Asymptotics of unitary matrix elements in canonical bases

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Abstract

We compute the asymptotics of matrix elements in canonical bases of irreducible representations of the unitary group as the highest weight goes to infinity, in terms of the symplectic geometry of the associated coadjoint orbit. This uses tools of Berezin-Toeplitz quantization, and recovers as a special case the asymptotics of Wigner's d-matrix elements for the spin representations in quantum mechanics.

1 Introduction

Let $n \in \mathbb{N}^*$, and consider the sequence of inclusions

$$U(1) \subset U(2) \subset \cdots \subset U(n) \tag{1.1}$$

of the unitary groups, where for any $k \in \mathbb{N}$ with $k \leqslant n$, the subgroup $U(k) \subset U(n)$ acts on the canonical decomposition $\mathbb{C}^n = \mathbb{C}^k \oplus \mathbb{C}^{n-k}$ in the usual way on the first factor and as the identity on the second. A classical result of Weyl, which we recall in Theorem 5.1, states that any unitary irreducible representation V of the unitary group $U(n)$ admits a unique decomposition

$$V|_{U(n-1)} = \bigoplus_{j \in J} V_j \tag{1.2}$$

into distinct irreducible representations V_j of $U(n-1) \subset U(n)$ by restriction, so that each irreducible representation of $U(n-1)$ appears with multiplicity at most 1 in this decomposition. Applying the same result to V_j for each $j \in J$, this further implies that V_j decomposes as a direct sum of distinct irreducible representations of $U(n-2) \subset U(n-1)$ as before. By downward iteration along the sequence of inclusions (1.1), we thus obtain a canonical decomposition

$$V = \bigoplus_{\nu \in \Gamma} \mathbb{C}_\nu \,, \tag{1.3}$$

where each component $\mathbb{C}_\nu \subset V$ is of complex dimension 1 as an irreducible representation of $U(1) \subset U(n)$. This decomposition was implicitly used by Gelfand and Zetlin

[1]partially supported by ANR-23-CE40-0021-01 JCJC project QCM

in [20] to establish explicit formulas for matrix elements of irreducible representations of $U(n)$, and a basis $\{e_\nu\}_{\nu\in\Gamma}$ of V compatible with the decomposition (1.3) is called a *Gelfand-Zetlin basis*.

The main results of this paper are Theorems 1.1 and 1.2, establishing the asymptotics of matrix elements in the Gelfand-Zetlin bases of unitary irreducible representations of $U(n)$ as the *highest weight* goes to infinity. We compute these asymptotics in terms of the symplectic geometry of the associated *coadjoint orbit* $X \subset \mathfrak{u}(n)^*$ for the coadjoint action of $U(n)$ on the dual $\mathfrak{u}(n)^*$ of its Lie algebra, endowed with its natural symplectic form $\omega \in \Omega^2(X,\mathbb{R})$ given in Proposition 3.7. Assuming for simplicity that X has maximal dimension among coadjoint orbits in $\mathfrak{u}(n)^*$, Guillemin and Sternberg constructed in [23] a remarquable integrable system on (X,ω), called the *Gelfand-Zetlin system* and given by a continuous map

$$M : X \longrightarrow \Delta \subset \mathbb{R}^{\frac{n(n-1)}{2}}\,, \tag{1.4}$$

which we describe in Theorem 5.9, whose image $\Delta := \mathrm{Im}(M)$ is a convex polytope, called the *Gelfand-Zetlin polytope*. The map (1.4) is smooth over the interior $\Delta^0 \subset \Delta$, and induces *action-angle coordinates* over the open dense subset $M^{-1}(\Delta_0) \subset X$. In particular, for any $v \in \Delta^0$, the fibre $M^{-1}(v) \subset X$ is a Lagrangian torus inside (X,ω). In [23, § 6], Guillemin and Sternberg establish a natural bijective correspondence

$$\Gamma \simeq \Delta \cap \mathbb{Z}^{\frac{n(n-1)}{2}} \tag{1.5}$$

between the components of the Gelfand-Zetlin decomposition (1.3) of an irreducible representation V of $U(n)$ and the integral points inside the image of the map (1.4) over the corresponding coadjoint orbit X. In the heuristics of *Geometric Quantization*, the coadjoint orbit represents a phase space of classical mechanics, while the irreducible representation represents the corresponding space of quantum states, called the *holomorphic quantization* of (X,ω). The classical integrable system (1.4) over (X,ω) then corresponds to the quantum integrable system on V given by the commutative algebra of diagonal operators in the decomposition (1.3), and the correspondence (1.5) can then be interpreted as the old quantum-classical correspondence between integral orbits of a classical integrable system and eigenstates of the corresponding quantum integrable system.

The general idea behind the main Theorems 1.1 and 1.2 of this paper sits in this context of Geometric Quantization, and consists in studying the so-called *semiclassical limit*, which is the regime where the scale becomes so large that one recovers the laws of classical mechanics from the laws of quantum mechanics. To describe this semiclassical limit, let us first recall another fundamental result of Weyl, which we recall in Theorem 3.6, giving a natural parametrization of the unitary irreducible representations of $U(n)$ by the n-tuples of integers $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{Z}^n$ satisfying $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$, called the *highest weight* of the corresponding representation, denoted by $V(\lambda)$. On the other hand, given such a weight $\lambda \in \mathbb{Z}^n$, one can consider the coadjoint orbit $X_\lambda \subset \mathfrak{u}(n)^*$ passing through the diagonal matrix $\mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathfrak{u}(n^*)$ under the natural identification (3.32) of $\mathfrak{u}(n)^*$ with the space of $n \times n$ Hermitian matrices. Let us assume that the weight $\lambda \in \mathbb{Z}^n$ is *regular*, so that $\lambda_1 > \lambda_2 > \cdots > \lambda_n$, and consider the integral part of the *half-sum of roots* of $\mathfrak{u}(n)$, defined by

$$\overline{\rho} := \left(\left\lceil \frac{n-1}{2} \right\rceil, \left\lceil \frac{n-3}{2} \right\rceil, \cdots, -\left\lceil \frac{n-3}{2} \right\rceil, -\left\lceil \frac{n-1}{2} \right\rceil \right) \in \mathbb{Z}^n . \tag{1.6}$$

As we explain at the end of Section 5.2, one gets that

$$\frac{1}{p}\Gamma_p := \frac{1}{p}\left(\Delta_{p\lambda+\overline{\rho}} \cap \mathbb{Z}^{\frac{n(n-1)}{2}} \right) \tag{1.7}$$

becomes dense in $\Delta_\lambda \subset \mathbb{R}^{\frac{n(n-1)}{2}}$ as $p \in \mathbb{N}$ tends to infinity, where $\Delta_\lambda \subset \mathbb{R}^{\frac{n(n-1)}{2}}$ denotes the Gelfand-Zetlin polytope (1.4) of the coadjoint orbit $X_\lambda \subset \mathfrak{u}(n)^*$. This type of behavior as $p \to +\infty$ is characteristic of a semiclassical limit in Geometric Quantization. Note from (1.5) that (1.7) parametrizes the components of the Gelfand-Zetlin decomposition (1.3) of the irreducible representation $V(p\lambda+\overline{\rho})$ of $U(n)$ with highest weight $p\lambda+\overline{\rho}$. The alternative parametrization of irreducible representations of $U(n)$ where one includes a shift by the half-sum of roots is sometimes called the *Harish-Chandra parametrization.*

For any $v \in \Delta_\lambda^0$, where $\Delta_\lambda^0 \subset \Delta_\lambda$ denotes the interior, write

$$\Lambda_v := M^{-1}(v) \subset X \tag{1.8}$$

for the associated Lagrangian fibre of the Gelfand-Zetlin system (1.4), and for each $1 \leqslant j \leqslant \frac{n(n-1)}{2}$, write $M_j \in \mathscr{C}^\infty(X, \mathbb{R})$ for the j^{th} component of the map (1.4) in $\mathbb{R}^{\frac{n(n-1)}{2}}$. Write $\{\cdot,\cdot\}$ for the Poisson bracket on $\mathscr{C}^\infty(X_\lambda, \mathbb{R})$ induced by the symplectic form of (X_λ, ω). The first main result of this paper is the following.

Theorem 1.1. *Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{Z}^n$ satisfy $\lambda_1 > \lambda_2 > \cdots > \lambda_n$, let $\{e_\nu\}_{\nu\in\Gamma_p}$ be a Gelfand-Zetlin basis of the unitary irreducible representation of $U(n)$ with highest weight $p\lambda+\overline{\rho}$ for all $p \in \mathbb{N}$, and let $(v_p,\, w_p \in \frac{1}{p}\Gamma_p)_{p\in\mathbb{N}}$ be two sequences respectively converging to some $v,\, w \in \Delta_\lambda^0 \subset \Delta_\lambda$.*

Then for any $g \in U(n)$ such that $g\Lambda_v \cap \Lambda_w = \emptyset$ and for any $k \in \mathbb{N}$, there exists $C_k > 0$ such that for all $p \in \mathbb{N}$, we have

$$\left| \langle g e_{pv_p}, e_{pw_p} \rangle \right| \leqslant C_k\, p^{-k} . \tag{1.9}$$

For any $g \in U(n)$ such that $g\Lambda_v$ intersects Λ_w transversally, there are scalars $z_p \in \mathbb{C}^*$ with $|z_p| = 1$ and points $x_p \in g\Lambda_{v_p} \cap \Lambda_{w_p}$ for each $p \in \mathbb{N}$ such that we have the following asymptotic expansion as $p \to +\infty$,

$$p^{\frac{n}{2}} \langle g e_{pv_p}, e_{pw_p} \rangle = z_p \sum_{x \in g\Lambda_{v_p} \cap \Lambda_{w_p}} \frac{\sqrt{-1}^{\kappa(x)} e^{2\pi\sqrt{-1}p\eta(x)}}{\left| \det\left(\{g_* M_j, M_k\}(x) \right)_{j,k} \right|^{\frac{1}{2}}} + O(p^{-1}) , \tag{1.10}$$

where for any $x \in g\Lambda_{v_p} \cap \Lambda_{w_p}$, the real number $\eta(x) \in \mathbb{R}$ is the symplectic area of a disk whose boundary is given by a path in $g\Lambda_{v_p}$ going from $x_p \in g\Lambda_{v_p} \cap \Lambda_{w_p}$ to any $x \in g\Lambda_{v_p} \cap \Lambda_{w_p}$ followed by a path in Λ_{w_p} returning to $x_p \in g\Lambda_{v_p} \cap \Lambda_{w_p}$, and $\kappa(x) \in \mathbb{Z}/4\mathbb{Z}$ is a Maslov index along this path.

Theorem 1.1 is proved in Section 5.3 for *regular weights* $\lambda \in \mathbb{Z}^n$, so that $\lambda_1 > \lambda_2 > \cdots > \lambda_n$, which corresponds to the fact that the associated coadjoint orbit $X_\lambda \subset \mathfrak{u}(n)^*$ is of maximal dimension among all coadjoint orbits in $\mathfrak{u}(n)^*$. In [23, p. 122] and [24, p. 229], Guillemin and Sternberg indicate how to extend their construction of the Gelfand-Zetlin system (1.4) to general coadjoint orbits, and the proof of Theorem 1.1 described in Section 5.3 should then readily extend to cover these cases. In this paper, we do not consider the general singular case for simplicity, since only the regular case is described precisely enough for our purposes in [23].

However, in the most singular case of $\lambda = (\lambda_1, 0, \cdots, 0) \in \mathbb{Z}^n$, the geometric picture greatly simplifies, and the second main result of our paper is an extension and refinement of Theorem 1.1 to that case. In fact, the Gelfand-Zetlin decomposition (1.3) of the irreducible representation of $U(n)$ with highest weight $(\lambda_1, 0, \cdots, 0)$ coincides with its *weight decomposition* into irreducible representations of the $(n-1)$-dimensional subtorus $T_0 \subset U(n)$ consisting of diagonal matrices with highest-left coefficient equal to 1. On the other hand, the coadjoint action of $T_0 \subset U(n)$ makes $X_\lambda \subset \mathfrak{u}(n)^*$ into a *toric manifold*, and the Gelfand-Zetlin system (1.4) coincide with the associated *moment map*

$$\mu : X_\lambda \longrightarrow \Delta_\lambda \subset \mathbb{R}^{n-1}\,, \tag{1.11}$$

so that the Gelfand-Zetlin polytope coincides with the *Delzant polytope* of (X_λ, ω) as a toric manifold. The correspondence (1.5) then translates into the well-known fact that the weights for the torus action of $T_0 \subset U(n)$ on the holomorphic quantization of a toric manifold are given by the integral points inside its Delzant polytope. In the case at hand, the coadjoint orbit (X_λ, ω) naturally identifies as a toric manifold with the projective space $\mathbb{CP}^{n-1}$ of complex lines inside $\mathbb{C}^n$ endowed with is canonical symplectic form of volume $\lambda_1 \in \mathbb{N}$.

