-1}(0)$, $k \in \mathbb{N}$ and $\delta > 0$, we can choose $\alpha > 0$ such that we have the following uniform estimate as $p \to \infty$, uniform in $x \in K$, in $\varepsilon \in]0, \alpha/2[$ and in $Z, Z' \in T^H_x X$ satisfying $|Z|, |Z'| < \varepsilon p^{\frac{-1+\varepsilon}{2}}$,

$$\begin{aligned}
p^{-n} P^{(-)}_{p,x}(Z, Z') &= p^{-n} \sum_{m \geqslant -\alpha p^{\frac{1+\alpha}{2}}}^{m=0} P^{(m)}_{p,x}(Z, Z') + O(p^{-\infty}) \\
&= p^{-\frac{1}{2}} \sum_{r=0}^{k-1} p^{-\frac{r}{2}} \sum_{m \geqslant -\alpha p^{\frac{1+\alpha}{2}}}^{m=0} Q_{r,x}\left(\frac{m}{\sqrt{p}}, \sqrt{p}Z, \sqrt{p}Z'\right) \mathscr{P}^{(m)}_x(\sqrt{p}Z, \sqrt{p}Z') + O(p^{-\frac{k}{2}+\delta})\,.
\end{aligned} \tag{3.35}$$

On the other hand, for any $r \in \mathbb{N}$, using the local model (3.20) for the equivariant Bergman kernel and recalling from Theorem 3.3 that the polynomial Q_r in formula (3.35) is a polynomial of degree less than $3r \in \mathbb{N}$ in $m/\sqrt{p}$, for any $\alpha > 0$ small enough, we get constants $C_r > 0$ and $c > 0$ such that for all $\varepsilon \in]0, \alpha/2[$, $x \in K$ and $Z, Z' \in T^H_x X$ as in (3.18) satisfying $|Z|, |Z'| < \varepsilon p^{\frac{\varepsilon}{2}}$, we have

$$\begin{aligned}
&\sum_{m < -\alpha p^{\frac{1+\alpha}{2}}} \left| Q_{r,x}\left(\frac{m}{\sqrt{p}}, Z, Z'\right) \mathscr{P}^{(m)}_x(Z, Z') \right| \\
&\leqslant C\, p^{\frac{3r}{2}\alpha} \sum_{m < -\alpha p^{\frac{1+\alpha}{2}}} \left|\frac{m}{\sqrt{p}}\right|^{3r} \exp\left(-2\pi\left(\frac{u+u'}{2} - \frac{m}{|\xi_x|\sqrt{p}}\right)^2\right) \\
&\leqslant C_r\, p^{\frac{3r}{2}\alpha+\frac{1}{2}} \exp\left(-c\, p^{\frac{\alpha}{2}}\right)\,,
\end{aligned} \tag{3.36}$$

which is decreasing faster than any power of $p \in \mathbb{N}$. In particular, we can apply (3.34) with $v = (u+u')/2$ and $a = |\xi_x|$ to get from (3.35) and (3.36) sequences of polynomials $\{P_r,\, R_{r,k} \in \mathbb{C}(Z, Z')\}_{r,\,k\in\mathbb{N}}$ such that for any compact subset $K \subset \mu^{-1}(0)$ and $\delta > 0$, there is $\alpha > 0$ such that we have the following asymptotic expansion as $p \to \infty$, for all

$\varepsilon \in]0, \alpha/2[$, $x \in K$ and $Z,\ Z' \in T^H_x X$ as in (3.18) satisfying $|Z|,\ |Z'| < \varepsilon p^{\frac{\varepsilon}{2}}$,

$$\begin{aligned}
\sum_{m \geqslant -\alpha p^{\frac{1+\alpha}{2}}}^{m=0} Q_{r,x}\left(\frac{m}{\sqrt{p}}, Z, Z'\right) \mathscr{P}^{(m)}_x(Z,Z') &= \sum_{m \leqslant 0} Q_{r,x}\left(\frac{m}{\sqrt{p}}, Z, Z'\right) \mathscr{P}^{(m)}_x(Z,Z') + O(p^{-\infty}) \\
&= \frac{\sqrt{2}}{|\xi_x|} \sum_{m \in \mathbb{N}} Q_{r,x}\left(-\frac{m}{\sqrt{p}}, Z, Z'\right) \exp\left(-2\pi \left(\frac{(u+u')}{2} + \frac{m}{|\xi_x|\sqrt{p}}\right)^2\right) \\
&\qquad \exp\left(-\frac{\pi}{2}(u-u')^2\right) \mathscr{P}(Z^\perp, Z'^\perp) + O(p^{-\infty}) \\
&= \sqrt{p}\, P_r(Z,Z') \mathscr{P}^{(-)}(Z,Z') + \sqrt{p}\, R_{r,0}(Z,Z') \mathscr{P}^{(0)}(Z,Z') \\
&\qquad + \sum_{k=0}^{j-1} p^{-\frac{k}{2}} R_{r,k}(Z,Z') \mathscr{P}^{(0)}(Z,Z') + O(p^{-\frac{j}{2}+\delta}),
\end{aligned} \tag{3.37}$$

where $P_0 \equiv 1$ and $R_{0,0} \equiv 0$ by equation (3.33) and the fact that $Q_0 \equiv 1$ in the asymptotic expansion of Theorem 3.3. Plugging (3.37) into (3.35) after the change of variables $(Z, Z') \mapsto (\sqrt{p}Z, \sqrt{p}Z')$, we then get the result. □

Remark 3.7. For any $x \in \mu^{-1}(0)$, by definition (3.16) of the bounded family of charts used in Theorem 3.3, we know that there exists $\varepsilon > 0$ such that for any $u \in]-\varepsilon, \varepsilon[$, we have $\phi_u(x) = \psi_x(-\frac{u}{|\xi_x|} e_1)$, where $\phi_u : \mu^{-1}(0) \to X$ is the flow of $-J\xi/|\xi|^2$ at time $u \in]-\varepsilon, \varepsilon[$ as defined in (3.14), which satisfies $\mu(\phi_u(x)) = u$. Then Theorems 3.5 and 3.6 applied to $Z = Z' = -\frac{u}{|\xi_x|} e_1$ recover the asymptotic expansion of the partial Bergman on the diagonal established by Ross and Singer in [26, Th. 1.2] and Zelditch and Zhou in [32, (8), Th. 4], while the general result recovers a result of Shabtai in [28, Th. 1.7]. In all these results, the asymptotic expansions hold in neighborhoods of size of order $\frac{1}{\sqrt{p}}$ around the boundary $\mu^{-1}(0) \subset X$ and outside neighborhoods of size of order 1 as $p \to \infty$, while Theorems 3.5 and 3.6 give a full asymptotic expansion in any neighborhood of $\mu^{-1}(0)$.

4 Determinantal point processes

In Section 4.1, we introduce the notion of a determinantal point process in our setting and recall its fundamental properties following Berman in [4], then explain how it recovers the usual Ginibre ensemble (1.4) in the case of $X = \mathbb{C}$ endowed with the action of $S^1 \subset \mathbb{C}$ by multiplication. In Section 4.2, we establish a law of large numbers for this determinantal point process, while in Section 4.3, we establish a central limit theorem, thus concluding the proof of Theorem 1.1.

4.1 Generalities

Let (X, ω, J) be a Kähler manifold prequantized by (L, h^L, ∇^L) in the sense of (1.6), with bounded geometry at infinity in the sense of Section 2.1 and endowed with a compatible

S^1-action as in Section 2.3. We assume in addition that S^1 acts freely on the level set $\mu^{-1}(0) \subset X$ of its Kostant moment map $\mu \in \mathscr{C}^\infty(X,\mathbb{R})$ introduced in Definition 2.7, and that it satisfies the following.

Definition 4.1. We say that $\mu \in \mathscr{C}^\infty(X,\mathbb{R})$ has *polynomial growth* if it is proper, bounded from below and and if there exists $\delta > 0$, $C > 0$ and $N \in \mathbb{N}$ such that for any $x \in X$

$$\mathrm{Vol}(\{\mu \leqslant \mu(x)\}) \leqslant C\,|\mu(x)|^N \quad \text{and} \quad |d\mu_x| \leqslant C\,|\mu(x)|^{1-\delta}\,. \tag{4.1}$$

Note that in the case $X = \mathbb{C}^n$, any positive polynomial in the coordinates of $\mathbb{C}^n$ has polynomial growth in the sense of Definition 4.1. We can then establish the following result from Proposition 3.2, which can be understood as a weak version of the principle of *Quantization commutes with Reduction* for non-compact manifolds established by Ma and Zhang in [22].

Proposition 4.2. *Assume that the Kostant moment map $\mu \in \mathscr{C}^\infty(X,\mathbb{R})$ has polynomial growth in the sense of Definition 4.1. Then there exists $p_0 \in \mathbb{N}$ and $M \in \mathbb{N}$ such that for any $p \geqslant p_0$ and any $m \in \mathbb{Z}$ satisfying $m < -p\,M$, we have*

$$H^0_{(2)}(X,L^p)_m = \{0\}\,. \tag{4.2}$$

In particular, the subspace $\mathscr{H}_p \subset H^0_{(2)}(X,L^p)$ defined by (1.11) satisfies

$$\mathscr{H}_p = \bigoplus_{m=-M\,p}^{m=0} H^0_{(2)}(X,L^p)_m\,, \tag{4.3}$$

and has finite dimension $N_p := \dim \mathscr{H}_p \in \mathbb{N}$.