In the following theorem, we assume without loss of generality that $\lambda = (1, 0, \cdots 0)$, so that as we explain in Section 4.1, for any $p \in \mathbb{N}$ we have

$$\frac{1}{p}\,\Gamma_p = \Delta \cap \left(\frac{1}{p}\,\mathbb{Z}^{n-1}\right), \tag{1.12}$$

where $\Gamma_p \subset \mathbb{Z}^n$ is the set of weights for the action of $T_0 \subset U(n)$ on the irreducible representation of $U(n)$ with highest weight $(p, 0, \cdots 0) \in \mathbb{Z}^n$ and $\Delta := \Delta_{(1,0,\cdots,0)} \subset \mathbb{R}^{n-1}$ is the Delzant polytope of the standard projective space $\mathbb{CP}^{n-1}$. For any $v \in \Delta$, we write

$$\Lambda_v := \mu^{-1}(v) \subset \mathbb{CP}^{n-1} \tag{1.13}$$

for the associated fibre of the corresponding moment map (1.11). The second main result of this paper is the following.

Theorem 1.2. *For every $p \in \mathbb{N}$, let $\{e_\nu\}_{\nu \in \Gamma_p}$ be a weight basis for the torus action $T_0 \subset U(n)$ on the irreducible representation of $U(n)$ with highest weight $(p, 0, \cdots 0) \in \mathbb{Z}^n$, and let $(v_p,\, w_p \in \frac{1}{p}\Gamma_p)_{p \in \mathbb{N}}$ be two sequences belonging to the same face of Δ after some rank and respectively converging to some $v,\, w \in \Delta$.*

Then for any $g \in U(n)$ such that $g\Lambda_v \cap \Lambda_w = \emptyset$ and for any $k \in \mathbb{N}$, there exists $C_k > 0$ such that for all $p \in \mathbb{N}$, we have

$$\left|\langle g e_{p v_p}, e_{p w_p}\rangle\right| \leqslant C_k\, p^{-k}\,. \tag{1.14}$$

For any $g \in U(n)$ such that $g\Lambda_v$ intersects Λ_w cleanly, writing $g\Lambda_v \cap \Lambda_w = \cup_{q=0}^m Y^{(q)}$ for the decomposition into connected components, there exist $\kappa_q \in \mathbb{Z}/4\mathbb{Z}$ and $b_p^{(q)} \in \mathbb{C}$ for each $0 \leqslant q \leqslant m$, as well as scalars $z_p \in \mathbb{C}^*$ with $|z_p| = 1$ for all $p \in \mathbb{N}$, such that the following asymptotic expansion holds as $p \to +\infty$,

$$\langle g e_{p v_p}, e_{p w_p}\rangle = z_p\, p^{-\frac{\dim \Lambda^{(1)} + \dim \Lambda^{(2)}}{4}} \sum_{q=1}^m p^{\frac{\dim Y_q}{2}} \sqrt{-1}^{\kappa_q} e^{2\pi\sqrt{-1} p \eta_p^{(q)}} \left(b_p^{(q)} + O(p^{-1})\right) \tag{1.15}$$

where $\eta_p^{(q)} \in \mathbb{R}$ is the symplectic area of a disk whose boundary is given by a path in $g\Lambda_{v_p}$ going from any point $x_p \in g\Lambda_{v_p} \cap \Lambda_{w_p}$ approaching $Y^{(0)}$ as $p \to +\infty$ to any point $y_p \in g\Lambda_{v_p} \cap \Lambda_{w_p}$ approaching $Y^{(q)}$ as $p \to +\infty$ followed by a path in Λ_{w_p} returning to $x_p \in g\Lambda_{v_p} \cap \Lambda_{w_p}$.

In case $n \in \mathbb{N}^*$ is even, let $\{e_\nu\}_{\nu \in \Gamma_{p-\frac{n}{2}}}$ be a weight basis of the irreducible representation of $U(n)$ with highest weight $(p - \frac{n}{2}, 0, \cdots 0) \in \mathbb{Z}^n$ for all $p \in \mathbb{N}$ instead, and let $(v_p,\, w_p \in \frac{1}{p}\,\Gamma_{p-\frac{n}{2}})_{p\in\mathbb{N}}$ be two sequences respectively converging to some $v,\, w \in \Delta^0 \subset \Delta$ Then for any $g \in U(n)$ such that $g\Lambda_v$ intersects Λ_w transversally, we have the following asymptotic expansion as $p \to +\infty$,

$$p^{\frac{n}{2}} \langle g e_{p v_p}, e_{p w_p}\rangle = z_p \sum_{q=1}^m \sqrt{-1}^{\kappa_q} \frac{e^{2\pi\sqrt{-1} p \eta_p^{(q)}}}{\left|\det\left(\alpha_p^{(q)}([\mathrm{Ad}_g\, \xi_j, \xi_k])\right)_{j,k}\right|^{\frac{1}{2}}} + O(p^{-1})\,, \tag{1.16}$$

where $\{\xi_j \in \mathfrak{t}_0\}_{j=1}^{n-1}$ is a basis of the integral lattice inside the Lie subalgebra $\mathfrak{t}_0 \subset \mathfrak{u}(n)$ of $T_0 \subset U(n)$ and where $\alpha_p^{(q)} \in \mathfrak{u}(n)^*$ is such that $Y_p^{(q)} = \{\alpha_p^{(q)}\}$ for all $0 \leqslant q \leqslant m$.

Theorem 1.2 is proved in Section 4.2. Note that every irreducible representation of $SU(n) \subset U(n)$ can be obtained from an irreducible representation of $U(n)$ by restriction, and that the weight basis considered in Theorem 1.2 induces by restriction to $SU(n) \subset U(n)$ a weight basis for the maximal torus of $SU(n)$, so that Theorem 1.2 can also be stated in terms of irreducible representations of $SU(n)$. In the case $n = 2$, the irreducible representation of $U(2)$ of highest weight $(p, 0)$ induces the *spin-$\frac{p}{2}$* representation of $SU(2) \subset U(2)$, and the elements of a weight basis for the 1-dimensional torus action $T_0 \subset U(n)$ coincide with the standard *spin states* in quantum mechanics. The matrix elements computed in Theorem 1.2 then coincide with *Wigner's d-matrix elements*, as described for instance in [32, § 3.1]. Furthermore, the associated coadjoint orbit (X_λ, ω) is naturally identified with the 2-dimensional sphere S^2 endowed with its standard volume form, and Theorem 1.2 then recovers the semiclassical *spherical area formula* for Wigner's d-matrix elements, as stated for instance in [32, § 3.2, (57)]. Here, the spherical area refers to the area delimited by the intersection of two circles on the sphere S^2, which is computed by the term $\eta_p^{(q)} \in \mathbb{R}$ in formula (1.15).

On the other hand, Theorem 1.2 gives a criterion from representation theory for the general problem of whether the intersection $g\Lambda_v \cap \Lambda_w \subset \mathbb{CP}^{n-1}$ is non-empty, for any given v, $w \in \Delta$. Already in the case when v is the *barycenter* of Δ, so that $\Lambda_v \subset \mathbb{CP}^{n-1}$ is the so-called *Clifford torus* of $\mathbb{CP}^{n-1}$ as a toric manifold, the only known proof of the fact that $g\Lambda_v \cap \Lambda_v \neq \emptyset$ for all $g \in U(n)$ follows from the Hamiltonian non-displaceability of Clifford tori due to Biran, Entov and Polterovich in [5] and Cho in [15], which is based on sophisticated tools of symplectic topology. We thus hope that Theorems 1.1 and 1.2 can shed light on the non-displaceability of remarkable Lagrangian submanifolds by automorphisms groups of Kähler manifolds.

In Sections 2 and 3, we recall the geometric picture behind Theorems 1.1 and 1.2, which is that of *Geometric Quantization*, following the initial insight of Guillemin and Sternberg in [23, § 1]. This picture is based on a result of Kostant, which we recall in Proposition 3.8, stating that for any $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{Z}^n$ with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$, the coadjoint action of $U(n)$ on the associated coadjoint orbit $X := X_\lambda$ lifts to a natural Hermitian line bundle (L, h^L) endowed with a Hermitian connection ∇^L satisfying the following *prequantization formula*,

$$\omega = \frac{\sqrt{-1}}{2\pi} R^L \,, \tag{1.17}$$

where $R^L \in \Omega^2(X, \mathbb{C})$ is the curvature of ∇^L. The *holomorphic quantization* of (X, ω) is then given by the space of its holomorphic sections $H^0(X, L)$, for the holomorphic structure on L induced by a natural compatible $U(n)$-invariant complex structure on (X, ω). By the celebrated *Borel-Weil theorem*, which we recall in Theorem 3.9, the induced action of $U(n)$ then makes $H^0(X, L)$ into an irreducible representation of $U(n)$ with highest weight λ. On the other hand, a submanifold $\Lambda \subset X$ is said to satisfy the *Bohr-Sommerfeld condition* if there exists a non-vanishing section $\zeta \in \mathscr{C}^\infty(\Lambda, L|_\Lambda)$ satisfying

$$\nabla^L_\xi \zeta \equiv 0 \ \text{ for all } \xi \in \mathscr{C}^\infty(\Lambda, T\Lambda) \,. \tag{1.18}$$

These submanifolds represent quantum states in the old *Bohr-Sommerfeld quantization* process, and we explain in Definition 2.5 how to associate to such a manifold an element of $H^0(X, L)$, which we call the associated *isotropic state* and which we interpret as the corresponding quantum state in holomorphic quantization. We then explain in Theorems 2.9 and 2.10 how the Hermitian product of two isotropic states associated with two submanifolds Λ_1 and Λ_2 satisfying the Bohr-Sommerfeld condition can be computed in terms of the geometry of the intersection $\Lambda_1 \cap \Lambda_2$ at the semiclassical limit $p \to +\infty$, where one replaces L by its p^{th} tensor power L^p. This is based on results of the author in [27], following the seminal work of Borthwick, Paul and Uribe in [8, Th. 3.2], and uses tools of *Berezin-Toeplitz quantization* first developped by Boutet de Monvel and Sjöstrand [10] and Boutet de Monvel and Guillemin [9], while the author follows in [27] the approach due to Ma and Marinescu in [33, 34]. The modern approach to Berezin-Toeplitz quantization is based on a classical result of Bordemann, Meinreken and Schlichenmaier in [7], which we recall in Theorem 2.2, establishing the quantum-classical correspondence between classical observables, represented by smooth function

$\mathscr{C}^\infty(X,\mathbb{R})$ over (X,ω), and quantum observables, represented by Hermitian operators acting on $H^0(X,L^p)$, at the semiclassical limit as $p\to+\infty$.

In Section 4, we consider first the projective case of $X=\mathbb{CP}^{n-1}$ treated in Theorem 1.2, in which case the prequantizing line bundle (L,h^L) coincides with the *dual of the tautological line bundle* over $\mathbb{CP}^{n-1}$ endowed with its associated Fubini-Study metric. Then for any $p\in\mathbb{N}$, the space of holomorphic sections $H^0(X,L^p)$ naturally identifies with the space of homogeneous polynomials of degree $p\in\mathbb{N}$ over $\mathbb{C}^n$, which is the classical realization of the irreducible representation of $U(n)$ with highest weight $(p,0,\cdots 0)\in\mathbb{Z}^n$ constructed by Weyl, as explained for instance in [19, § 6.1]. On the other hand, from general considerations on toric manifolds, the fibres of the moment map (1.11) satisfying the Bohr-Sommerfeld condition are exactly those above integral points inside $p\Delta\subset\mathbb{R}^{n-1}$, where Δ is the Delzant polytope of $\mathbb{CP}^{n-1}$, and we show in Proposition 4.2 that the associated istropic states provide basis elements for the weight decomposition of $H^0(X,L^p)$ for the torus action $T_0\subset U(n)$ with corresponding weight. This correspondence holds as such for all toric manifolds, and provides the most precise instance of the expected correspondence between Bohr-Sommerfeld quantization and holomorphic quantization. The proof of Theorem 1.2, that we give in Section 4.2, then follows as an application of the results of the author in [27] described above. Note that the previous result of Borthwick, Paul and Uribe in [8] only holds for Bohr-Sommerfeld Lagrangian submanifolds, which corresponds to the case when $w\in\Delta^0\subset\Delta$ in Theorem 1.2, while the case of $w\in\Delta$ belonging to a general face requires the more general case of Bohr-Sommerfeld isotropic submanifolds considered in [27], since in that case the dimension of the fibre $\Lambda_v\subset\mathbb{CP}^{n-1}$ is strictly lower than $n-1$. On the other hand, Borthwick, Paul and Uribe compute in [8, Th. 4.4] a special case of their formula when $\dim_{\mathbb{C}}X=1$, from which one readily recovers the spherical area formula for Wigner's d-matrix elements associated with great circles of S^2, as explained in [3, Th. 16].