Proof. Assume that the moment map $\mu \in \mathscr{C}^\infty(X,\mathbb{R})$ has polynomial growth in the sense of Definition 4.1. In particular, it is bounded from below, and we can pick $M \in \mathbb{N}$ such that for all $x \in X$, we have $\mu(x) > -M+1$. Note also from (1.7) and (2.14) that the fundamental vector field (2.12) satisfies $|\xi_x| = |d\mu_x|$. Then Definition 4.1 and Proposition 3.2 imply the existence of constants δ, $C > 0$ such that for any $k \in \mathbb{N}$, there is a constant $C_k > 0$ such that for any $p \in \mathbb{N}$, any $m \in \mathbb{N}$ satisfying $m < -p\,M$ and any $q \in \mathbb{Z}$ satisfying $q \geqslant -M+1$, we have

$$\begin{aligned}\int_{\{q\leqslant\mu<q+1\}} |P_p^{(m)}(x,x)|_p\,dv_X(x) &\leqslant C_k\,p^{n-\frac{k}{2}} \int_{\{q\leqslant\mu<q+1\}} |\mu+M|^{-k}\,|d\mu|^k\,dv_X \\ &\leqslant C_k\,C\,p^{n-\frac{k}{2}}|q+1|^N|q+M|^{-k}|q+1|^{k(1-\delta)}\,.\end{aligned} \tag{4.4}$$

Taking $k > (N+2)/\delta$ and summing over all $q \in \mathbb{Z}$ satisfying $q \geqslant -M+1$, by standard properties of smooth Schwartz kernels of the orthogonal projection $P_p^{(m)} : L^2(X,L^p) \to H^0_{(2)}(X,L^p)_m$, for any $k \in \mathbb{N}$, we get a constant $C_k > 0$ such that for all $p \in \mathbb{N}$ and all $m \in \mathbb{Z}$ satisfying $m < -p\,M$, we have

$$\begin{aligned}\dim H^0_{(2)}(X,L^p)_m &= \int_X P_p^{(m)}(x,x)\,dv_X(x) \\ &= \sum_{q=-M+1}^{\infty} \int_{\{q\leqslant\mu<q+1\}} P_p^{(m)}(x,x)\,dv_X(x) \leqslant C_k\,p^{n-\frac{k}{2}}\,.\end{aligned} \tag{4.5}$$

Taking $k > 2n$, we can choose $p_0 \in \mathbb{N}$ such that for all $p \geqslant p_0$ and all $m \in \mathbb{Z}$ satisfying $m < -M\,p$, we have $C_k\,p^{n-\frac{k}{2}} < 1$, so that $\dim H^0_{(2)}(X, L^p)_m = 0$. This implies the identities (4.2) and (4.3) by definition (1.11) of the subspace $\mathscr{H}_p \subset H^0_{(2)}(X, L^p)$.

Finally, to show that the space (1.11) is finite-dimensional, it suffices to show that for any $m \leqslant 0$, the weight space $H^0_{(2)}(X, L^p)_m$ is finite-dimensional. Fix then $m \in \mathbb{N}$ such that $m \leqslant 0$. For any $k \in \mathbb{N}$, Proposition 3.2 and Definition 4.1 imply as in (4.4) the existence of a constant $\delta > 0$ such that for any $k \in \mathbb{N}$, there exists a constant $C_k > 0$ such that for any $p \in \mathbb{N}$ and any $q \in \mathbb{N}$ satisfying $q \geqslant 1$, we have

$$\int_{\{q \leqslant \mu < q+1\}} P^{(m)}_p(x, x)\, dv_X(x) \leqslant C_k\, p^{-k}(q+1)^N q^{-\delta k}\,. \tag{4.6}$$

Taking $k > (N+2)/\delta$, we obtain that the sum of (4.6) for all $q \in \mathbb{N}$ satisfying $q \geqslant 1$ converges, so that $\dim H^0_{(2)}(X, L^p)_m < +\infty$ by the first line of (4.5) and using the fact that $\{\mu \leqslant 1\} \subset X$ is compact. □

Let us now fix $p \in \mathbb{N}$ large enough so that the conclusions of Proposition 4.2 hold, set $N_p := \dim \mathscr{H}_p \in \mathbb{N}$, and consider an orthonormal basis $\{s_j \in \mathscr{H}_p\}_{j=0}^{N_p}$ with respect to the L^2-Hermitian product (2.1). We define the *Slater determinant* $\det s_p \in \mathscr{C}^\infty(X^{N_p}, (L^p)^{\boxtimes N_p})$ as the section of $(L^p)^{\boxtimes N_p}$ over the N_p-fold product X^{N_p} given for any $(x_1, x_2, \cdots x_{N_p}) \in X^{N_p}$ by

$$(\det s_p)(x_1, x_2, \cdots x_{N_p}) := \det(s_j(x_i))_{i,j=1}^{N_p}\,, \tag{4.7}$$

which does not depend on the choice of orthonormal basis of $\mathscr{H}_p$. Let us endow $(L^p)^{\boxtimes N_p}$ with the Hermitian metric induced by h^L, and write $|\cdot|_p$ for the induced pointwise norm. The following fundamental Lemma from the basic theory of *determinantal point processes* can be found for instance in [2, §. 4.2.3], following the adaptation to prequantized Kähler manifolds due to Berman in [4, § 5.1].

Lemma 4.3. *For any $p \in \mathbb{N}$ and any $(x_1, x_2, \cdots x_{N_p}) \in X^{N_p}$, the Slater determinant (4.7) satisfies the following formula,*

$$|(\det s_p)(x_1, x_2, \cdots x_{N_p})|_p^2 = \det(P^{(-)}_p(x_i, x_j))_{i,j=1}^{N_p}\,, \tag{4.8}$$

and the measure $d\nu_{N_p}$ over X_{N_p} given by

$$d\nu_{N_p} := \frac{1}{N_p!}\,|\det s_p|_p^2\;dv_X^{N_p}\,, \tag{4.9}$$

defines a probability measure over X_{N_p}, called the determinantal point process *associated with the Slater determinant (4.7).*

Proof. Let us first recall that, by standard properties of the Schwartz kernel of the orthogonal projection $P^{(-)}_p : L^2(X, L^p) \to \mathscr{H}_p$ on the subspace $\mathscr{H}_p \subset L^2(X, L^p)$ of finite dimension $N_p \in \mathbb{N}$, for any orthonormal basis $\{s_j \in \mathscr{H}_p\}_{j=1}^{N_p}$ and any $x, y \in X$, we have

$$P^{(-)}_p(x, y) = \sum_{j=1}^{N_p} s_j(x) \otimes s_j(y)^* \in L^p_x \otimes (L^p_y)^*\,. \tag{4.10}$$

One then readily compute for any $(x_1, x_2, \cdots x_{N_p}) \in X^{N_p}$ that

$$|(\det s_p)(x_1, x_2, \cdots x_{N_p})|_p^2 = \det(s_j(x_i))_{i,j=1}^{N_p} \overline{\det(s_j(x_i))_{i,j=1}^{N_p}} = \det(P_p^{(-)}(x_i, x_j))_{i,j=1}^{N_p}, \tag{4.11}$$

which establishes (4.8).

On the other hand, the fact that (4.9) defines a probability measure is a straightforward consequence of the following *Andreief formula* from [4, (5.8)], valid for any basis $\{s_j \in \mathscr{H}_p\}_{j=1}^{N_p}$ and which can be adapted for instance from the proof of [2, Lem. 3.2.3],

$$\int_{X^{N_k}} \left|\det(s_j(x_i))_{i,j=1}^{N_p}\right|_p dv_X(x_1) \cdots dv_X(x_{N_k}) = N_p! \det\left(\langle s_i, s_j \rangle_p\right)_{i,j=1}^{N_p}. \tag{4.12}$$

Since the basis $\{s_j \in \mathscr{H}_p\}_{j=1}^{N_p}$ is orthonormal, this establishes the result. □

For any bounded measurable function $f \in L^\infty(X, \mathbb{R})$ and any $p \in \mathbb{N}$, let us consider the associated *linear statistics*, which is the random variable over $(X^{N_p}, d\nu_{N_p})$ defined by the function

$$\begin{aligned} \mathcal{N}_p[f] : X^{N_p} &\longrightarrow \mathbb{R} \\ (x_j)_{j=1}^{N_p} &\longmapsto \sum_{j=1}^{N_p} f(x_j). \end{aligned} \tag{4.13}$$

Recall that for all $x \in X$, we have $P_p(x, x) > 0$ through the canonical identification $L_x^p \otimes (L_x^p)^* \simeq \mathbb{C}$. The following explicit formulas for the expectation $\mathbb{E}[\mathcal{N}_p[f]]$ and the variance $\mathbb{V}[\mathcal{N}_p[f]]$ of the random variable (4.13) are straightforward concequences of the explicit formula for correlations functions of determinantal point processes as defined for instance in [12, Def. 4.2.1], which can be obtained following the proof of [12, Lem. 4.5.1].