In Section 5, we proceed to consider the case of a general regular weight $\lambda=(\lambda_1,\lambda_2,\cdots,\lambda_n)\in\mathbb{Z}^n$ with $\lambda_1>\lambda_2>\cdots>\lambda_n$, so that the associated coadjoint orbit $X_\lambda\subset\mathfrak{u}(n)^*$ has maximal dimension among coadjoint orbits in $\mathfrak{u}(n)^*$. In Section 5.1, we introduce the tools of representation theory of compact Lie groups needed for the proof of Theorem 1.1. In particular, the Gelfand-Zetlin decomposition (1.3) is the decomposition into common eigenspaces of the commutative *Gelfand-Zetlin subalgebra*

$$\mathscr{A}_n:=\langle Z[U(\mathfrak{u}(k))]\mid 1\leqslant k\leqslant n\rangle\subset U(\mathfrak{u}(n))\,,\tag{1.19}$$

which we introduce in Definition 5.4 as the commutative subalgebra of the *universal enveloping algebra* $U(\mathfrak{u}(n))$ of $\mathfrak{u}(n)$ generated by the centers $Z[U(\mathfrak{u}(k))]\subset U(\mathfrak{u}(k))$ of the universal enveloping algebra of $\mathfrak{u}(k)$ for each $k\in\mathbb{N}$ with $k\leqslant n$ via the sequence of inclusions (1.1). We study the common eigenvalues of the Gelfand-Zetlin subalgebra (1.19) in the decomposition (1.3), in order to link it with the parametrization (1.5) due to Guillemin and Sternberg in [23, Prop. 5.4]. In particular, the half-sum of roots inducing (1.6) plays an essential role in the *Harish-Chandra isomorphism* (3.58) used to determine these eigenvalues. In Section 5.3, we then give the proof of Theorem 1.1, by showing that a natural generating family for the Gelfand-Zetlin subalgebra (1.19) can be expressed in terms of the Berezin-Toeplitz quantization of the Gelfand-Zetlin system

(1.4) over (X_λ, ω) introduced by Guillemin and Sternberg in [23], then using results of Charles in [12, 13] on the semiclassical limit of quantum integrable systems in Berezin-Toeplitz quantization, to show that Gelfand-Zetlin bases elements are Lagrangian states associated with regular Bohr-Sommerfed fibres of the Gelfand-Zetlin system (1.4) in an appropriate sense. Note that the integral part of the half-sum of roots (1.6) corresponds here to the *metaplectic correction* appearing in [13], which corresponds to the choice of a square root of the *canonical line bundle* (2.5) of X_λ, and which is needed in order to provide the purely symplectic formula (1.10) for the first order term of the asymptotics of matrix elements established in the main Theorem 1.1. Note on the other hand that the projective space $\mathbb{CP}^{n-1}$ considered in Theorem 1.2 admits a metaplectic correction if and only if $n \in \mathbb{N}^*$ is even, which explains the shift by $n/2$ needed to obtain the purely symplectic formula (1.16) in the second main Theorem 1.2.

Let us finally point out that one of our motivation for Theorem 1.1 comes from the study of $SU(2)$-character varieties of surfaces, which admit a remarquable classical integrable system due to Jeffrey and Weitsman in [28], as well as a natural action of the mapping class group of the surface. On the other hand, character varieties admit a natural quantization endowed with a natural action of the mapping class group, provided by the TQFT of Witten in [39], Reshetikhin and Turaev in [36] and Blanchet, Habegger, Masbaum and Vogel in [6]. This quantization also comes with a natural basis, and the matrix elements for the mapping class group action in this basis have been computed by Detcherry in [16, Th. 11.1] using the results of Charles in [12, 13], obtaining the asymptotics (1.10) in this setting. Note also that the same method was used by Charles in [14, Th. 7.1] to compute the asymptotics of the classical $6j$-symbols, which as explained in [32, § 4.1, (68)] are closely related to the asymptotics of Wigner's d-matrix elements.

2 Geometric quantization

In this Section, we describe the general set-up of Geometric quantization used in the proofs of Theorems 1.1 and 1.2, and in particular that of Berezin-Toeplitz quantization, following [27] and [33].

2.1 Berezin-Toeplitz quantization

Let (X, ω) be a compact symplectic manifold of dimension $2n \in \mathbb{N}^*$ together with a Hermitian line bundle (L, h^L) endowed with a Hermitian connection ∇^L satisfying the prequantization formula (1.17). Let also X be equipped with a complex structure $J \in \operatorname{End}(TX)$ compatible with ω, making (X, ω, J) into a *Kähler manifold*, and write g^{TX} for the *Kähler metric* of (X, ω, J), which is the Riemannian metric defined over X by the formula

$$g^{TX}(\cdot, \cdot) = \omega(\cdot, J\cdot)\,. \tag{2.1}$$

The Riemannian volume form dv_X of (X, g^{TX}) then coincides with the *Liouville volume form* of (X, ω), that is

$$dv_X = \frac{\omega^n}{n!}\,. \tag{2.2}$$

Let us write

$$T_{\mathbb{C}}X = T^{(1,0)}X \oplus T^{(0,1)}X \tag{2.3}$$

for the splitting of the complexification $T_{\mathbb{C}}X$ of the tangent bundle of X into the eigenspaces of J corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$ respectively. For any $v \in \mathscr{C}^\infty(X, TX)$, we write $v = v^{(1,0)} + v^{(0,1)}$ for its decomposition into this splitting. Under the canonical isomorphism of complex vector bundles $(TX, J) \simeq T^{(1,0)}X$ induced by the splitting (2.3), the Hermitian metric on $T^{(1,0)}X$ induced by the restriction of g^{TX} via (2.3) corresponds to the Hermitian metric h^{TX} on (TX, J) defined by

$$h^{TX} = g^{TX} - \sqrt{-1}\omega\,. \tag{2.4}$$

The *canonical line bundle* of (X, J, ω) is the holomorphic line bundle

$$K_X = \det(T^{(1,0)}X^*) \tag{2.5}$$

endowed with the Hermitian structure h^{K_X} and Hermitian connection ∇^{K_X} respectively induced by g^{TX} and ∇^{TX} via the splitting (2.3). By the prequantization formula (1.17), the curvature $R^L \in \Omega^2(X, \mathbb{C})$ of ∇^L is J-invariant, which implies that the $(0,1)$-part of ∇^L under the splitting (2.3) defines a Cauchy-Riemann $\overline{\partial}$-operator, inducing a holomorphic structure on L.

For any holomorphic Hermitian line bundle (K, h^K) over X, we write $|\cdot|_{h^K}$ for the pointwise norm on K induced by h^K. For any $p \in \mathbb{N}$, write $L^p := L^{\otimes p}$ for the p^{th} tensor power of L, and h^p, ∇^p for the Hermitian metric and connection on the tensor product $L^p \otimes K$ respectively induced by h^L, h^K and ∇^L, ∇^K. We denote by $\mathscr{C}^\infty(X, L^p \otimes K)$ the space of smooth sections of $L^p \otimes K$, endowed with the L^2-*Hermitian product* $\langle\cdot,\cdot\rangle_{L^2}$ given for any $s_1, s_2 \in \mathscr{C}^\infty(X, L^p \otimes K)$ by the formula

$$\langle s_1, s_2\rangle_{L^2} = \int_X h^p(s_1(x), s_2(x))\, dv_X(x)\,. \tag{2.6}$$

We write $\|\cdot\|_{L^2}$ for the associated L^2-norm, and $L^2(X, L^p \otimes K)$ for the completion of $\mathscr{C}^\infty(X, L^p \otimes K)$ with respect to $\|\cdot\|_{L^2}$. We also write $\|\cdot\|_{op}$ for the operator norm on bounded operators acting on $L^2(X, L^p \otimes K)$.

Following for instance [33, Th. 1.4.1], the subspace $H^0(X, L^p \otimes K) \subset L^2(X, L^p \otimes K)$ of holomorphic sections of $L^p \otimes K$ is finite-dimensional, so that the orthogonal projection

$$P_p : L^2(X, L^p \otimes K) \longrightarrow H^0(X, L^p \otimes K) \tag{2.7}$$

with respect to the L^2-product (2.6) has finite rank, and hence admits a smooth Schwartz kernel. In the following definition, we introduce a fundamental tool of this paper.

Definition 2.1. For any $f \in \mathscr{C}^\infty(X,\mathbb{R})$ and $p \in \mathbb{N}$, the associated *Berezin-Toeplitz operator* is the operator $T_p(f)$ acting on $L^2(X, L^p \otimes K)$ defined by

$$T_p(f) := P_p\, f\, P_p\,, \tag{2.8}$$

where f denotes the operator of multiplication by f.

The main interest of Berezin-Toeplitz operators is that they provide *quantum observables* for the holomorphic quantization of the symplectic manifold (X,ω). This is described by the following fundamental theorem due to Bordemann, Meinrenken and Schlichenmaier [7], which we present here in a guise due to Ma and Marinescu in [34].

Theorem 2.2. *[7, 34] For any $f \in \mathscr{C}^\infty(X, L^p \otimes L)$, we have*

$$\|T_p(f)\|_{op} \xrightarrow{p\to+\infty} |f|_{\mathscr{C}^0}\,, \tag{2.9}$$

where $|f|_{\mathscr{C}^0}$ denotes the uniform norm of f.

Furthermore, for any $f,\ g \in \mathscr{C}^\infty(X, L^p \otimes K)$, we have the following estimates in operator norm as $p \to +\infty$,

$$T_p(f)T_p(g) = T_p(fg) + O(p^{-1})\,, \tag{2.10}$$

and

$$[T_p(f), T_p(g)] = \frac{\sqrt{-1}}{2\pi p} T_p(\{f,g\}) + O(p^{-2})\,, \tag{2.11}$$

where $\{\cdot,\cdot\}$ denotes the Poisson bracket on $\mathscr{C}^\infty(X,\mathbb{R})$ induced by ω.

The approach of Ma and Marinescu in [34] to establish Theorem 2.2 is based on the off-diagonal asymptotic expansion as $p \to +\infty$ of the smooth Schwarz kernel of the orthogonal projection (2.7), which we introduce in the following definition.

Definition 2.3. For any $p \in \mathbb{N}$, the *Bergman kernel* $P_p(x,y) \in (L^p \otimes K)_x \otimes (L^p \otimes K)^*_y$, for all $x,\ y \in X$, is the Schwarz kernel of the orthogonal projection (2.7), characterized for any $s \in \mathscr{C}^\infty(X, L^p \otimes K)$ and $x \in X$ by the formula

$$(P_p s)(x) = \int_X P_p(x,y).s(y)\, dv_X(y). \tag{2.12}$$

2.2 Isotropic states

In this Section, we describe the semiclassical asymptoptics of *isotropic states* in Berezin-Toeplitz quantization, which are at the basis of the asymptotics described in Theorems 1.1 and 1.2.

Recall that a properly embedded submanifold $\iota : \Lambda \hookrightarrow X$ in a symplectic manifold (X,ω) is said to be *isotropic* if $\iota^*\omega = 0$. If in addition $\dim \Lambda = n$, it is said to be *Lagrangian*. We write dv_Λ for the Riemannian volume form of $(\Lambda, \iota^* g^{TX})$.

Let $\nabla^{\iota^* L}$, $h^{\iota^* L}$ be the connection and Hermitian metric induced by ∇^L, h^L on the pullback line bundle $\iota^* L$ over Λ. Note that by (1.17), the condition $\iota^*\omega = 0$ implies that $\nabla^{\iota^* L}$ is *flat*. This observation motivates the following definition.

Definition 2.4. For any $p \in \mathbb{N}$, a properly embedded oriented submanifold $\iota : \Lambda \hookrightarrow X$ is said to satisfy the *Bohr-Sommerfeld condition* at level $p \in \mathbb{N}$ if there exists a non-vanishing smooth section $\zeta^p \in \mathscr{C}^\infty(\Lambda, \iota^* L^p)$ satisfying

$$\nabla^{\iota^* L^p} \zeta^p = 0 \,. \tag{2.13}$$

A *sequence of Bohr-Sommerfeld submanifolds* is a sequence $\{(\Lambda_p, \zeta^p, f_p)\}_{p\in\mathbb{N}}$ of submanifolds $\iota_p : \Lambda_p \hookrightarrow X$ satisfying the Bohr-Sommerfeld condition at level $p \in \mathbb{N}$, of sections $\zeta^p \in \mathscr{C}^\infty(\Lambda_p, \iota_p^* L^p)$ with $|\zeta^p|_{\iota_p^* L^p} \equiv 1$ satisfying (2.13), and of sections $f_p \in \mathscr{C}^\infty(\Lambda_p, \iota_p^* K)$, for all $p \in \mathbb{N}$.

In the case $K = \mathbb{C}$ and $f_p = 1$ for all $p \in \mathbb{N}$, we will simply write $\{(\Lambda_p, \zeta^p)\}_{p\in\mathbb{N}}$ for a sequence of Bohr-Sommerfeld submanifolds. When $\dim \Lambda_p = n$ for all $p \in \mathbb{N}$, we talk about a sequence of *Bohr-Sommerfled Lagrangian submanifolds*.

As explained in the Introduction, a Bohr-Sommerfeld submanifold represents a quantum state in the old Bohr-Sommerfeld quantization scheme, and there should be a corresponding state in the holomorphic quantization scheme described in this Section. Following [8, (12)] and [27, Def. 3.3], this is provided by the following definition.

Definition 2.5. The *isotropic state* associated with a sequence of Bohr-Sommerfeld submanifolds $\{(\Lambda_p, \zeta^p, f_p)\}_{p\in\mathbb{N}}$ is the sequence of sections $\{s_{\Lambda_p} \in H^0(X, L^p \otimes K)\}_{p\in\mathbb{N}}$ defined for any $x \in X$ by the formula

$$s_{\Lambda_p}(x) = \int_{\Lambda_p} P_p(x,y).\zeta^p f_p(y)\, dv_{\Lambda_p}(y). \tag{2.14}$$

An isotropic state associated with a sequence of Bohr-Sommerfeld Lagrangian submanifolds is called a *Lagrangian state*. Istropic states are characterized by the following reproducing property, which readily follows from their Definition 2.5.

Proposition 2.6. *[27, Prop. 3.4] For any $p \in \mathbb{N}$ and $s \in H^0(X, L^p \otimes K)$, we have*

$$\langle s, s_{\Lambda_p}\rangle_{L^2} = \int_{\Lambda_p} h^p(s(x), \zeta^p f_p(x))\, dv_{\Lambda_p}(x). \tag{2.15}$$

Proof. By Definition 2.3 of the Bergman kernel, the fact that the orthogonal projection (2.7) is self-adjoint translates into the following formula, for any $x, y \in X$ and $\eta \in \mathscr{C}^\infty(X, L^p \otimes K)$,

$$h^p\left(\eta(x), P_p(x,y).\eta(y)\right) = h^p\left(P_p(y,x).\eta(x), \eta(y)\right) \,. \tag{2.16}$$

Furthermore, for any $s \in H^0(X, L^p \otimes K)$, the fact that the orthogonal projection (2.7) acts as the identity on $H^0(X, L^p)$ translates into the following reproducing formula, for all $x \in X$,

$$s(x) = \int_X P_p(x,y).s(y)\, dv_X(y) \,. \tag{2.17}$$

Then using Definition 2.5 and Fubini's theorem, we compute

$$\begin{aligned}\langle s, s_{\Lambda_p}\rangle_{L^2} &= \int_X \int_{\Lambda_p} h^p\left(s(y), P_p(y,x).\zeta^p f_p(x)\right) dv_{\Lambda_p}(x)\, dv_X(y) \\ &= \int_{\Lambda_p} \int_X h^p\left(P_p(x,y).s(y), \zeta^p f_p(x)\right) dv_X(y)\, dv_{\Lambda_p}(x) \\ &= \int_{\Lambda_p} h^p(s(x), \zeta^p f_p(x))\, dv_{\Lambda_p}(x).\end{aligned} \tag{2.18}$$

This shows the result. □

The following Definition is purely technical, and will be used to deal with the varying sequences of Bohr-Sommerfeld submanifolds appearing in the main Theorems 1.1 and 1.2.

Definition 2.7. We say that a sequence $\{\iota_p : \Lambda_p \hookrightarrow X\}_{p\in\mathbb{N}}$ of embedded submanifolds of X *converges smoothly* towards a properly embedded submanifold $\iota : \Lambda \hookrightarrow X$ if there exists $p_0 \in \mathbb{N}$ and a finite collection of charts $\{U_j \subset X\}_{j=1}^m$ covering both Λ and Λ_p for all $p \geqslant p_0$, together with diffeomorphisms $\phi_j : U_j \longrightarrow V \subset \mathbb{R}^{2n}$ with $\phi_j(U_j \cap \Lambda) = V \cap \mathbb{R}^{\dim\Lambda}$ and diffeomorphisms $\phi_j^{(p)} : U_j \longrightarrow V \subset \mathbb{R}^{2n}$ with $\phi_j^{(p)}(U_j \cap \Lambda_p) = V \cap \mathbb{R}^{\dim\Lambda}$, for all $1 \leqslant j \leqslant m$ and $p \geqslant p_0$, such that $\phi_j^{(p)} \circ \phi_j^{-1} : V \to V$ converges smoothly towards the identity as $p \to +\infty$.

We say that a sequence $\{(\Lambda_p, \zeta^p, f_p)\}_{p\in\mathbb{N}}$ of Bohr-Sommerfeld submanifolds *converges smoothly* towards the couple (Λ, f), where $f \in \mathscr{C}^\infty(\Lambda, \iota^*K)$ is a section over Λ, if furthermore, there exist $f_j \in \mathscr{C}^\infty(U_j, K)$ for all $1 \leqslant j \leqslant m$ such that $f|_{\Lambda\cap U_j} = \iota^* f_j$ and $f_p|_{\Lambda\cap U_j} = \iota_p^* f_j$ for all $p \geqslant p_0$.

Note that if a sequence of submanifolds $\{\Lambda_p \subset X\}_{p\in\mathbb{N}}$ converges smoothly towards a compact submanifold $\Lambda \subset X$, then Λ_p is diffeomorphic to Λ for all $p \in \mathbb{N}$ big enough. We will also need the following standard notion.

Definition 2.8. We say that two submanifolds $\Lambda^{(1)}, \Lambda^{(2)} \subset X$ are intersecting *cleanly* if the intersection $\Lambda^{(1)} \cap \Lambda^{(2)}$ is a submanifold of X such that for any $x \in \Lambda^{(1)} \cap \Lambda^{(2)}$, we have $T_x\Lambda^{(1)} \cap T_x\Lambda^{(2)} = T_x(\Lambda^{(1)} \cap \Lambda^{(2)})$.

Note that if two respective sequences of submanifolds $\{\Lambda_p^{(1)}, \Lambda_p^{(2)} \subset X\}_{p\in\mathbb{N}}$ converge smoothly towards $\Lambda^{(1)}, \Lambda^{(2)} \subset X$ intersecting cleanly, then the sequence $\{\Lambda_p^{(1)} \cap \Lambda_p^{(2)}\}_{p\in\mathbb{N}}$ converges smoothly towards $\Lambda^{(1)} \cap \Lambda^{(2)}$.

The following result, adapted from [27], gives an asymptotic estimate on the norm of an isotropic state in terms of the volume of the associated isotropic submanifold as $p \to +\infty$.

Theorem 2.9. *[27, Th. 3.6] Let $\iota : \Lambda \hookrightarrow X$ be an isotropic submanifold endowed with $f \in \mathscr{C}^\infty(\Lambda, \iota^*K)$, and let $\{(\Lambda_p, \zeta^p, f_p)\}_{p\in\mathbb{N}}$ be a sequence of Bohr-Sommerfeld submanifolds converging smoothly towards (Λ, f). Then the L^2-norm of the associated isotropic state $\{s_{\Lambda_p} \in H^0(X, L^p \otimes K)\}_{p\in\mathbb{N}}$ satisfy the following asymptotic expansion as $p \to +\infty$,*

$$\left\| s_{\Lambda_p} \right\|_{L^2}^2 = 2^{\frac{\dim\Lambda}{2}} p^{n-\frac{\dim\Lambda}{2}} \left(\int_\Lambda |f_p|_K^2 \, dv_{\Lambda_p} + O(p^{-1}) \right) . \tag{2.19}$$

Proof. This is a straightforward consequence of the proof of [27, Th. 3.6], where all estimates are uniform with respect to deformation of parameters, so that it readily extends to cover the case of sequences of smoothly converging submanifolds in the sense of Definition 2.7 instead of a fixed Bohr-Sommerfeld submanifold for all $p \in \mathbb{N}$. □

The following result, also adapted from [27], gives an asymptotic expansion as $p \to +\infty$ on the Hermitian product of two isotropic states in terms of the intersection of the associated isotropic submanifolds.

Theorem 2.10. *[27, Th. 4.3] Let $\iota_j : \Lambda^{(j)} \hookrightarrow X$ be isotropic submanifolds endowed with $f^{(j)} \in \mathscr{C}^\infty(\Lambda^{(j)}, \iota_j^* K)$, let $\{(\Lambda_p^{(j)}, \zeta_j^p, f_p^{(j)})\}_{p\in\mathbb{N}}$ be sequences of Bohr-Sommerfeld submanifolds converging smoothly towards $(\Lambda^{(j)}, f^{(j)})$, and write $\{s_{\Lambda_p^{(j)}} \in H^0(X, L^p \otimes K)\}_{p\in\mathbb{N}}$ for the associated associated isotropic states, for each $j = 1, 2$.*

If $\Lambda_1 \cap \Lambda_2 = \emptyset$, then for any $k \in \mathbb{N}$, there exists $C_k > 0$ such that for all $p \in \mathbb{N}$, we have

$$\left|\left\langle s_{\Lambda_p^{(1)}}, s_{\Lambda_p^{(2)}} \right\rangle_{L^2}\right| \leqslant C_k\, p^{-k} . \tag{2.20}$$

If Λ_1 and Λ_2 intersect cleanly, writing $\Lambda^{(1)} \cap \Lambda^{(2)} = \cup_{q=0}^m Y^{(q)}$ for the decompositions into connected components, there exist $b_p^{(q)} \in \mathbb{C}$ for each $0 \leqslant q \leqslant m$ and $p \in \mathbb{N}$, such that for any $k \in \mathbb{N}$ and as $p \to +\infty$,

$$\left\langle s_{\Lambda_p^{(1)}}, s_{\Lambda_p^{(2)}} \right\rangle_{L^2} = p^{n-\left(\frac{\dim \Lambda^{(1)}}{2} + \frac{\dim \Lambda^{(2)}}{2}\right)} \sum_{q=1}^m p^{\frac{\dim Y^{(q)}}{2}} \lambda_p^{(q)} \left(b_p^{(q)} + O(p^{-1})\right) , \tag{2.21}$$

where $\lambda_p^{(q)} \in \mathbb{C}$ is the value of the constant function on $Y_p^{(q)}$ defined for any $x \in Y_p^{(q)}$ by $\lambda_p^{(q)}(x) = h^{L^p}(\zeta_1^p(x), \zeta_2^p(x))$, with $\Lambda_p^{(1)} \cap \Lambda_p^{(2)} = \cup_{q=0}^m Y_p^{(q)}$ a decomposition into connected components for all $p \in \mathbb{N}$ such that $\{Y_p^{(q)}\}_{p\in\mathbb{N}}$ converges smoothly towards $Y^{(q)}$ for each $1 \leqslant q \leqslant m$.

Furthermore, if $\dim \Lambda_1 = n$, we have

$$b_p^{(q)} = 2^{n/2} \int_{Y_p^{(q)}} h^K \left(f_p^{(1)}, f_p^{(2)}\right) \det{}^{-\frac{1}{2}} \left(\sum_{r=1}^{n-\dim Y^{(q)}} \sqrt{-1}\, h^{TX}(e_i^{(2)}, e_r^{(1)})\, \omega(e_k^{(2)}, e_r^{(1)}) \right)_{i,k} |dv|_{Y^{(q)}}, \tag{2.22}$$

for some choice of square root for the determinant, where $\{e_k^{(j)}\}_k$ are local orthonormal frames of the normal bundle of $Y_p^{(q)}$ inside $\Lambda_p^{(j)}$ for each $j = 1, 2$, and $|dv|_{Y_p^{(q)}}$ is the Riemannian density associated with the restriction of g^{TX} to $Y_p^{(q)}$.

Proof. This is a straightforward consequence of the proof of [27, Th. 4.3], where all estimates are uniform with respect to deformations of parameters, so that it readily extends to cover the case of sequences of converging submanifolds in the sense of Definition 2.7 instead of a fixed Bohr-Sommerfeld manifold for all $p \in \mathbb{N}$. Note however that the determinant term in the integrand of formula (2.22) is actually the conjugate of the corresponding term in [27, Th. 4.2, (4.4)], which is due to the wrong sign appearing in front of the $\sqrt{-1}$-term in the second equality of [27, (4.17)]. Then [27, Th. 6.3, (6.9)] should also be corrected accordingly with a minus sign in front of $\sqrt{-1}$ appearing inside the exponential term. □

Let us now consider the important special case when one takes $K = K_X^{1/2}$ to be a square root of the canonical line bundle (2.5) of X. As explained in [31, § D], such a square root exists only if the first Chern class $c_1(TX)$ of is even in $H^2(X, \mathbb{Z})$, and is not unique in general. We endow $K_X^{1/2}$ with the Hermitian structure induced by the Hermitian metric (2.4). Now if $\iota : \Lambda \hookrightarrow X$ is a properly embedded Lagrangian submanifold, we have a canonical isomorphism

$$\begin{aligned} \iota^* K_X &\xrightarrow{\sim} \det(T^*_{\mathbb{C}}\Lambda) \\ \beta &\longmapsto \Big[e_1 \wedge \cdots \wedge e_n \mapsto \beta(e_1^{(1,0)}, \cdots, e_n^{(1,0)})\Big]\,. \end{aligned} \tag{2.23}$$

We then have the following consequence of Theorem 2.10.