Proposition 4.4. *[4, Lem. 6.2] For any $f \in L^\infty(X, \mathbb{R})$ and $p \in \mathbb{N}$, the expectation and the variance of the random variable (4.13) with respect to the determinantal point process (4.9) are given by*

$$\mathbb{E}[\mathcal{N}_p[f]] = \int_X P_p^{(-)}(x, x)\, f(x)\, dv_X(x) \quad \textit{and}$$

$$\mathbb{V}[\mathcal{N}_p[f]] = \frac{1}{2} \int_X \int_X \left|P_p^{(-)}(x, y)\right|_p^2 (f(x) - f(y))^2\, dv_X(x)\, dv_X(y). \tag{4.14}$$

Example 4.5. Let us consider the model case $X = \mathbb{C}$ equipped with the trivial Hermitian line bundle $(L, h^L) = (\mathbb{C}, |\cdot|)$, endowed with the Hermitian connection ∇^L defined for all $z \in \mathbb{C}$ by

$$\nabla^L := d + \frac{\pi}{2}(z d\bar{z} - \bar{z} dz). \tag{4.15}$$

Its curvature satisfies the prequantization formula (1.6) for the standard symplectic form $\omega = \frac{\sqrt{-1}}{2} dz \wedge d\bar{z}$ on $X = \mathbb{C}$, and for any $p \in \mathbb{N}$, the Cauchy-Riemann operator associated with the induced holomorphic structure on $L^p = \mathbb{C}$ is given by $\nabla^{L^p}_{\frac{\partial}{\partial \bar{z}}} = \frac{\partial}{\partial \bar{z}} + p\,\frac{\pi}{2} z$, so that the global holomorphic sections of L^p are exactly the smooth sections $s \in \mathscr{C}^\infty(X, L^p)$ of the form $s(z) = f(z)\, e^{-p\frac{\pi}{2}|z|^2}$ for all $z \in \mathbb{C}$, where $f \in H^0(\mathbb{C}, \mathbb{C})$ is holomorphic. Let now $S^1 \subset \mathbb{C}$ act both on $X = \mathbb{C}$ and on $L = \mathbb{C}$ by complex multiplication, so that for any holomorphic section $s \in H^0(X, L^p)$ as above, for all $t \in S^1 \simeq \mathbb{R}/\mathbb{Z}$ and $z \in \mathbb{C}$, we have

$$\varphi_t^* s(z) = e^{-2\pi\sqrt{-1}pt} f(e^{2\pi\sqrt{-1}t} z) e^{-p\frac{\pi}{2}|z|^2}\,. \tag{4.16}$$

We thus see that for all $m \in \mathbb{Z}$ satisfying $m \geqslant -p$, the associated weight space (1.10) is the subspace $H^0_{(2)}(X, L^p)_m \subset H^0_{(2)}(X, L^p)$ generated by the section $s_m \in H^0_{(2)}(X, L^p)$ defined for all $z \in \mathbb{C}$ by

$$s_m(z) = z^{m-p}\, e^{-p\frac{\pi}{2}|z|^2}\,, \tag{4.17}$$

while $H^0_{(2)}(X, L^p)_m = \{0\}$ for all $m \in \mathbb{Z}$ satisfying $m < -p$. This shows that the finite dimensional space (1.11) used to define the determinantal point process (1.8) considered in Theorem 1.1 coincides with the subspace $\mathscr{H}_p \subset \mathscr{C}^\infty(X, L^p)$ generated by the orthonormal families of functions (1.4) used to define the *Ginibre ensemble*, after the change of variable $z \mapsto z/\sqrt{\pi}$ and setting $N := p + 1$.

On the other hand, the associated Kostant moment map $\mu \in \mathscr{C}^\infty(X, \mathbb{R})$ as in Definition 2.7 is given for any $z \in \mathbb{C}$ by $\mu(z) = \pi\,|z|^2 - 1$. Thus after the change of variable $z \mapsto z/\sqrt{\pi}$, the law of large numbers (1.13) in Theorem 1.1 recovers the classical result of Ginibre in [11, § 1], stating that this determinantal point process admits the measure $d\nu := \mathbb{1}_{\mathbb{D}}\, dv_{\mathbb{C}}$ as an equilibrium measure, while the asymptotics (1.14) for its variance recovers the asymptotics (1.5) of Rider and Virag in [25] describing the behavior of its fluctuations, as well as the corresponding central limit theorem.

4.2 Law of large Numbers

Consider the setting of Section 4.1, and for any $f \in L^\infty(X, \mathbb{R})$, let us consider the linear statistics (4.13) as a random variable with respect to the determinantal point process introduced in Lemma 4.3. The first part of Theorem 1.1 is a consequence of the following *law of large numbers*, which is the first main result of this section.

Theorem 4.6. *For any $f \in L^\infty(X, \mathbb{R})$, the expectation of the linear statistics* (4.13) *is finite for all $p \in \mathbb{N}$, and satisfies the following asymptotics as $p \to \infty$,*

$$\mathbb{E}\left[\mathcal{N}_p[f]\right] = p^n \int_{\{\mu<0\}} f(x)\, dv_X(x) + o(p^n)\,. \tag{4.18}$$

Furthermore, we have the following convergence in probability as $p \to \infty$,

$$\frac{1}{N_p}\mathcal{N}_p[f] \xrightarrow{p\to\infty} \frac{1}{\mathrm{Vol}\left(\{\mu < 0\}\right)} \int_{\{\mu<0\}} f\, dv_X\,. \tag{4.19}$$

If in addition $f \in L^\infty(X, \mathbb{R})$ is continuous with compact support, there exists $C > 0$ such that for any $\varepsilon > 0$ and $p \in \mathbb{N}$, we have

$$\mathbb{P}\left(\left\{(x_j)_{j=1}^{N_p} \in X^{N_p} \;\middle|\; \left|\frac{1}{N_p}\mathcal{N}_p[f] - \frac{1}{\mathrm{Vol}\left(\{\mu < 0\}\right)} \int_{\{\mu<0\}} f\, dv_X\right| > \varepsilon\right\}\right) \leqslant \frac{C}{\varepsilon p^n}\,. \tag{4.20}$$

Proof. Recall that $\mu \in \mathscr{C}^\infty(X,\mathbb{R})$ has polynomial growth in the sense of Definition 4.1, let $\varepsilon_0 > 0$ be as in definition (3.16) of the smooth family of charts used in Sections 3.2 and 3.3, and pick $\varepsilon \in]0,\varepsilon_0[$. Recall also that for any $p \in \mathbb{N}$ and all $x \in X$, we have $P_p(x,x) > 0$ through the canonical identification $L_x^p \otimes (L_x^p)^* \simeq \mathbb{C}$. By Proposition 4.4, for any $p \in \mathbb{N}$ and any positive function $f \in L^\infty(X,\mathbb{R})$, we have the following identity in $[0,+\infty]$,

$$\begin{aligned}\mathbb{E}[\mathcal{N}_p[f]] &= \int_X P_p^{(-)}(x,x)\, f(x)\, dv_X(x)\\ &= \int_{\{\mu<-\varepsilon p^{\frac{-1+\varepsilon}{2}}\}} P_p^{(-)}(x,x)\, f(x)\, dv_X(x) + \int_{\{|\mu|\leqslant\varepsilon p^{\frac{-1+\varepsilon}{2}}\}} P_p^{(-)}(x,x)\, f(x)\, dv_X(x)\\ &\qquad + \int_{\{\mu>\varepsilon p^{\frac{-1+\varepsilon}{2}}\}} P_p^{(-)}(x,x)\, f(x)\, dv_X(x)\,. \end{aligned} \tag{4.21}$$

We will show that all terms of (4.21) are finite, and that the last one is negligible as $p \to \infty$. Let us first deal with the last term. By Theorem 3.5 and the properness of $\mu \in \mathscr{C}^\infty(X,\mathbb{R})$, for any $k \in \mathbb{N}$, we get a constant $C_k > 0$ such that for any $p \in \mathbb{N}$, we have

$$\int_{\{\varepsilon p^{\frac{-1+\varepsilon}{2}}<\mu<2\}} P_p^{(-)}(x,x)\, f(x)\, dv_X(x) \leqslant C_k\, p^{-k} \sup_{x\in X} f(x)\,, \tag{4.22}$$

while on the other hand, using Proposition 4.2, we can apply Proposition 3.2 as in formula (4.6) to show that there exist M, $N \in \mathbb{N}$ and $\delta > 0$ such that for any q, $p \in \mathbb{N}$ satisfying $q \geqslant 1$, we have

$$\begin{aligned}\int_{\{q\leqslant\mu<q+1\}} P_p^{(-)}(x,x)\, f(x)\, dv_X(x) &= \sum_{m=-p\,M}^{m=0} \int_{\{q\leqslant\mu<q+1\}} P_p^{(m)}(x,x)\, f(x)\, dv_X(x)\\ &\leqslant M\, C_k\, p^{-k+1}(q+1)^N q^{-\delta k} \sup_{x\in X} f(x), \end{aligned} \tag{4.23}$$

Taking $k \in \mathbb{N}$ large enough and summing (4.22) with (4.23) over all $q \in \mathbb{N}$ satisfying $q > 2$, for any $k \in \mathbb{N}$, we get a constant $C_k > 0$ such that for any $p \in \mathbb{N}$, we have the following estimate,

$$\int_{\{\mu>\varepsilon p^{\frac{-1+\varepsilon}{2}}\}} P_p^{(-)}(x,x)\, f(x)\, dv_X(x) \leqslant C_k\, p^{-k} \sup_{x\in X} f(x)\,. \tag{4.24}$$

This implies that the third term of formula (4.21) for the expectation $\mathbb{E}[\mathcal{N}_p[f]] \in [0,+\infty]$ is negligible as $p \to \infty$, and since $\{\mu < \varepsilon p^{\frac{-1+\varepsilon}{2}}\} \subset X$ is compact by Definition 4.1, it implies in particular that the expectation $\mathbb{E}[\mathcal{N}_p[f]] > 0$ is finite.

To deal with the second term of (4.21), recall that $\{|\mu| \leqslant \varepsilon p^{\frac{-1+\varepsilon}{2}}\} \subset X$ is compact and formed of regular values of $\mu \in \mathscr{C}^\infty(X,\mathbb{R})$ by definition of $\varepsilon > 0$ from (3.14) and (3.16). Then Theorem 3.6 implies the existence of a constant $C > 0$ such that for all $p \in \mathbb{N}$ and all $x \in X$, we have

$$\int_{\{|\mu|\leqslant\varepsilon p^{\frac{-1+\varepsilon}{2}}\}} P_p^{(-)}(x,x)\, f(x)\, dv_X(x) \leqslant C\, p^n p^{\frac{-1+\varepsilon}{2}} \sup_{x\in X} f(x)\,, \tag{4.25}$$

Hence from Corollary 2.6, Theorem 3.5 and the estimate (4.24) for the third term of (4.21), we get the following asymptotic estimate as $p \to \infty$,

$$\begin{aligned} p^{-n}\,\mathbb{E}[\mathcal{N}_p[f]] &= p^{-n}\int_{\{\mu<-\varepsilon p^{\frac{-1+\varepsilon}{2}}\}} P_p(x,x)\,f(x)\,dv_X(x) + O(p^{\frac{-1+\varepsilon}{2}}) \\ &= \int_{\{\mu<-\varepsilon p^{\frac{-1+\varepsilon}{2}}\}} f(x)\,dv_X(x) + O(p^{\frac{-1+\varepsilon}{2}})\,. \end{aligned} \tag{4.26}$$

Using once more that $\{|\mu| \leqslant \varepsilon p^{\frac{-1+\varepsilon}{2}}\} \subset X$ is compact and formed of regular values of $\mu \in \mathscr{C}^\infty(X,\mathbb{R})$, this implies the asymptotics (4.18) for the expectation of the linear statistics (4.13) of a bounded positive function $f \in L^\infty(X,\mathbb{R})$, hence of any $f \in L^\infty(X,\mathbb{R})$ by linearity.

Let us now establish the law of large numbers (4.19). Using the fact that for all $p \in \mathbb{N}$, we have $N_p = \mathbb{E}[\mathcal{N}_p[f]]$ by definition of the linear statistics (4.13), formula (4.18) gives

$$\mathbb{E}\left[\frac{1}{N_p}\mathcal{N}_p[f]\right] \xrightarrow{p\to\infty} \frac{1}{\mathrm{Vol}\left(\{\mu<0\}\right)}\int_{\{\mu<0\}} f\,dv_X\,. \tag{4.27}$$

On the other hand, the identity for the variance of $\mathcal{N}_p[f]$ given in Proposition 4.4 together with elementary properties of the smooth Schwartz kernel of the orthogonal projection $P_p^{(-)} : L^2(X,L^p) \to \mathscr{H}_p$ gives the following estimates for all $p \in \mathbb{N}$,

$$\begin{aligned} \mathbb{V}[\mathcal{N}_p[f]] &= \frac{1}{2}\int_X\int_X \left|P_p^{(-)}(x,y)\right|_p^2 (f(x)-f(y))^2\,dv_X(x)\,dv_X(y) \\ &\leqslant 2\sup_{x\in X}|f(x)|^2\int_X\int_X P_p^{(-)}(x,y).P_p^{(-)}(y,x)\,dv_X(y)\,dv_X(x) \\ &\leqslant 2\sup_{x\in X}|f(x)|^2\int_X P_p^{(-)}(x,x)\,dv_X(x) \leqslant 2\sup_{x\in X}|f(x)|^2\,N_p\,. \end{aligned} \tag{4.28}$$

Together with formula (4.18) applied to $N_p = \mathbb{E}[\mathcal{N}_p[1]]$ and elementary properties of the variance, this implies that the existence of a constant $C > 0$ such that for all $p \in \mathbb{N}$, we have

$$\begin{aligned} \mathbb{V}\left[\frac{1}{N_p}\mathcal{N}_p[f]\right] &= \frac{1}{N_p^2}\mathbb{V}\left[\mathcal{N}_p[f]\right] \\ &\leqslant C\,p^{-n}\,. \end{aligned} \tag{4.29}$$

The convergence in probability (4.19) then follows from the classical Chebyshev inequality, as in the standard proof of the weak law of large numbers for random variables with finite variance.

To get the more precise result (4.20) when $f \in \mathscr{C}^0_c(X,\mathbb{R})$ is continuous with compact support, let us use the identity for the variance of $\mathcal{N}_p[f]$ given in Proposition 4.4 to split it into two parts in the following way, for all $p \in \mathbb{N}$,

$$\begin{aligned} \mathbb{V}\left[\mathcal{N}_p[f]\right] &= \frac{1}{2}\int_X\int_X |P_p^{(-)}(x,y)|_p^2\,(f(x)-f(y))^2\,dv_X(x)\,dv_X(y) \\ &= \frac{1}{2}\int_{\{\mu<\varepsilon p^{\frac{-1+\varepsilon}{2}}\}}\int_X |P_p^{(-)}(x,y)|_p^2\,(f(x)-f(y))^2\,dv_X(x)\,dv_X(y) \\ &+ \frac{1}{2}\int_{\{\mu>\varepsilon p^{\frac{-1+\varepsilon}{2}}\}}\int_X |P_p^{(-)}(x,y)|_p^2\,(f(x)-f(y))^2\,dv_X(x)\,dv_X(y)\,. \end{aligned} \tag{4.30}$$

To deal with the second term, we can apply the estimate (4.24) and use elementary properties of the Schwartz kernel of the orthogonal projection $P_p^{(-)} : L^2(X, L^p) \to \mathscr{H}_p$ to get for any $k \in \mathbb{N}$ a constant $C_k > 0$ such that for all $p \in \mathbb{N}$, we have

$$\begin{aligned}
&\frac{1}{2}\int_{\{\mu > \varepsilon p^{\frac{-1+\varepsilon}{2}}\}}\int_X |P_p^{(-)}(x,y)|_p^2\,(f(x)-f(y))^2\,dv_X(x)\,dv_X(y) \\
&\leqslant 2\sup_{x\in X}|f(x)|^2\int_{\{\mu > \varepsilon p^{\frac{-1+\varepsilon}{2}}\}}\left(\int_X P_p^{(-)}(x,y).P_p^{(-)}(y,x)\,dv_X(y)\right)dv_X(x) \\
&\leqslant 2\sup_{x\in X}|f(x)|^2\int_{\{\mu > \varepsilon p^{\frac{-1+\varepsilon}{2}}\}} P_p^{(-)}(x,x)\,dv_X(x) \leqslant 2\,C_k\,p^{-k}\sup_{x\in X}|f(x)|^2\,.
\end{aligned} \tag{4.31}$$

On the other hand, for any $x \in X$, we can apply Theorem 2.2 and use the uniform continuity of $f \in \mathscr{C}_c^0(X, \mathbb{R})$ over its compact support to get, for any $\delta > 0$ and $k \in \mathbb{N}$, the existence of constants C, $C_k > 0$ and $p_0 \in \mathbb{N}$ such that for any $p \geqslant p_0$ and $x \in X$, we have

$$\begin{aligned}
\frac{1}{2}\int_X |P_p(x,y)|_p^2\,(f(x)-f(y))^2\,dv_X(y) &= \frac{1}{2}\int_{B^X(x,\varepsilon p^{\frac{-1+\varepsilon}{2}})}|P_p(x,y)|_p^2\,(f(x)-f(y))^2\,dv_X(y) \\
&\quad + \frac{1}{2}\int_{X\setminus B^X(x,\varepsilon p^{\frac{-1+\varepsilon}{2}})}|P_p(x,y)|_p^2\,(f(x)-f(y))^2\,dv_X(y) \\
&\leqslant \frac{1}{2}\left(\sup_{y\in B^X(x,\varepsilon p^{\frac{-1+\varepsilon}{2}})}|f(x)-f(y)|^2\right)\int_X |P_p(x,y)|_p^2\,dv_X(y) \\
&\quad + C_k\,p^{-k}\int_X |P_p(x,y)|_p\,|f(x)-f(y)|^2\,dv_X(y) \\
&\leqslant \delta\,P_p(x,x) + C\,C_k\,p^{n-k}\,2\sup_{x\in X}|f(x)|^2\ \mathrm{Vol}(\mathrm{Supp}\ f) \\
&\leqslant Cp^n\left(\delta + C_k\,p^{-k}\,2\sup_{x\in X}|f(x)|^2\ \mathrm{Vol}(\mathrm{Supp}\ f)\right).
\end{aligned} \tag{4.32}$$