Theorem 2.11. *Let $\iota_j : \Lambda^{(j)} \to X$ be Lagrangian submanifolds intersecting transversally endowed with $f^{(j)} \in \mathscr{C}^\infty(\Lambda_j, \iota_j^* K_X^{1/2})$, and let $\{(\Lambda_p^{(j)}, \zeta_j^p, f_p^{(j)})\}_{p\in\mathbb{N}}$ be sequences of Bohr-Sommerfeld submanifolds converging smoothly towards $(\Lambda^{(j)}, f^{(j)})$, for each $j = 1, 2$. Then as $p \to +\infty$, we have*

$$\begin{aligned} \Big\langle s_{\Lambda_p^{(1)}}, s_{\Lambda_p^{(2)}} \Big\rangle_{L^2} = 2^{\frac{n}{2}} \sum_{x \in \Lambda_p^{(1)} \cap \Lambda_p^{(2)}} & h^{L^p}(\zeta_1^p(x), \zeta_2^p(x)) \\ & \det{}_x^{-1/2}\Big(\sqrt{-1}\omega(\xi_i^{(2)}, \xi_k^{(1)})\Big)_{i,k} + O(p^{-1})\,, \end{aligned} \tag{2.24}$$

for some choices of square root of the determinant, where for all $p \in \mathbb{N}$ big enough, all $x \in \Lambda_p^{(1)} \cap \Lambda_p^{(2)}$ and each $j = 1, 2$, the vectors $\{\xi_k^{(j)}\}_{k=1}^n$ are bases of $T_x\Lambda^{(j)}$ satisfying

$$(f_p^{(j)})^2(\xi_1^{(1)}, \cdots \xi_n^{(1)}) = 1\,, \tag{2.25}$$

via the isomorphism (2.23).

Proof. Set $K = K_X^{1/2}$, and recall that the Hermitian metric (2.4) on (TX, J) coincides with the metric on $T^{(1,0)}X$ induced by g^{TX} via the splitting (2.3). Letting $x \in \Lambda_p^{(1)} \cap \Lambda_p^{(2)}$, if $\{\xi_k^{(j)}\}_{k=1}^n$ is a basis of $T_x\Lambda^{(j)}$ satisfying (2.25) and $\{e_k^{(j)}\}_{i=1}^n$ is an orthonormal basis of $(T_x\Lambda^{(j)}, \iota_j^* g^{TX})$ for each $j = 1, 2$, we get

$$\begin{aligned} h_x^K(f_p^{(1)}, f_p^{(2)}) &= \det{}^{-1/2}\Big(h_x^{TX}(\xi_i^{(1)}, \xi_k^{(2)})\Big)_{i,k} \\ &= \det{}^{-1/2}\Big(h_x^{TX}(e_i^{(1)}, e_k^{(2)})\Big)_{i,k} \det{}^{-1/2}(g_x^{TX}(e_i^{(1)}, \xi_k^{(1)}))_{i,k} \\ &\qquad \det{}^{-1/2}(g_x^{TX}(e_i^{(2)}, \xi_k^{(2)}))_{i,k}\,. \end{aligned} \tag{2.26}$$

Thus under the assumption that $\dim \Lambda_1 = \dim \Lambda_2 = n$ and using the multiplicativity property of the determinant, we get

$$\begin{aligned} h^K_x(f^{(1)}_p, f^{(2)}_p) \det{}^{-\frac{1}{2}}\Big(\sum_{r=1}^{n-\dim Y^{(q)}} \sqrt{-1}\, h^{TX}_x(e^{(2)}_i, e^{(1)}_r)\, \omega_x(e^{(2)}_k, e^{(1)}_r)\Big)_{i,k} \\ = \left|\det{}^{-1/2}\left(h^{TX}_x(e^{(1)}_i, e^{(2)}_k)\right)_{i,k}\right|^2 \det{}^{-1/2}(g^{TX}_x(e^{(1)}_i, \xi^{(1)}_k))_{i,k} \\ \det{}^{-1/2}(g^{TX}_x(e^{(2)}_i, \xi^{(2)}_k))_{i,k} \det{}^{-\frac{1}{2}}\left(\sqrt{-1}\omega_x(e^{(2)}_i, e^{(1)}_k)\right)_{i,k} \\ = \det{}^{-1/2}(\sqrt{-1}\omega_x(\xi^{(2)}_i, \xi^{(1)}_k))_{i,k}\,, \quad (2.27) \end{aligned}$$

where we also used the fact that orthonormal bases of $(T_x\Lambda^{(j)}, \iota_j^* g^{TX})$ for each $j = 1, 2$ induce orthonormal bases of $T^{(1,0)}X$ for the induced Hermitian metric via the splitting (2.3), and that a change of orthonormal bases of a Hermitian vector space is unitary, hence has determinant of complex modulus 1. We then get formula (2.24) from formula (2.22). Note that due to the fact the determinant term appearing in the integrand of [27, Th. 4.2, (4.4)] should be replaced by its conjugate, formula (2.24) differs from the corresponding formula in [27, Rmk. 4.5, (4.38)], where the term $\det\{\Lambda_1, \Lambda_2\}$ should not appear. □

3 Quantization of group actions

In this section, we describe the behavior of Berezin-Toeplitz quantization described in Section 2 with respect to the action of a compact Lie group G, and its applications to the representation theory of G via the symplectic geometry of coadjoint orbits.

3.1 Kostant-Souriau quantization

Let X be a smooth manifold endowed with a Hermitian line bundle (L, h^L) with Hermitian connection ∇^L, and let G be a compact Lie group acting on L over X, preserving the Hermitian metric h^L and the Hermitian connection ∇^L. We write $\mathfrak{g} := \mathrm{Lie}(G)$ for the Lie algebra of G. We consider the action of G on the space of smooth sections $\mathscr{C}^\infty(X, L)$ defined for all $g \in G$, $s \in \mathscr{C}^\infty(X, L)$ and $x \in X$ by

$$(gs)(x) := g.s(g^{-1}x)\,. \quad (3.1)$$

For any $\xi \in \mathfrak{g}$, we write L_ξ for the induced *Lie derivative* acting on $s \in \mathscr{C}^\infty(X, L)$ by

$$L_\xi\, s := \frac{d}{dt}\Big|_{t=0} e^{-t\xi} s\,, \quad (3.2)$$

and denote by $\xi^X \in \mathscr{C}^\infty(X, TX)$ the vector field over X induced by the infinitesimal action of ξ, defined by the formula $\xi^X := \frac{d}{dt}\big|_{t=0} e^{t\xi}x$. The following definition is taken from [4, Def. 7.5].

Definition 3.1. The *moment* $\mu^L : X \to \mathfrak{g}^*$ of the action of G on (L, h^L, ∇^L) over X is defined by the following formula, for all $\xi \in \mathfrak{g}$ and $s \in \mathscr{C}^\infty(X, L)$,

$$\langle \mu^L, \xi \rangle\, s = \frac{\sqrt{-1}}{2\pi} \left(L_\xi\, s - \nabla^L_{\xi^X} s \right) . \tag{3.3}$$

Formula (3.3) is called the *Kostant formula*. One readily checks that the right-hand side of formula (3.3) is $\mathscr{C}^\infty(X)$-linear, so that the left-hand side defines a function $\langle \mu^L, \xi \rangle \in \mathscr{C}^\infty(X, \mathbb{R})$, and that the application $\mu^L : X \to \mathfrak{g}^*$ thus defined is linear and equivariant with respect to the *coadjoint action* of G on $\mathfrak{g}^*$, defined for any $g \in G$ and $\lambda \in \mathfrak{g}^*$ by

$$\langle \operatorname{Ad}^*_g \lambda, \xi \rangle := \langle \lambda, \operatorname{Ad}_{g^{-1}} \xi \rangle . \tag{3.4}$$

In case (X, ω) is a symplectic manifold prequantized by (L, h^L, ∇^L) in the sense of (1.17), one readily checks that Definition 3.1 defines a *moment map* $\mu := \mu^L : X \to \mathfrak{g}^*$ for the symplectic action of G on (X, ω), so that

$$d\langle \mu, \xi \rangle = \iota_{\xi^X} \omega \,, \tag{3.5}$$

where ι_v denotes the interior product with respect to $v \in \mathscr{C}^\infty(X, TX)$. Note now that for any $p \in \mathbb{N}$, Definition 3.1 applied to the induced action of G on L^p and to $s^p := s^{\otimes p} \in \mathscr{C}^\infty(X, L^p)$ readily gives that $\mu^{L^p} = p\,\mu$, so that for any $s_p \in \mathscr{C}^\infty(X, L^p)$ and $\xi \in \mathfrak{g}$, we get

$$\frac{\sqrt{-1}}{2\pi} L_\xi s_p = \Big(p \, \langle \mu, \xi \rangle + \frac{\sqrt{-1}}{2\pi} \nabla^{L^p}_{\xi^X} \Big) s_p \,. \tag{3.6}$$

More generally, if G acts on a complex line bundle K preserving a Hermitian metric h^K and a Hermitian connection ∇^K, Definition 3.1 applied $L^p \otimes K$ gives $\mu^{L^p \otimes K} = p\mu + \mu^K$ for any $p \in \mathbb{N}$, so that for any $s_p \in \mathscr{C}^\infty(X, L^p \otimes K)$, the Kostant formula (3.3) gives

$$\frac{\sqrt{-1}}{2\pi} L_\xi s_p = \Big(p \, \langle \mu, \xi \rangle + \langle \mu^K, \xi \rangle + \frac{\sqrt{-1}}{2\pi} \nabla^p_{\xi^X} \Big) s_p \,. \tag{3.7}$$

Let us now assume that (X, ω) is equipped with a compatible G-invariant complex structure $J \in \operatorname{End}(TX)$, making (X, ω, J) into a Kähler manifold and (L, h^L), (K, h^K) into holomorphic Hermitian line bundles on which the action of G lifts holomorphically. In particular, the action of G on sections preserves the finite-dimensional subspace $H^0(X, L^p \otimes K) \subset \mathscr{C}^\infty(X, L^p \otimes K)$ of holomorphic sections, making $H^0(X, L^p \otimes K)$ into a finite-dimensional representation of G, and the Lie derivative L_ξ defined as in (3.2) preserves $H^0(X, L^p \otimes K)$. Recalling the Definition 2.1 of Berezin-Toeplitz quantization, we then have the following classical formula of Tuynman.

Proposition 3.2. [38] *For any $\xi \in \mathfrak{g}$ and any $s \in H^0(X, L^p \otimes K)$, we have*

$$\frac{\sqrt{-1}}{2\pi p} L_\xi s = T_p(\langle \mu, \xi \rangle) s + \frac{1}{p} T_p \Big(\langle \mu^K, \xi \rangle + \frac{1}{4\pi} \Delta^X \langle \mu, \xi \rangle \Big) s \,, \tag{3.8}$$

where Δ^X is the Laplace-Beltrami operator of (X, g^{TX}) acting on $\mathscr{C}^\infty(X, \mathbb{R})$.

Proof. For any $s_1,\, s_2 \in H^0(X, L^p \otimes K)$, the Kostant formula (3.7) implies

$$\frac{\sqrt{-1}}{2\pi}\langle L_\xi s_1, s_2\rangle_{L^2} = \langle (p\,\langle \mu, \xi\rangle + \langle \mu^K, \xi\rangle) s_1, s_2\rangle_{L^2} + \frac{\sqrt{-1}}{2\pi}\langle \nabla^p_{\xi^X} s_1, s_2\rangle_{L^2}\,. \tag{3.9}$$

Recall on the other hand that by definition of a holomorphic section, we have $\nabla^p_{v^{(0,1)}} s = 0$ for all $v \in \mathscr{C}^\infty(X, TX)$ and $s \in H^0(X, L^p \otimes K)$. Thus by definition (2.6) of the L^2-product and using that the connection ∇^p induced by ∇^L, ∇^K on $L^p \otimes K$ preserves the Hermitian product h^p induced by h^L, h^K, we get

$$\begin{aligned} \langle \nabla^p_v s_1, s_2\rangle_{L^2} &= \int_X h^p(\nabla^p_{v^{(1,0)}} s_1, s_2)\, dv_X \\ &= \int_X L_{v^{(1,0)}} h^p(s_1, s_2)\, dv_X \\ &= -\int_X h^p(s_1, s_2)\, d\iota_{v^{(1,0)}} \frac{\omega^n}{n!}\,, \end{aligned} \tag{3.10}$$

where we also used Stokes theorem, the definition (2.2) of the Liouville volume form and Cartan's formula $L_\xi = \iota_\xi d + d\iota_\xi$. Using now the fact that $v^{(1,0)} = v - iJv$, which readily follows from the definition of the splitting (2.3), and that ω is J-invariant, we get for any $w \in \mathscr{C}^\infty(X, TX)$ that

$$\iota_{v^{(1,0)}}\omega(w) = \omega(v, w) + i\omega(v, Jw) = \iota_v \omega(w^{(0,1)})\,. \tag{3.11}$$

By the fundamental property (3.5) of a moment map and using that ω is closed, we then compute for all $\xi \in \mathfrak{g}$ that

$$\begin{aligned} d\iota_{(\xi^X)^{(1,0)}} \frac{\omega^n}{n!} &= d\left(\frac{\overline{\partial}\langle \mu, \xi\rangle \wedge \omega^{n-1}}{(n-1)!}\right) \\ &= -\frac{\sqrt{-1}}{2}\Delta^X \langle \mu, \xi\rangle\ \frac{\omega^n}{n!}\,, \end{aligned} \tag{3.12}$$

where we also used the classical formula of Kähler geometry, which can be deduced for instance from [37, Ex. 1.8],

$$\frac{1}{2}\Delta^X f\ \frac{\omega^n}{n!} = \sqrt{-1}\frac{\overline{\partial}\partial f \wedge \omega^{n-1}}{(n-1)!}\,. \tag{3.13}$$

Using that the orthogonal projection (2.7) with respect to the L^2-product (2.6) restricts to the identity on $H^0(X, L^p \otimes K)$, we thus get from equation (3.9) that

$$\begin{aligned} \frac{\sqrt{-1}}{2\pi}\langle L_\xi s_1, s_2\rangle_{L^2} &= \left\langle \Big(p\,\langle \mu, \xi\rangle + \langle \mu^K, \xi\rangle + \frac{1}{4\pi}\Delta^X\langle \mu, \xi\rangle\Big) s_1, s_2 \right\rangle_{L^2} \\ &= \left\langle T_p\Big(p\,\langle \mu, \xi\rangle + \langle \mu^K, \xi\rangle + \frac{1}{4\pi}\Delta^X\langle \mu, \xi\rangle\Big) s_1, s_2 \right\rangle_{L^2}\,, \end{aligned} \tag{3.14}$$

which is true for all $s_1,\, s_2 \in H^0(X, L^p \otimes K)$. This proves the result. □

Recall now the natural ring isomorphism

$$\begin{aligned} S(\mathfrak{g}) &\xrightarrow{\sim} \mathbb{R}[\mathfrak{g}^*] \\ \xi_1 \cdots \xi_j &\longmapsto [\alpha \mapsto \langle \alpha, \xi_1 \rangle \cdots \langle \alpha, \xi_j \rangle] \end{aligned} \tag{3.15}$$

between the real symmetric algebra $S(\mathfrak{g})$ of $\mathfrak{g}$ and the ring $\mathbb{R}[\mathfrak{g}^*]$ of polynomials with real coefficients over $\mathfrak{g}^*$. We then have the following definition, which will play a central role for the proof of Theorem 1.1 in Section 5.