Recalling that $\mu \in \mathscr{C}^\infty(X, \mathbb{R})$ is proper and bounded from below, using formula (4.18) applied to $N_p = \mathbb{E}[\mathcal{N}_p[1]]$ and elementary properties of the variance, we can then plug the estimates (4.31) and (4.32) inside (4.30) to get a constant $C > 0$ such that for any $\delta > 0$, there exists $p_0 \in \mathbb{N}$ such that for all $p \geqslant p_0$, we have

$$\mathbb{V}\left[\frac{1}{N_p}\mathcal{N}_p[f]\right] \leqslant C\,p^{-n}\,\mathrm{Vol}(\{\mu < \varepsilon\})\,\delta\,. \tag{4.33}$$

The inequality (4.20) in then a direct consequence of the classical Chebyshev inequality.

□

4.3 Central Limit Theorem

Consider the setting of Section 4.1, and for any $f \in L^\infty(X, \mathbb{R})$, let us consider the linear statistics (4.13) as a random variable with respect to the determinantal point process introduced in (1.1). Together with Theorem 4.6, the following result concludes the proof of Theorem 1.1.

Theorem 4.7. *For any $f \in \mathscr{C}^\infty_c(X,\mathbb{R})$, the variance of the associated linear statistics (4.13) satisfies the following asymptotics as $p \to \infty$,*

$$\lim_{p\to\infty} p^{-n+1}\mathbb{V}\left[\mathcal{N}_p[f]\right] = \frac{1}{4\pi}\int_{\{\mu<0\}} |df|^2\, dv_X(x) + \frac{1}{2}\int_{X_0}\sum_{k\in\mathbb{Z}} |k|\,|\hat{f}_k(x)|^2\, dv_{X_0}(x)\,, \quad (4.34)$$

and the random variable $N_p^\alpha(\mathcal{N}_p[f]-\mathbb{E}[\mathcal{N}_p[f]])$ with $\alpha = \frac{1}{2n}-\frac{1}{2}$ converges in distribution to a centered normal random variable with variance (4.34) as $p\to\infty$.

Proof. Recall that $\mu \in \mathscr{C}^\infty(X,\mathbb{R})$ has polynomial growth in the sense of Definition 4.1, and fix $f \in \mathscr{C}^\infty_c(X,\mathbb{R})$. Let $\varepsilon_0 > 0$ be as in definition (3.16) of the smooth family of charts used in Sections 3.2 and 3.3, and pick $\varepsilon \in]0,\varepsilon_0[$. Recalling the decomposition (4.30) of the formula for the variance of $\mathcal{N}_p[f]$ given by Proposition 4.4, the estimate (4.31) implies the following asymptotics as $p\to\infty$,

$$\begin{aligned} p^{-n+1}\,\mathbb{V}\left[\mathcal{N}_p[f]\right] &= \frac{1}{2}\,p^{-n+1}\int_{\{\mu<-\varepsilon p^{\frac{-1+\varepsilon}{2}}\}}\int_X |P_p^{(-)}(x,y)|_p^2\,(f(x)-f(y))^2\,dv_X(x)\,dv_X(y)\\ &+\frac{1}{2}\,p^{-n+1}\int_{\{|\mu|\leqslant\varepsilon p^{\frac{-1+\varepsilon}{2}}\}}\int_X |P_p^{(-)}(x,y)|_p^2\,(f(x)-f(y))^2\,dv_X(x)\,dv_X(y) + O(p^{-\infty})\,. \end{aligned} \quad (4.35)$$

We claim that the first and second terms of the decomposition (4.35) respectively converge to the first and second terms of the limit (4.34) as $p\to\infty$.

To deal with the first term of decomposition (4.35), first note from Theorem 3.5 that, since $f\in\mathscr{C}^\infty_c(X,\mathbb{R})$ has compact support, we have the following estimate as $p\to\infty$,

$$\begin{aligned} &\int_{\{\mu<-\varepsilon p^{\frac{-1+\varepsilon}{2}}\}}\int_X |P_p^{(-)}(x,y)|_p^2\,(f(x)-f(y))^2\,dv_X(x)\,dv_X(y)\\ &\qquad = \int_{\{\mu<-\varepsilon p^{\frac{-1+\varepsilon}{2}}\}}\int_X |P_p(x,y)|_p^2\,(f(x)-f(y))^2\,dv_X(x)\,dv_X(y) + O(p^{-\infty})\,. \end{aligned} \quad (4.36)$$

Consider now a bounded family of charts $\psi : B^{TX}(0,\varepsilon_0)\to X$ in the sense of Definition 2.3, and note that the Riemannian volume form satisfies $\psi_x^* dv_X = (1+O(|Z|))\,dZ$ uniformly in $x\in\operatorname{Supp} f$ for all $Z\in T_xX$ satisfying $|Z|<\varepsilon_0$, where dZ is the Lebesgue measure of $(T_xX,|\cdot|)$. Let now $x\in X$, pick an orthonormal basis $(e_j\in T_xX)_{j=1}^n$, and for any $Z\in T_xX$, write $Z=\sum_{j=1}^n Z_j\,e_j$ with $Z_j\in\mathbb{R}$ for all $1\leqslant j\leqslant n$. We can then apply Theorem 2.2 as in Corollary 2.4 together with Theorem 2.5 to get $\varepsilon_0>0$ such that for all $\varepsilon\in]0,\varepsilon_0[$, the following estimates hold as $p\to\infty$,

$$\begin{aligned} &\frac{1}{2}\,p^{-n+1}\int_X |P_p(x,y)|_p^2\,(f(x)-f(y))^2\,dv_X(y)\\ &= \frac{1}{2}\,p^{-n+1}\int_{B^{T_xX}(0,\varepsilon p^{\frac{-1+\varepsilon}{2}})} |P_{p,x}(0,Z)|_p^2\,(f_x(0)-f_x(Z))^2\,dv_X(y) + O(p^{-\infty})\\ &= \frac{1}{2}\,p^{n+1}\int_{B^{T_xX}(0,\varepsilon p^{\frac{\varepsilon}{2}})} |P_{p,x}(0,Z/\sqrt{p})|_p^2\left(\sum_{j=1}^{2n}\left(\frac{Z_j}{\sqrt{p}}\frac{\partial f_x}{\partial Z_j}(0)\right)^2 + O(|Z/\sqrt{p}|^3)\right)dZ + O(p^{-\infty})\\ &= \frac{1}{2}\sum_{j=1}^{2n}\left(\frac{\partial f_x}{\partial Z_j}(0)\right)^2\int_{\mathbb{R}^{2n}}\exp(-\pi\,|Z|^2)\,Z_j^2\,dZ + o(1) = \frac{1}{4\pi}\,|df_x|^2 + o(1)\,, \end{aligned} \quad (4.37)$$

and this estimate is uniform in $x \in X$ since $f \in \mathscr{C}_c^\infty(X,\mathbb{R})$ has compact support. Using further that for all $\varepsilon \in]0,\varepsilon_0[$ and $p \in \mathbb{N}$, the subset $\{|\mu| \leqslant \varepsilon p^{\frac{-1+\varepsilon}{2}}\} \subset X$ is compact and formed of regular values of $\mu \in \mathscr{C}^\infty(X,\mathbb{R})$, we deduce from (4.36) and (4.37) the following estimates for the first term in the decomposition (4.35) as $p \to \infty$,

$$\frac{1}{2}\,p^{-n+1}\int_{\{\mu<-\varepsilon p^{\frac{-1+\varepsilon}{2}}\}}\int_X |P_p^{(-)}(x,y)|_p^2\,(f(x)-f(y))^2\,dv_X(x)\,dv_X(y) = \frac{1}{4\pi}\int_{\{\mu<0\}}|df|^2\,dv_X(x)+o(1)\,. \tag{4.38}$$

This identifies the first term of the right-hand side of the asymptotics (4.35) with the first term of the right-hand side of the asymptotics (4.34).

Let us now focus on the second term in the decomposition (4.35) for the variance. To that end, let $x_0 \in \mu^{-1}(0)$, and recall the local section $V_0 \subset \mu^{-1}(0)$ of the S^1-principal bundle $\pi : \mu^{-1}(0) \to X_0 := \mu^{-1}(0)/S^1$ defined in (3.13). Then the image $V \subset X$ of the embedding $\phi : I_0 \times V_0 \to X$ defined by (3.14) is a local section of the quotient map

$$\pi : U \to B := U/S^1\,, \tag{4.39}$$

where the open set $U \subset X$ is the image of the open embedding $\Phi : S^1 \times I_0 \times V_0 \to X$ defined by (3.15). From the definition (3.17) of the horizontal tangent bundle, we see that for any $x \in U$, the differential $d\pi_x : T_xX \to T_{\pi(x)}B$ induces an isomorphism $T_x^HX \simeq T_{\pi(x)}B$ by restriction. Since the S^1-action preserves the Riemannian metric g^{TX}, there is a unique Riemannian metric g^B on B such that $\pi^*g^B = g^{TX}|_{T^HX}$, and recalling the fundamental vector field (2.12), the volume form dv_B of (B, g^B) satisfies

$$dv_X = |\xi|\,dt\,\pi^*dv_B\,, \tag{4.40}$$

where dt is the Lebesgue measure of $S^1 \simeq \mathbb{R}/\mathbb{Z}$.

Let now $\varrho \in \mathscr{C}^\infty(X_0,\mathbb{R})$ have compact support in $\pi(V_0) \subset X_0$, and let $\widetilde{\varrho} \in \mathscr{C}_c^\infty(X,\mathbb{R})$ be the S^1-invariant function with compact support in $U \subset X$ defined for any $t \in S^1$, $u \in I_0$ and $x \in V_0$ by

$$\widetilde{\varrho}\,(\Phi(t,u,x)) := \varrho\,(\pi(x))\,. \tag{4.41}$$

For any $\varepsilon \in]0,\varepsilon_0[$, write

$$V(\varepsilon) := V \cap \{|\mu| \leqslant \varepsilon\} \subset U\,. \tag{4.42}$$

Then for any $\varepsilon \in]0,\varepsilon_0[$ and $p \in \mathbb{N}$, since S^1 acts unitarily on (L, h^L) via $\varphi_{t,p} : L^p \to L^p$ for all $t \in S^1 \simeq \mathbb{R}/\mathbb{Z}$, we get that

$$\begin{aligned}\int_{\{|\mu|\leqslant\varepsilon p^{\frac{-1+\varepsilon}{2}}\}}\int_X |P_p^{(-)}(x,y)|_p^2\,(f(x)-f(y))^2\,\widetilde{\varrho}(x)\,dv_X(x)\,dv_X(y) \\ = \int_{V(\varepsilon p^{\frac{-1+\varepsilon}{2}})}\int_V\int_{S^1}\int_{S^1}|\varphi_{t,p}^{-1}P_p^{(-)}(\varphi_t(x),\varphi_u(y))\varphi_{u,p}|_p^2 \\ (f(\varphi_t(x))-f(\varphi_u(y)))^2\,\widetilde{\varrho}(x)\,|\xi_x|\,|\xi_y|\,dt\,du\,\pi^*dv_B(x)\,\pi^*dv_B(y)\,.\end{aligned} \tag{4.43}$$

Now for all $x,\, y \in X$ and $t,\, u \in \mathbb{R}$, through the canonical isomorphism $L^p_x \otimes (L^p_x)^* \simeq \mathbb{C}$, Propositions 3.1 and 4.2 imply the existence of $M \in \mathbb{N}$ such that

$$\begin{aligned}|\varphi_{t,p}^{-1} P_p^{(-)}(\varphi_t(x), \varphi_u(y))\varphi_{u,p}|_p^2 \\ &= \left(\varphi_{t,p}^{-1} P_p^{(-)}(\varphi_t(x), \varphi_u(y))\varphi_{u,p}\right) . \left(\varphi_{t,p}^{-1}\, P_p^{(-)}(\varphi_t(y), \varphi_u(x))\varphi_{u,p}\right) \\ &= \sum_{m,\, r=-Mp}^{m,\, r=0} e^{2\pi\sqrt{-1}t(m-r)} e^{2\pi\sqrt{-1}u(r-m)}\, P^{(m)}(x,y).P^{(r)}(y,x)\,. \end{aligned} \tag{4.44}$$

By standard Fourier theory from notation (1.12) for the Fourier coefficients, this implies that for any functions $g,\, h \in \mathscr{C}^\infty(X, \mathbb{C})$ and any $x,\, y \in U$, we have

$$\begin{aligned}\int_{S^1}\int_{S^1} |\varphi_{t,p}^{-1} P_p^{(-)}(\varphi_t(x), \varphi_s(y))\varphi_{u,p}|_p^2\, g(\varphi_t(x))\, h(\varphi_u(y))\, dt\, du \\ = \sum_{m,\, r=-Mp}^{m,\, r=0} P^{(m)}(x,y).P^{(r)}(y,x)\, \hat{g}_{r-m}(x)\, \hat{h}_{m-r}(y)\,. \end{aligned} \tag{4.45}$$

Furthermore, since $f \in \mathscr{C}_c^\infty(X, \mathbb{R})$ is smooth and compactly supported for any $r \in \mathbb{Z}$, we have that $\widehat{f^2_r}(x) = \sum_{k\in\mathbb{Z}} \hat{f}_k(x)\hat{f}_{r-k}(x)$ with uniform convergence in $x \in U$, and that $\widehat{f}_r(x) = \overline{\widehat{f}_{-r}(x)}$ since f assumes real values. Then applying formula (4.45) to each term of $(f(x) - f(y))^2 = f(x)^2 - 2f(x)f(y) + f(y)^2$ for any $x,\, y \in U$ and using the fact from Proposition 4.2 that $P_p^{(m)} \equiv 0$ for all $m < -Mp$, one gets

$$\begin{aligned}\int_{S^1}\int_{S^1} |\varphi_{t,p}^{-1} P_p^{(-)}(\varphi_t(x), \varphi_u(y))\varphi_{u,p}|_p^2\, \left(f(\varphi_t(x)) - f(\varphi_u(y))\right)^2\, dt\, du \\ = \sum_{m=-Mp}^{m=0} |P^{(m)}(x,y)|_p^2 \sum_{k\in\mathbb{Z}} \left(|\hat{f}_k(x)|^2 + |\hat{f}_k(y)|^2\right) \\ - 2 \sum_{m,\, r=-Mp}^{m,\, r=0} P^{(m)}(x,y).P^{(r)}(y,x)\, \hat{f}_{r-m}(x)\, \overline{\hat{f}_{m-r}(y)} \\ = \sum_{m=-Mp}^{m=0} |P^{(m)}(x,y)|_p^2 \sum_{k\in\mathbb{Z}} \left|\hat{f}_k(x) - \hat{f}_k(y)\right|^2 \\ - 2\sum_{k\leqslant 0}\sum_{m=-Mp}^{m=0} \left(P^{(m)}(x,y).P^{(m+k)}(y,x) - |P^{(m)}(x,y)|_p^2\right)\, \hat{f}_k(x)\, \overline{\hat{f}_k(y)} \\ - 2\sum_{k<0}\sum_{m=-Mp}^{m=0} \left(P^{(m+k)}(x,y).P^{(m)}(y,x) - |P^{(m)}(x,y)|_p^2\right)\, \hat{f}_{-k}(x)\, \overline{\hat{f}_{-k}(y)}\,, \end{aligned} \tag{4.46}$$

where all sums in $k \in \mathbb{Z}$ converge uniformly in $x,\, y \in U$. Plugging (4.46) into the expression (4.43), we then get the following identities,

$$\begin{aligned}\frac{1}{2}\,p^{-n+1}&\int_{\{|\mu|<\varepsilon p^{\frac{-1+\varepsilon}{2}}\}}\int_X |P_p^{(-)}(x,y)|_p^2\,(f(x)-f(y))^2\,\widetilde{\varrho}(x)\,dv_X(x)\,dv_X(y)\\
&= p^{-n+1}\int_{V(\varepsilon p^{\frac{-1+\varepsilon}{2}})}\int_V \Bigg(\frac{1}{2}\sum_{m=-Mp}^{m=0}|P^{(m)}(x,y)|_p^2\sum_{k\in\mathbb{Z}}\left|\hat{f}_k(x)-\hat{f}_k(y)\right|^2\\
&-\sum_{k\leqslant 0}\sum_{m=-Mp}^{m=0}\Big(P^{(m)}(x,y).P^{(m+k)}(y,x)-|P^{(m)}(x,y)|_p^2\Big)\,\hat{f}_{-k}(x)\,\overline{\hat{f}_{-k}(y)}\\
&-\sum_{k<0}\sum_{m=-Mp}^{m=0}\Big(P^{(m+k)}(x,y).P^{(m)}(y,x)-|P^{(m)}(x,y)|_p^2\Big)\,\hat{f}_k(x)\,\overline{\hat{f}_k(y)}\Bigg)\\
&\qquad\qquad\widetilde{\varrho}(x)\,|\xi_x|\,|\xi_y|\,\pi^*dv_B(x)\,\pi^*dv_B(y)\,. \qquad (4.47)\end{aligned}$$