Definition 3.3. For any $p \in \mathbb{N}$, the *symmetric quantization* of the action of G on $H^0(X, L^p \otimes K)$ is the linear map defined through the identification (3.15) by

$$\begin{aligned} Q_p : \mathbb{R}[\mathfrak{g}^*] &\longrightarrow \mathrm{End}(H^0(X, L^p \otimes K)) \\ \xi_1 \cdots \xi_j &\longmapsto \frac{(-1)^j}{j!} \sum_{\sigma \in \Sigma_j} L_{\xi_{\sigma(1)}} \cdots L_{\xi_{\sigma(j)}} . \end{aligned} \tag{3.16}$$

Note that by definition of the action (3.1) and of the Lie derivative (3.2), the operator $Q_p(\xi) \in \mathrm{End}(H^0(X, L^p \otimes K))$ coincides with the infinitesimal action of $\xi \in \mathfrak{g}$ on $H^0(X, L^p \otimes K)$. The following fundamental semiclassical property of the symmetric quantization of Definition 3.3 readily follows from the tools of Berezin-Toeplitz quantization described in Section 2.

Proposition 3.4. *For any homogeneous polynomial $P \in \mathbb{R}[\mathfrak{g}^*]$ of degree $j \in \mathbb{N}$, we have the following asymptotic expansion in the sense of the operator norm as $p \to +\infty$,*

$$\frac{1}{(2\pi\sqrt{-1}p)^j} Q_p(P) = T_p(\mu^* P) + O(p^{-1}) . \tag{3.17}$$

Proof. Using Proposition 3.2 and applying repeatedly formula (2.10) of Theorem 2.2, we readily get the following asymptotic formula in the sense of the operator norm as $p \to +\infty$, for any $\xi_1, \cdots, \xi_k \in \mathfrak{g}$,

$$\frac{\sqrt{-1}^k}{(2\pi p)^k} L_{\xi_1} \cdots L_{\xi_2} = T_p(\langle \mu, \xi_1 \rangle \cdots \langle \mu, \xi_n \rangle) + O(p^{-1}) , \tag{3.18}$$

where we also used formula (2.9) of Theorem 2.2 on the terms of order at most p^{-1}. As the right-hand side does not depend on the ordering of $\xi_1, \cdots, \xi_k \in \mathfrak{g}$ in the left-hand side, this implies the result by Definition 3.3 via the identification (3.15). □

3.2 Structure and representation theory of compact Lie groups

Let G be a compact Lie group, and fix a *maximal torus* $T \subset G$, that is an abelian subgroup of G which is maximal for the inclusion among abelian subgroups. Recall for instance from [11, Chap. IV, (1.6)] that any two maximal tori in G are conjugate to each other, and that every $g \in G$ is contained in a maximal torus. Write $N(T) \subset G$ for the normalizer of T inside G, and note in particular that $T \subset N(T)$. Following [11, Chap. IV], let us write

$$W := N(T)/T \tag{3.19}$$

for the *Weyl group* of G, which is a finite group with an action on T by automorphisms induced from the action of $N(T)$ by conjugation, so that two elements of T are conjugate in G if and only if they belong to the same orbit of W.

Write $\mathfrak{g} := \mathrm{Lie}(G)$ for the Lie algebra of G and $\mathfrak{t} := \mathrm{Lie}(T) \subset \mathfrak{g}$ for the Lie subalgebra of $T \subset G$. Let us fix once and for all a G-invariant scalar product $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ on $\mathfrak{g}$, inducing identifications $\mathfrak{g} \simeq \mathfrak{g}^*$ and $\mathfrak{t} \simeq \mathfrak{t}^*$ which will be understood in the whole sequel. In particular, we will often consider the inclusion $\mathfrak{t}^* \subset \mathfrak{g}^*$ induced by $\mathfrak{t} \subset \mathfrak{g}$. We write $\langle \cdot, \cdot \rangle_{\mathfrak{g}^*}$ for the scalar product on $\mathfrak{g}^*$ induced by $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$. Following [11, Chap.V, (2.15)], let $\Gamma^* \subset \mathfrak{t}^*$ be the *lattice of integral forms*, defined by

$$\Gamma^* := \{\alpha \in \mathfrak{t}^* \mid \langle \alpha, \xi \rangle \in \mathbb{Z}, \text{ for all } \xi \in \mathfrak{t} \text{ such that } \exp(\xi) = 1\}\,, \tag{3.20}$$

Following [11, Chap.V, §2], there exists a finite subset $R \subset \Gamma^*$, called the *set of roots*, such that we have a decompostion

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in R} (\mathfrak{g}_{\mathbb{C}})_\alpha\,, \tag{3.21}$$

of the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes \mathbb{C}$ into the common eigenspaces

$$(\mathfrak{g}_{\mathbb{C}})_\alpha := \{\xi \in \mathfrak{g} \mid [\eta, \xi] = 2\pi\sqrt{-1}\langle \alpha, \eta \rangle\, \xi \text{ for all } \eta \in \mathfrak{t}\}\,, \tag{3.22}$$

for the adjoint action of $\mathfrak{t}$ on $\mathfrak{g}$, which satisfy $\dim_{\mathbb{C}} \mathfrak{g}_\alpha = 1$ for all $\alpha \in R$ and where $\mathfrak{t}_{\mathbb{C}} := \mathfrak{t} \otimes \mathbb{C}$. Let us define the open set of *regular points* of $\mathfrak{t}^*$ by

$$\mathfrak{t}^*_{\mathrm{reg}} := \{\lambda \in \mathfrak{t}^* \mid \langle \alpha, \lambda \rangle_{\mathfrak{g}^*} \neq 0 \text{ for all } \alpha \in R\}\,. \tag{3.23}$$

The connected components of $\mathfrak{t}^*_{\mathrm{reg}} \subset \mathfrak{t}^*$ are called *Weyl chambers*, and the Weyl group W acts simply and transitively on the set of Weyl chambers via the action on $\mathfrak{t}^*$ induced by its action on T. We fix once and for all a set of *positive roots* $R_+ \subset R$, which by definition satisfies

$$R = R_+ \cup (-R_+) \text{ and } R_+ \cap (-R_+) = \emptyset\,. \tag{3.24}$$

This allows to single out a Weyl chamber, called the fundamental or *positive Weyl chamber* corresponding to $R_+ \subset R$, defined by

$$\mathfrak{t}^*_+ := \{\lambda \in \mathfrak{t}^* \mid \langle \alpha, \lambda \rangle_{\mathfrak{g}^*} > 0 \text{ for all } \alpha \in R_+\}\,. \tag{3.25}$$

Following [11, Chap.V, (4.11)], let us finally introduce the *half-sum of positive roots*

$$\rho := \frac{1}{2} \sum_{\alpha \in R_+} \alpha \in \mathfrak{t}^*\,, \tag{3.26}$$

which plays a key role in the structure theory of compact Lie groups, as well as their representation theory described below.

Let now V be a finite-dimensional unitary representation of G, let $T_0 \subset T$ be a subtorus of the maximal torus of G, and write $\mathfrak{t}_0 := \mathrm{Lie}(T_0) \subset \mathfrak{t}$ for its Lie subalgebra. Following [29, Chap. III, § 2], the action of $T_0 \subset G$ induced on V by restriction leads to a unique orthogonal decomposition

$$V = \bigoplus_{\lambda \in \Gamma^* \cap \mathfrak{t}_0^*} V_\lambda \,, \tag{3.27}$$

preserved by the action of T_0, where

$$V_\lambda := \{ v \in V \mid \exp(\xi).v = e^{2\pi\sqrt{-1}\langle\lambda,\xi\rangle} v \text{ for all } \xi \in \mathfrak{t}_0 \} \,. \tag{3.28}$$

The finite subset

$$\Gamma_V^{T_0} := \{ \lambda \in \Gamma^* \cap \mathfrak{t}_0 \mid V_\lambda \neq 0 \} \subset \Gamma^* \tag{3.29}$$

is called the *set of weights* of the action of T_0 on V. If $T_0 = T$, we simply write $\Gamma_V := \Gamma_V^{T_0}$, and call it the *set of weights* of V. Note that it is independant of the choice of the maximal torus $T \subset G$ since two maximal tori are conjugate, and that the finite-dimensional representation $\mathfrak{g}_\mathbb{C}$ of G by adjoint action has $\Gamma_{\mathfrak{g}_\mathbb{C}} = R \cup \{0\}$ as set of weights.

Following for instance [29, Th. 4.28], we introduce the following notion, which plays a fundamental role in representation theory.

Definition 3.5. An irreducible representation V of G is said to have *highest weight* $\lambda \in \Gamma_V$ if for all $\gamma \in \Gamma_V$, we have

$$\lambda - \gamma \in \mathfrak{t}_+^* \,. \tag{3.30}$$

Write $\overline{\mathfrak{t}_+^*} \subset \mathfrak{t}^*$ for the closure of the positive Weyl chamber (3.25) inside $\mathfrak{t}^*$. The following fundamental classification of finite-dimensional representations of G is due to Weyl.

Theorem 3.6. *[29, Th. 4.28] There is a bijective correspondence between the set of unitary irreducible representations of G and the discrete set*

$$\overline{\Gamma}_+^* := \Gamma^* \cap \overline{\mathfrak{t}_+^*} \,, \tag{3.31}$$

in such a way that for any $\lambda \in \Gamma_+^*$, there exists a unique finite-dimensional irreducible representation $V(\lambda)$ of G with highest weight λ.

The discrete set (3.31) parameterizing the irreducible representations of G is called the *lattice of integral dominant weights*.

Let us finish this Section by describing the case of the unitary group $G = U(n)$ for any $n \in \mathbb{N}^*$, with Lie algebra $\mathfrak{u}(n) := \mathrm{Lie}(U(n))$, which we adapt from [23, p. 120] to fit with our conventions. Following [11, Chap. IV, § 3], we make the natural choice of maximal torus $T \subset U(n)$ consisting of the diagonal matrices, so that the associated Weyl group (3.19) coincides with the n^{th} symmetric group $\mathfrak{S}_n$ acting on diagonal matrices in $T \subset G$ by permutation of the diagonal entries. Recall that $\mathfrak{u}(n)$ is naturally identified with the space of $n \times n$ anti-Hermitian matrices, and consider the $U(n)$-equivariant identification of its dual $\mathfrak{u}(n)^*$ with the space $\mathrm{Herm}(\mathbb{C}^n)$ of $n \times n$ Hermitian matrices given by

$$\begin{aligned} \mathrm{Herm}(\mathbb{C}^n) &\xrightarrow{\sim} \mathfrak{u}(n)^* \\ 2\pi\sqrt{-1}A &\longmapsto \Big[B \mapsto \mathrm{Tr}[AB]\Big] \,. \end{aligned} \tag{3.32}$$

We endow $\mathfrak{u}(n)^*$ with the $U(n)$-invariant scalar product $\langle\cdot,\cdot\rangle_{\mathfrak{g}^*}$ induced by the trace product on $\mathrm{Herm}(\mathbb{C}^n)$ under this identification. Then the subspace $\mathfrak{t}^* \subset \mathfrak{g}^*$ induced by $\mathfrak{t} \subset \mathfrak{g}$ coincides under this identification with the subspace of $\mathrm{Herm}(\mathbb{C}^n)$ consisting of diagonal matrices with real coefficients, so that we have a natural identification

$$\mathfrak{t}^* \simeq \mathbb{R}^n \,, \tag{3.33}$$

on which the natural action of the Weyl group $W = \mathfrak{S}_n$ acts by permutations of the coordinates. Then the lattice of integral weights (3.31) is given by the lattice of integral points $\Gamma^* \simeq \mathbb{Z}^n$. Following [11, Chap.V, § 6], there exists a natural choice of positive roots (3.24) for which the the associated positive Weyl chamber (3.25) is given by

$$\mathfrak{t}^*_+ = \{(\lambda_1, \cdots \lambda_n) \in \mathbb{R}^n \mid \lambda_1 > \lambda_2 > \cdots > \lambda_n\} \,, \tag{3.34}$$

so that the lattice of dominant weights (3.31) is given by

$$\overline{\Gamma}^*_+ = \{(\lambda_1, \cdots \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n\} \,. \tag{3.35}$$

By [11, Chap.V, Prop. 6.2.(vi)], the half-sum of positive roots (3.26) is given by

$$\rho_n = \left(\frac{n-1}{2}, \frac{n-3}{2}, \cdots, -\frac{n-3}{2}, -\frac{n-1}{2}\right) \,. \tag{3.36}$$

3.3 Coadjoint orbits

Recall that we fixed a G-invariant scalar product $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ on $\mathfrak{g}$, inducing an identification $\mathfrak{g} \simeq \mathfrak{g}^*$. Via this identification, the usual adjoint action of G on $\mathfrak{g}$ corresponds to the coadjoint action (3.4) of G on $\mathfrak{g}^*$. Recall also that we chose a maximal torus $T \subset G$, and that we have an inclusion $\mathfrak{t}^* \subset \mathfrak{g}^*$ induced by the G-invariant scalar product on $\mathfrak{g}$. For any $\lambda \in \mathfrak{t}^*$, we write $X_\lambda \subset \mathfrak{g}^*$ for the associated *coadjoint orbit*, defined by

$$X_\lambda := \{\mathrm{Ad}^*_g \lambda \in \mathfrak{g}^* \mid g \in G\} \,. \tag{3.37}$$

Note that since any element in G is conjugate to an element in $T \subset G$, all coadjoint orbits in $\mathfrak{g}^*$ are of the form $X_\lambda \subset \mathfrak{g}^*$ for some $\lambda \in \mathfrak{t}^*$, and that two elements $\lambda,\, \gamma \in \mathfrak{t}^*$ belong to the same coadjoint orbit $X_\lambda = X_\gamma \subset \mathfrak{g}^*$ if and only if they belong to the same orbit of the Weyl group (3.19).

Note also that the coadjoint action of G on X_λ is transitive by definition, so that its tangent space $T_\alpha X_\lambda$ at any point $\alpha \in X_\lambda$ is spanned by the vector fields $\xi^{X_\lambda} \in \mathscr{C}^\infty(X_\lambda, TX_\lambda)$ induced by the infinitesimal action of $\mathfrak{g}$, for all $\xi \in \mathfrak{g}$. The following fundamental property of coadjoint orbits can be found in [4, Lem. 7.22].

Proposition 3.7. *For any $\lambda \in \mathfrak{t}^*$, the associated coadjoint orbit X_λ is endowed with a natural G-invariant symplectic form $\omega \in \Omega^2(X_\lambda, \mathbb{R})$, defined for any $\xi_1, \xi_2 \in \mathfrak{g}$ and $\alpha \in X_\lambda$ by*

$$\omega_\alpha(\xi_1^{X_\lambda}, \xi_2^{X_\lambda}) = \alpha([\xi_1, \xi_2]) \,, \tag{3.38}$$

and the action of G on (X_λ, ω) is Hamiltonian, with moment map

$$\mu : X_\lambda \hookrightarrow \mathfrak{g}^* \tag{3.39}$$

given by the inclusion.

The symplectic form (3.38) is called the *Kirillov-Kostant-Souriau* symplectic form over the coadjoint orbit X_λ.

Let us write $G_\lambda \subset G$ for the stabilizer of $\lambda \in \mathfrak{t}^* \subset \mathfrak{g}^*$ by the coadjoint action (3.4), so that its Lie algebra $\mathfrak{g}_\lambda := \mathrm{Lie}(G_\lambda)$ is given by

$$\mathfrak{g}_\lambda = \left\{ \xi \in \mathfrak{g} \;\middle|\; \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}^*_{e^{t\xi}} \lambda = 0 \right\}. \tag{3.40}$$

The transitive action of G on X_λ then induces a natural identification

$$\begin{aligned} G/G_\lambda &\xrightarrow{\sim} X_\lambda \\ [g] &\longmapsto \mathrm{Ad}^*_g \lambda \,. \end{aligned} \tag{3.41}$$

Since the coadjoint action (3.4) corresponds to the adjoint action via the identification $\mathfrak{g} \simeq \mathfrak{g}^*$, we see that the coadjoint action of $T \subset G$ on $\mathfrak{g}^*$ restricts to a trivial action on $\mathfrak{t}^* \subset \mathfrak{g}^*$, so that formula (3.40) implies the inclusion $T \subset G_\lambda$ as a maximal torus of G_λ. Furthermore, it readily follows from the definitions (3.22) of the roots and (3.23) of the regular points that

$$T = G_\lambda \;\text{ if and only if }\; \lambda \in \mathfrak{t}^*_{\mathrm{reg}} \,. \tag{3.42}$$

In this case, the coadjoint orbit X_λ is called a *regular coadjoint orbit*, and via the identification (3.41), we see that regular coadjoint orbits are the orbits of maximal dimension among coadjoint orbits in $\mathfrak{g}^*$.

Let us now assume that $\lambda \in \Gamma^* \subset \mathfrak{t}^*$ belongs to the lattice of integral forms (3.20). Then following [30, Th. 5.7.1], the induced character

$$\begin{aligned} \sigma_\lambda : T &\longrightarrow S^1 \subset \mathbb{C}^* \\ \xi &\longmapsto e^{2\pi\sqrt{-1}\langle \lambda, \xi \rangle} \,, \end{aligned} \tag{3.43}$$

extends uniquely to a character $\sigma_\lambda : G_\lambda \to S^1 \subset \mathbb{C}^*$ of G_λ, and the G-equivariant line bundle

$$L_\lambda := G \times_{\sigma_\lambda} \mathbb{C} \longrightarrow G/G_\lambda \tag{3.44}$$

is naturally endowed with a G-equivariant connection ∇^{L_λ} preserving the Hermitian metric h^{L_λ} induced by the standard Hermitian metric of $\mathbb{C}$. Via the identification (3.41), we then get a natural Hermitian line bundle with connection $(L_\lambda, h^{L_\lambda}, \nabla^{L_\lambda})$ over the coadjoint orbit X_λ on which the G-action lifts, preserving Hermitian metric and connection. The following classical result is due to Kostant, and can be found in [4, Prop. 8.6].

Proposition 3.8. *[30, Th. 5.7.1, Cor. 1] For any $\lambda \in \Gamma^*$, the Hermitian connection ∇^{L_λ} on the Hermitian line bundle line $(L_\lambda, h^{L_\lambda})$ over the coadjoint orbit X_λ prequantizes the symplectic form (3.38) in the sense of (1.17), and the moment map (3.39) satisfies the Kostant formula (3.3) for the action of G on $(L_\lambda, h^{L_\lambda}, \nabla^{L_\lambda})$.*

Recall now the choice (3.24) of a set of positive roots $R_+ \subset R$. For any $\lambda \in \mathfrak{t}^* \subset \mathfrak{g}^*$, recall that $T \subset G$ is included as a maximal torus in the stabilizer $G_\lambda \subset G$, and write $R_\lambda \subset \mathfrak{t}^*$ for the set of roots of G_λ. By definition (3.22) of the set of roots and by definition (3.40) of the Lie algebra $\mathfrak{g}_\lambda = \mathrm{Lie}(G_\lambda)$, one readily checks that we have $R_\lambda \subset R$, and that $R_\lambda^+ := R_\lambda \cap R^+$ defines a set of positive roots of R_λ. In particular,in the decomposition (3.21) of $\mathfrak{g}_{\mathbb{C}}$, the complexified subalgebra $\mathfrak{g}_\lambda \otimes \mathbb{C} \subset \mathfrak{g}_{\mathbb{C}}$ coincides with

$$\mathfrak{g}_\lambda \otimes \mathbb{C} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in R_\lambda} (\mathfrak{g}_{\mathbb{C}})_\alpha \,. \tag{3.45}$$

which induces an identification

$$T_{\mathbb{C}}\, G/G_\lambda = \bigoplus_{\alpha \in R \setminus R_\lambda} (\mathfrak{g}_{\mathbb{C}})_\alpha \,. \tag{3.46}$$

Following [4, p. 257], we then get a natural G-invariant complex structure on G/G_λ defined through the decomposition (2.3) by

$$T^{(1,0)} G/G_\lambda = \bigoplus_{\alpha \in R^+ \setminus R_\lambda^+} (\mathfrak{g}_{\mathbb{C}})_\alpha \,. \tag{3.47}$$

Via the identification (3.41), this induces a G-invariant complex structure on X_λ compatible with the natural symplectic form of Proposition 3.7. As explained in Section 2.1, in the case $\lambda \in \Gamma^*$, we get from Proposition 3.8 a natural holomorphic structure on the complex line bundle (3.44) over X_λ induced by ∇^{L_λ}. Note that if $\lambda \in \mathfrak{t}^*_{\mathrm{reg}}$, so that $G_\lambda = T$ by (3.42), then for any $\gamma \in \mathfrak{t}^*_{\mathrm{reg}}$, the identification (3.41) induces an identification $X_\lambda \simeq X_\gamma$, and this identification is biholomorphic if and only if λ and γ belong to the same Weyl chamber.

Let now $\lambda \in \mathfrak{t}^*_{\mathrm{reg}}$, and recall the definition (3.26) of the half-sum of positive roots. Then by definition (2.5) of the canonical line bundle K_{X_λ} of X_λ associated with the complex structure (3.47) and the fact that $\dim_{\mathbb{C}} \mathfrak{g}_\alpha = 1$ for all $\alpha \in R$, we readily compute that

$$(K_{X_\lambda}, h^{K_{X_\lambda}}) = (L_{-2\rho}, h_{-2\rho}) \tag{3.48}$$

as Hermitian holomorphic line bundles over $X_\lambda \simeq G/T$ via the identification (3.41). In particular, in case the half-sum of roots (3.26) satisfies $\rho \in \Gamma^*$, the line bundle $L_{-\rho}$ over G/T as in (3.44) is well defined, and via the identification (3.41), we get a canonical choice of a square root of K_{X_λ} over X_λ given by

$$(K_{X_\lambda}^{1/2}, h^{K_{X_\lambda}^{1/2}}) = (L_{-\rho}, h_{-\rho}) \,. \tag{3.49}$$

3.4 Borel-Weil theorem and the orbit method

Let now $\lambda \in \Gamma^*_+ \subset \mathfrak{t}^*$ belong to the set of dominant integral weights (3.31). Following Proposition 3.8, consider the holomorphic Hermitian line bundle $(L_\lambda, h^{L_\lambda})$ with Hermitian connection ∇^{L_λ} prequantizing (X_λ, ω) endowed with the compatible complex structure induced by (3.47). The natural action of G induced by (3.44) is holomorphic, so that one can consider the unitary representation of G given by the associated space of holomorphic sections $H^0(X_\lambda, L_\lambda)$ endowed with the L^2-product (2.6). We then have the following version due to Kostant of a famous result due to Borel-Weil and Bott, in a form that can be found for instance in [22, Th. 3.7] and which constitutes the building block of the so-called *orbit method* in representation theory.

Theorem 3.9. *[1, §4.3] For any $\lambda \in \overline{\Gamma}^*_+$, we have the following identity of unitary representations of G,*

$$V(\lambda) = \left(H^0(X_\lambda, L_\lambda), \langle\cdot,\cdot\rangle_{L^2}\right), \tag{3.50}$$

where $V(\lambda)$ is the irreducible representation of G with highest weight λ.