We will show that the integral of the first term in the integrand of (4.47) is negligible as $p\to\infty$, while the integrals of the two other terms lead to the boundary term in formula (4.34). To do so, consider the trivialization in normal coordinates described in Section 3.2, let us fix $k\leqslant 0$, and recall that $\mu^{-1}(0)\subset X$ is compact by properness of $\mu\in\mathscr{C}^\infty(X,\mathbb{R})$. By a computation strictly analogous to (3.29) and using the estimates (3.35) and (3.36), Theorem 3.3 implies that for any $\delta>0$, there is $\varepsilon_0>0$ such that for all $\varepsilon\in\,]0,\varepsilon_0[$, we have the following asymptotics as $p\to\infty$, uniform in $x\in\mu^{-1}(0)$,in $k\in\mathbb{Z}$ satisfying $k\leqslant 0$ and in $Z,\,Z'\in T_x^HX$ satisfying $|Z|,\,|Z'|<\varepsilon p^{\frac{-1+\varepsilon}{2}}$,

$$\begin{aligned}p^{-2n+1}&\sum_{m=-Mp}^{m=0}P_{p,x}^{(m)}(Z,Z')\,P_{p,x}^{(m+k)}(Z',Z)\\
&=\sum_{m\leqslant 0}\mathscr{P}_x^{(m)}(\sqrt{p}Z,\sqrt{p}Z')\mathscr{P}_x^{(m+k)}(\sqrt{p}Z',\sqrt{p}Z)+O(p^{-\frac{1}{2}+\delta})\,,\end{aligned}\qquad (4.48)$$

Now, to deal with the first term of (4.47), first note from (1.12) that by standard Fourier theory, we have the following uniform convergence in $x,\,y\in U$,

$$\sum_{k\in\mathbb{Z}}\left|\hat{f}_k(x)-\hat{f}_k(y)\right|^2=\int_0^1\Big(f(\varphi_t(x))-f(\varphi_t(y))\Big)^2\,dt\,. \qquad (4.49)$$

Recall the bounded family of charts (3.16) and formula (4.40) for the volume form dv_B of the base (B,g^B) defined by (4.39). Recalling the embedding $\phi:I_0\times V_0\xrightarrow{\sim}V\subset X$ defined by (3.14), Let $h\in\mathscr{C}^\infty(V,\mathbb{R})$ be the positive function satisfying

$$\phi^*\pi^*dv_B|_V=:h\,du\,\pi^*dv_{X_0}\,, \qquad (4.50)$$

where du is the Lebesgue measure of $\mathbb{R}$, so that $h|_{V_0}\equiv 1$ by construction. Finally, note from (4.42) that for any $\varepsilon\in\,]0,\varepsilon_0[$, we have

$$V(\varepsilon)=\{\psi_x(u\,e_1)\in X\mid x\in V_0,\,|u|<\varepsilon\}\,. \qquad (4.51)$$

Setting $\varepsilon' > 4\,\varepsilon\,\sup_{x\in\mu^{-1}(0)}|\xi_x|$, using Theorem 2.2, the asymptotics (4.48) with $k=0$ and the definition (4.41) of $\widetilde{\varrho}\in\mathscr{C}^\infty(X,\mathbb{R})$, we then get constants C, C_1, $C_2>0$ such that for all $p\in\mathbb{N}$, the following estimates hold,

$$
\begin{aligned}
& p^{-n+1}\int_{V(\varepsilon p^{\frac{-1+\varepsilon}{2}})}\int_V \sum_{m=-Mp}^{m=0} |P^{(m)}(x,y)|_p^2 \\
& \int_0^1 \Big(f(\varphi_t(x))-f(\varphi_t(y))\Big)^2\,dt\,\widetilde{\varrho}(x)\,|\xi_x|\,|\xi_y|\,\pi^*dv_B(y)\,\pi^*dv_B(x) \\
& = p^{-n+1}\int_{V_0}\varrho(\pi(x))\int_{-|\xi_x|\varepsilon p^{\frac{-1+\varepsilon}{2}}}^{|\xi_x|\varepsilon p^{\frac{-1+\varepsilon}{2}}}\bigg(\int_{B^{T_x^HX}(0,\varepsilon' p^{\frac{-1+\varepsilon}{2}})}\sum_{m=-Mp}^{m=0}|P_{p,x}^{(m)}(u\,e_1,Z)|_p^2 \\
& \int_{S^1}(\varphi_t^*f_x(u\,e_1)-\varphi_t^*f_x(Z))^2\,dt\,|\xi_{\psi_x(u\,e_1)}|\,|\xi_{\psi_x(Z)}|\,h_x(u\,e_1)\,h_x(Z)\,dZ\bigg)du\,\pi^*dv_{X_0}(x)+O(p^{-\infty}) \\
& \leqslant Cp^{-2n+1}\int_{V_0}\varrho(\pi(x))\int_{-|\xi_x|\varepsilon p^{\frac{\varepsilon}{2}}}^{|\xi_x|\varepsilon p^{\frac{\varepsilon}{2}}} \\
& \left(\int_{B^{T_x^HX}(0,\varepsilon' p^{\frac{\varepsilon}{2}})}\sum_{m=-Mp}^{m=0}|\mathscr{P}_x^{(m)}((u/\sqrt{p})e_1,Z/\sqrt{p})|^2\ \frac{|u\,e_1-Z|^2}{p}\,dZ\right)du\,\pi^*dv_{X_0}(x) \\
& \leqslant C_1\,p^{-\frac{1}{2}}\int_{V_0}\varrho(\pi(x))\int_{-|\xi_x|\varepsilon p^{\frac{\varepsilon}{2}}}^{|\xi_x|\varepsilon p^{\frac{\varepsilon}{2}}}\bigg(\int_{T_x^HX}\exp\left(-\pi|u\,e_1-Z|^2\right)\ |u\,e_1-Z|^2\,dZ\bigg)du\,\pi^*dv_{X_0}(x) \\
& \leqslant C_2\,p^{\frac{-1+\varepsilon}{2}}\,.
\end{aligned}
\tag{4.52}
$$

This shows that the integral of the first term appearing in the integrand of (4.47) tends to 0 as $p\to\infty$.

To deal with the other terms in formula (4.47), we will apply the Euler-Maclaurin formula to the asymptotics (4.48) as in the proof of Theorem 3.6. Namely, letting $I\subset]0,+\infty[$ be a compact interval, we compute the following Taylor expansion in $\frac{k}{\sqrt{p}}$ in the Euler-Maclaurin formula (3.32), uniform in $v\in\mathbb{R}$, $k\in\mathbb{Z}$, $p\in\mathbb{N}$ and $a\in I$,

$$
\begin{aligned}
& \sum_{m\in\mathbb{N}} e^{-2\pi\left[\left(v-\frac{m}{a\sqrt{p}}\right)^2+\left(v-\frac{m-k}{a\sqrt{p}}\right)^2\right]} \\
& = \int_0^\infty e^{-2\pi\left[\left(v-\frac{t}{a\sqrt{p}}\right)^2+\left(v-\frac{t-k}{a\sqrt{p}}\right)^2\right]}\,dt+a_0\,e^{-2\pi\left[v^2+\left(v-\frac{k}{a\sqrt{p}}\right)^2\right]}+O\Big(\frac{k}{\sqrt{p}}\Big) \\
& = a\sqrt{p}\int_{-\infty}^v e^{-4\pi t^2}\left(1-4\pi t\frac{k}{a\sqrt{p}}+O\Big(\frac{k^2}{p}\Big)\right)dt+a_0\,e^{-4\pi v^2}+O\Big(\frac{k}{\sqrt{p}}\Big) \\
& = a\sqrt{p}\int_{-\infty}^v e^{-4\pi t^2}\,dt+\frac{k}{2}\,e^{-4\pi v^2}+a_0\,e^{-4\pi v^2}+k\,O\Big(\frac{k}{\sqrt{p}}\Big)\,.
\end{aligned}
\tag{4.53}
$$

Substracting from formula (4.53) the same formula for $k=0$, we get the following asymptotics as $p \to \infty$, uniform in $k \in \mathbb{Z}$, $v \in \mathbb{R}$ and $a \in I$,

$$\sum_{m \in \mathbb{N}} e^{-2\pi\left[\left(v - \frac{m}{a\sqrt{p}}\right)^2 + \left(v - \frac{m-k}{a\sqrt{p}}\right)^2\right]} - \sum_{m \in \mathbb{N}} e^{-4\pi\left(v - \frac{m}{a\sqrt{p}}\right)^2} = \frac{k}{2}\, e^{-4\pi v^2} + k^2\, O(p^{-\frac{1}{2}})\,. \tag{4.54}$$

Recalling the local model (3.20) for the equivariant Bergman kernels, taking $v = (u + u')/2$ and $a = |\xi_x|$ in (4.54) and after a change of variable $m \mapsto -m$, for any $\delta > 0$, the asymptotics (4.48) give $\varepsilon_0 > 0$ such that we get the following asymptotics as $p \to \infty$, uniform in $x \in \mu^{-1}(0)$, in $k \in \mathbb{Z}$ satisfying $k \leqslant 0$ and in Z, $Z' \in T^H_x X$ as in (3.18) with $|Z|\,|Z'| < \varepsilon p^{\frac{-1+\varepsilon}{2}}$,

$$\begin{aligned}
&p^{-2n+1} \sum_{m=-Mp}^{m=0} P^{(m)}_{p,x}(Z, Z')\, P^{(m+k)}_{p,x}(Z', Z) - \left|P^{(m)}_{p,x}(Z, Z')\right|^2 \\
&= \sum_{m \in \mathbb{N}} \left(\mathscr{P}^{(-m)}_x(Z, Z')\mathscr{P}^{(-(m-k))}_x(Z', Z) - |\mathscr{P}^{(-m)}_x(Z, Z')|^2\right) + O(p^{-\frac{1}{2}+\delta}) \\
&= \frac{k}{|\xi_x|^2} \exp(-\pi p(u + u')^2) \exp\left(-\pi p(u - u')^2\right) \left|\mathscr{P}_x(\sqrt{p}Z^\perp, \sqrt{p}Z'^\perp)\right|^2 + k^2\, O(p^{-\frac{1}{2}+\delta})\,.
\end{aligned} \tag{4.55}$$

Setting $\varepsilon' > 4\,\varepsilon \sup_{x\in\mu^{-1}(0)} |\xi_x|$ as in (4.52), writing $Z' = u'\, e_1 + Z'^\perp \in T^H_x X$ as in (3.18), taking Taylor expansions of f, $|\xi|$ and h and recalling from (4.50) that $h|_{V_0} \equiv 1$, Theorem 2.2 together with (4.55) imply the following asymptotics as $p \to \infty$,

$$\begin{aligned}
&p^{-n+1} \int_{V(\varepsilon p^{\frac{-1+\varepsilon}{2}})} \int_V \widetilde{\varrho}(x)\, |\xi_x|\, |\xi_y| \\
&\sum_{m=-Mp}^{m=0} \left(P^{(m)}(x, y).P^{(m+k)}(y, x) - |P^{(m)}(x, y)|^2_p\right) \hat{f}_k(x)\, \overline{\hat{f}_k(y)}\, \pi^* dv_B(y)\, \pi^* dv_B(x) \\
&= p^n \int_{V_0} \varrho(\pi(x)) \int_{-|\xi_x|\varepsilon p^{\frac{-1+\varepsilon}{2}}}^{|\xi_x|\varepsilon p^{\frac{-1+\varepsilon}{2}}} \int_{B^{T^H_x X}(0,\varepsilon' p^{\frac{-1+\varepsilon}{2}})} |\xi_{\psi_x(u\, e_1)}|\, |\xi_{\psi_x(Z')}| \\
&\sum_{m=-Mp}^{m=0} \left(P^{(m)}_{p,x}(u\, e_1, Z')\, P^{(m+k)}_{p,x}(Z', u\, e_1) - |P^{(m)}_{p,x}(u\, e_1, Z')|^2_p\right) \\
&\hat{f}_{k,x}(u\, e_1)\, \overline{\hat{f}_{k,x}(Z')}\, h_x(u\, e_1)\, h_x(Z')\, dZ'\, du\, \pi^* dv_{X_0}(x) + O(p^{-\infty}) \\
&= k \int_{V_0} \left(\int_{\mathbb{R}} \int_{T^H_x X} \exp(-\pi(u + u')^2)\, \exp(-\pi(u - u')^2)\, \exp(-\pi|Z'^\perp|^2)\, dZ'\, du\right) \\
&\varrho(\pi(x))\, \hat{f}_k(x)\, \overline{\hat{f}_k(x)}\, \pi^* dv_{X_0}(x) + k^2\, O(p^{-\frac{1}{2}+\delta}) \\
&= \frac{k}{2} \int_{X_0} \varrho(x)\, |\hat{f}_k(x)|^2\, dv_{X_0}(x) + k^2\, O(p^{-\frac{1}{2}+\delta})\,.
\end{aligned} \tag{4.56}$$

Recall now the standard fact from Fourier theory of a smooth compactly supported function $f \in \mathscr{C}^\infty_c(X, \mathbb{R})$ that for any $N \in \mathbb{N}$, there exists $C_N > 0$ such that for all $x \in U$ and all $k \in \mathbb{Z}$, the associated Fourier coefficient (1.12) satisfies $|\hat{f}_k(x)| < C_N\, |k|^{-N}$.

Using Corollary 2.6 and Proposition 3.1, we then get a constant $C > 0$ such that for all $N \geqslant 2$, x, $y \in U$ and $p \in \mathbb{N}$, through the canonical isomorphism $L_x^p \otimes (L_x^p)^* \simeq \mathbb{C}$, we have

$$p^{-n+1} \sum_{|k| \geqslant p^{\frac{1-\varepsilon}{8}}} \sum_{m=-Mp}^{m=0} \left| \left(P^{(m)}(x,y)\, P^{(m+k)}(y,x) - |P^{(m)}(x,y)|_p^2 \right) \hat{f}_k(x)\, \overline{\hat{f}_k(y)} \right| \\ \leqslant C\, C_N\, p^{2n+1} \sum_{|k| \geqslant p^{\frac{1-\varepsilon}{8}}} |k|^{-2N} \leqslant C\, C_N\, p^{2n+1-(2N-1)\frac{1-\varepsilon}{8}} . \quad (4.57)$$

Taking $N \in \mathbb{N}$ large enough, we then get that (4.57) tends to 0 as $p \to \infty$, while on the other hand, we get from (4.56) the following asymptotics as $p \to \infty$,

$$\begin{aligned} & p^{-n+1} \sum_{k \geqslant -p^{\frac{1-\varepsilon}{8}}}^{k=0} \int_{V(\varepsilon p^{\frac{-1+\varepsilon}{2}})} \int_V \widetilde{\varrho}(x)\, |\xi_x|\, |\xi_y|\, \hat{f}_k(x)\, \overline{\hat{f}_k(y)} \\ & \sum_{m=-Mp}^{m=0} \left(P^{(m)}(x,y).P^{(m+k)}(y,x) - |P^{(m)}(x,y)|_p^2 \right) \pi^* dv_B(x)\, \pi^* dv_B(y) \\ & = \frac{1}{2} \sum_{k \geqslant -p^{\frac{1-\varepsilon}{8}}}^{k=0} k \int_{X_0} \varrho(x)\ |\hat{f}_k(x)|^2 dv_{X_0}(x) + \left(\sum_{k \geqslant -p^{\frac{1-\varepsilon}{8}}}^{k=0} k^2 \right) O(p^{-\frac{1}{2}+\delta}) \\ & = \frac{1}{2} \sum_{k \leqslant 0} k \int_{X_0} \varrho(x)\ |\hat{f}_k(x)|^2 dv_{X_0}(x) + O(p^{-\frac{1}{8}+\frac{3}{8}\varepsilon+\delta}) . \end{aligned} \quad (4.58)$$

The same reasonning applied to the last term of formula (4.46) gives in turn the following asymptotic expansion as $p \to +\infty$,

$$\begin{aligned} & \sum_{k \geqslant -p^{\frac{1-\varepsilon}{8}}}^{k=-1} p^{-n+1} \int_{V(\varepsilon p^{\frac{-1+\varepsilon}{2}})} \int_{V(\varepsilon p^{\frac{-1+\varepsilon}{2}})} \int_V \widetilde{\varrho}(x)\, |\xi_x|\, |\xi_y|\, \hat{f}_{-k}(x)\, \overline{\hat{f}_{-k}(y)} \\ & \sum_{m=-Mp}^{m=0} \left(P^{(m+k)}(x,y)\, P^{(m)}(y,x) - |P^{(m)}(x,y)|_p^2 \right) \pi^* dv_B(x)\, \pi^* dv_B(y) \\ & = \frac{1}{2} \sum_{k<0} k \int_{X_0} \varrho(x)\ |\hat{f}_{-k}(x)|^2\, dv_{X_0}(x) + O(p^{-\frac{1}{8}+\frac{3}{8}\varepsilon+\delta}) . \end{aligned} \quad (4.59)$$

Since $\mu^{-1}(0) \subset X$ is compact by properness of $\mu \in \mathscr{C}^\infty(X, \mathbb{R})$, we can pick a finite cover of $X_0 := \mu^{-1}(0)$ by open sets $\pi(V_0) \subset X_0$ with $V_0 \subset \mu^{-1}(0)$ of the form (3.13) and consider an adapted partition of unity. Then choosing $\delta > 0$ and $\varepsilon > 0$ small enough and summing the estimates (4.52) and (4.57) to (4.59) with $\varrho \in \mathscr{C}^\infty(X_0, \mathbb{R})$ running over this partition of unity, this identifies the second term of the right-hand side of the asymptotics (4.35) with the second term of the right-hand side of the asymptotics (4.34), concluding the proof of the asymptotics (4.34) for the variance.

The convergence of the random variable $N_p^\alpha(\mathcal{N}_p[f] - \mathbb{E}[\mathcal{N}_p[f]])$ with $\alpha = \frac{1}{2n} - \frac{1}{2}$ to a centered normal random variable is a consequence of the asymptotics (4.34) for the variance and of the asymptotics (4.18) for the expectation established in Theorem 4.6, thanks to a general argument adapted from the result of Soshnikov in [29, Th. 1] by Berman in [4, § 6.5]. This concludes the proof of the Theorem.

□

References

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