Note that if $\lambda \in \Gamma^*_+$ belongs to the set of *regular dominant weights*

$$\Gamma^*_+ := \Gamma^* \cap \mathfrak{t}^*_+ \,, \tag{3.51}$$

so that we have $X_\lambda \simeq G/T$ by (3.42), then for any other $\gamma \in \Gamma^*_+$, as it belongs to the same Weyl chamber (3.25), the G-equivariant identification $X_\lambda \simeq X_\gamma$ induced by (3.41) is biholomorphic, so that (3.44) defines a holomorphic Hermitian line bundle with connection $(L_\gamma, h^{L_\gamma}, \nabla^{L_\gamma})$ over (X_λ, ω), and by Theorem 3.9 we get

$$V(\gamma) = \left(H^0(X_\lambda, L_\lambda), \langle\cdot,\cdot\rangle_{L^2}\right), \tag{3.52}$$

where $V(\gamma)$ is the irreducible representation of G with highest weight $\gamma \in \Gamma^*_+$. Note also from (3.43) that for any $\gamma_1,\, \gamma_2 \in \Gamma^*_+$, we have $L_{\gamma_1+\gamma_2} = L_{\gamma_1} \otimes L_{\gamma_2}$. Furthermore, for any $\lambda \in \overline{\Gamma}^*_+$ and $p \in \mathbb{N}$, we get from the same argument the following consequence of Theorem 3.9,

$$V(p\lambda) = \left(H^0(X_\lambda, L^p_\lambda), \langle\cdot,\cdot\rangle_{L^2}\right). \tag{3.53}$$

Following now [26, § 17.2], let us consider the *universal enveloping algebra* $U(\mathfrak{g})$ of $\mathfrak{g}$, with associated Lie algebra morphism

$$j : \mathfrak{g} \longrightarrow U(\mathfrak{g}) \,, \tag{3.54}$$

which is characterized by the universal property that for any vector space V and any Lie algebra morphism $\varphi : \mathfrak{g} \to \mathrm{End}(V)$, there exists a unique algebra morphism

$$\Phi : U(\mathfrak{g}) \longrightarrow \mathrm{End}(V) \tag{3.55}$$

such that $\varphi = \Phi \circ j$. In particular, for any representation V of G, one gets an induced representation of $U(\mathfrak{g})$ on V by taking $\varphi : \mathfrak{g} \to \mathrm{End}(V)$ to be the associated infinitesimal action. Under the identification (3.15) of the ring of polynomials $\mathbb{R}[\mathfrak{g}^*]$ with real coefficients over $\mathfrak{g}^*$ with the symmetric algebra of $\mathfrak{g}$, we can then state the celebrated *Poincaré-Birkoff-Witt theorem.*

Theorem 3.10. *[17, Prop. 2.4.10] The* symmetrization map

$$\begin{aligned} sym : \mathbb{R}[\mathfrak{g}^*] &\xrightarrow{\sim} U(\mathfrak{g}) \\ \xi_1 \cdots \xi_k &\longmapsto \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} j(\xi_{\sigma(1)}) \cdots j(\xi_{\sigma(k)}) , \end{aligned} \tag{3.56}$$

is a bijective $\mathfrak{g}$*-equivariant linear map for the natural adjoint action on both spaces.*

Note that Theorem 3.10 implies in particular that the map $j : \mathfrak{g} \to U(\mathfrak{g})$ of (3.54) is injective, and we will call it as the *inclusion map* of $\mathfrak{g}$ into $U(\mathfrak{g})$. Theorem 3.10 also induces an adjoint action of G on $U(\mathfrak{g})$ by exponentiating the adjoint action of $\mathfrak{g}$ induced by the inclusion map (3.54), and for which the symmetrization map (3.56) is G-equivariant. Note also that the symmetrization map (3.56) is not an isomorphism of algebras in general, as the first space is commutative, while the second is not commutative in general since the inclusion map (3.54) is a Lie algebra morphism. Finally, in the setting of Section 3.1, if one considers the representation of G given by $V = H^0(X, L^p \otimes K)$, then by Definition 3.3 and definition (3.56) of the symmetrization map, for any $P \in \mathbb{R}[\mathfrak{g}^*]$ one gets

$$Q_p(P) = \Phi(\mathrm{sym}(P)) , \tag{3.57}$$

where $\Phi : U(\mathfrak{g}) \to \mathrm{End}(H^0(X, L^p \otimes K))$ is the induced action (3.55) of $U(\mathfrak{g})$ on $H^0(X, L^p \otimes K)$.

Let us write $Z[U(\mathfrak{g})] \subset U(\mathfrak{g})$ for the center of the universal envelopping algebra. Following for instance [26, § 6.1], the usual Schur lemma readily implies that $Z[U(\mathfrak{g})]$ acts by scalar multiplication on any irreducible representation of G. Recall on the other hand the action on $\mathfrak{t}^*$ of the Weyl group W defined by (3.19), and denote by $\mathbb{R}[\mathfrak{t}^*]^W$ the algebra of W-invariant polynomials with real coefficients over $\mathfrak{t}^*$. We then have the following result due to Harish-Chandra, which can be found in [17, Prop. 7.4.4, Th. 7.4.5].

Theorem 3.11. *[25] There is an isomorphism*

$$\gamma : Z[U(\mathfrak{g})] \xrightarrow{\sim} \mathbb{R}[\mathfrak{t}^*]^W , \tag{3.58}$$

characterised by the fact that for any $z \in Z[U(\mathfrak{g})]$ *and* $\lambda \in \overline{\Gamma}^*_+$*, we have*

$$\Phi(z).v = \gamma(z)(2\pi\sqrt{-1}(\lambda + \rho))\, v , \tag{3.59}$$

for all $v \in V(\lambda)$*, where* $\Phi : U(\mathfrak{g}) \to \mathrm{End}(V(\lambda))$ *is the induced action* (3.55) *of* $U(\mathfrak{g})$ *on the irreducible representation* $V(\lambda)$ *of* G *with highest weight* λ*, and where* $\rho \in \mathfrak{t}^*$ *denotes the half-sum of roots defined by* (3.26)*.*

Note that the $2\pi\sqrt{-1}$ comes from our convention (3.29) for the weights, and that we are using the fact that a polynomial over a real vector space extends to its complexification in the obvious way. Note also that, since $\frac{1}{p}\Gamma^*_+$ becomes dense inside the open cone $\overline{\mathfrak{t}^*_+} \subset \mathfrak{t}^*$ as $p \to +\infty$ by (3.20) and (3.31), one readily infers that a polynomial in $\mathbb{R}[\mathfrak{t}^*]$ is uniquely determined by its values on $\overline{\Gamma}^*_+ \subset \mathfrak{t}^*$, so that formula (3.59) in fact characterizes $\gamma(z) \in \mathbb{R}[\mathfrak{t}^*]^W$ for all $z \in Z[U(\mathfrak{g})]$.

Since the inclusion map (3.54) is injective and its image generates $U(\mathfrak{g})$ by the Poincaré-Birkhoff-Witt Theorem 3.10, the center $Z[U(\mathfrak{g})] \subset U(\mathfrak{g})$ coincides with the subalgebra of $U(\mathfrak{g})$ of $\mathfrak{g}$-invariant elements by the adjoint action, and since the symmetrization map (3.56) is $\mathfrak{g}$-equivariant, the Poincaré-Birkhoff-Witt Theorem 3.10 implies that it restricts to an isomorphism of vector spaces

$$\mathrm{sym} : \mathbb{R}[\mathfrak{g}^*]^G \xrightarrow{\sim} Z[U(\mathfrak{g})] \,, \tag{3.60}$$

where $\mathbb{R}[\mathfrak{g}^*]^G \subset \mathbb{R}[\mathfrak{g}^*]$ denotes the space of real polynomials over $\mathfrak{g}^*$ which are invariant with respect to the coadjoint action (3.4) of G. Together with the Borel-Weil Theorem 3.9, Proposition 3.4 then has the following important consequence, which is typical of a semiclassical result in the orbit method.

Corollary 3.12. *For any homogeneous G-invariant polynomial $P \in \mathbb{R}[\mathfrak{g}^*]^G$ of degree $j \in \mathbb{N}$ and any $\lambda \in \mathfrak{t}^*$, we have the following asymptotic estimate as $p \to +\infty$,*

$$\frac{1}{p^j}\gamma(\mathrm{sym}(P))(p\lambda) = P(\lambda) + O(p^{-1}) \,, \tag{3.61}$$

so that $\gamma(\mathrm{sym}(P)) \in \mathbb{R}[\mathfrak{t}^*]^W$ has degree j and its homogeneous part of highest degree coincide with the restriction of $P \in \mathbb{R}[\mathfrak{g}^*]^G$ to $\mathfrak{t}^* \subset \mathfrak{g}^*$.

Proof. Let $\lambda \in \overline{\Gamma}^*_+$, and recall from Proposition 3.7 that the moment map $\mu : X_\lambda \to \mathfrak{g}^*$ associated with the coadjoint action of G on the coadjoint orbit (X_λ, ω) is given by the inclusion, so that since the restriction of a homogeneous G-invariant polynomial $P \in \mathbb{R}[\mathfrak{g}^*]^G$ to a coadjoint orbit is constant, we have $\mu^*P \equiv P(\lambda)$. Now Proposition 3.8 states that the Hermitian line bundle with connection $(L_\lambda, h^{L_\lambda}, \nabla^{L_\lambda})$ prequantizes (X_λ, ω), and Definition 2.1 for $X = X_\lambda$, $L = L_\lambda$ and $K = \mathbb{C}$ implies that $T_p(\mu^*P) = P(\lambda)\mathrm{Id}$ for all $p \in \mathbb{N}$, so that Proposition 3.4 gives the following asymptotic as $p \to +\infty$, for any sequence $\{s_p \in H^0(X, L^p)\}_{p\in\mathbb{N}}$ with $\|s_p\|_{L^2} = 1$,

$$\frac{1}{(2\pi\sqrt{-1}p)^k} Q_p(P)s_p = P(\lambda)s_p + O(p^{-1}) \,. \tag{3.62}$$

On the other hand, by (3.57), the operator $Q_p(P) \in \mathrm{End}(H^0(X, L^p))$ correspond to the action of $\mathrm{sym}(P) \in U(\mathfrak{g})$ on the irreducible representation of G withhighest weight $p\lambda$ via (3.53), so that by Theorem 3.11, for all $p \in \mathbb{N}$ we get

$$Q_p(P)s_p = \gamma(\mathrm{sym}(P))(2\pi\sqrt{-1}(p\lambda + \rho))s_p \,. \tag{3.63}$$

As $\gamma(\mathrm{sym}(P)) \in \mathbb{R}[\mathfrak{t}^*]^W$ is a polynomial, formula (3.61) then follows from comparing the asymptotic expansion (3.62) with the expansion in powers of $p \in \mathbb{N}$ of the right-hand side of formula (3.63). Since a polynomial in $\mathbb{R}[\mathfrak{t}^*]$ is uniquely determined by its values on $\overline{\Gamma}^*_+ \subset \mathfrak{t}^*$ as in Theorem 3.11, this concludes the proof.

□

Note that Corollary 3.12 can also be seen as a consequence of the explicit description of the Harish-Chandra isomorphism (3.58) given for instance in [35, § 9.4].

4 Quantization of the projective space

In this section, we fix $n \in \mathbb{N}^*$ and consider first the case of irreducible representations of $G = U(n)$ with highest weight $\lambda = (p, 0, \cdots, 0) \in \overline{\Gamma}^*_+$ for all $p \in \mathbb{N}$ in the sense of Definition 3.5 and under the identification (3.35). In particular, we establish Theorem 1.2 using the tools of Section 2.2.

4.1 Projective space as a coadjoint orbit

Write $\langle \cdot, \cdot \rangle$ for the natural Hermitian product of $\mathbb{C}^n$ and $|\cdot|$ for the associated norm. Under the identification (3.35), let $\lambda := (1, 0, \cdots, 0) \in \overline{\Gamma}^*_+ \subset \mathbb{Z}^n$, and note that it corresponds in the identification (3.32) to the Hermitian matrix $\Pi_{e_1} \in \mathrm{Herm}(\mathbb{C}^n)$ of orthogonal projection on the first vector $e_1 \in \mathbb{C}^n$ in the canonical basis of $\mathbb{C}^n$. Since any vector $z = (z_1, \cdots, z_n) \in \mathbb{C}^n$ of norm 1 is of the form $z = ge_1$ for some $g \in U(n)$, we see that under the identification (3.32), the coadjoint orbit X_λ is given by

$$X_\lambda = \{\Pi_z \in \mathrm{Herm}(\mathbb{C}^n) \mid |z| = 1\} . \tag{4.1}$$

We thus get a natural $U(n)$-equivariant identification of X_λ with the *complex projective space* $\mathbb{CP}^{n-1}$ of complex lines inside $\mathbb{C}^n$ via the map

$$\begin{aligned} X_\lambda &\xrightarrow{\sim} \mathbb{CP}^{n-1} \\ \Pi_z &\longmapsto [z] . \end{aligned} \tag{4.2}$$

Using the explicit description of the decomposition (3.21) given for instance in [11, Chap. V, § 6], one readily checks that this identification is biholomorphic for the complex structure induced by (3.47) on X_λ. Furthermore, the natural symplectic form (3.38) on X_λ is sent to the canonical Fubini-Study symplectic form of $\mathbb{CP}^{n-1}$ of volume 1, and the prequantizing line bundle $(L_\lambda, h^{L_\lambda}, \nabla^{L_\lambda})$ of Proposition 3.8 corresponds to the dual of the tautological line bundle with associated Fubini-Study Hermitian metric and connection. From now on, we will set $X := X_\lambda$ and $L := L_\lambda$, and use freely the identification (4.2).

Let now $T_0 \subset U(n)$ be the $(n-1)$-dimensional torus of diagonal matrices with highest-left coefficient equal to 1, and write $\mathfrak{t}_0 := \mathrm{Lie}(T_0)$ for its Lie algebra. The associated moment map $\mu_0 : X \to \mathfrak{t}_0^*$ by the Kostant formula (3.3) satisfies $\mu_0 = \mu \circ \pi$, where $\mu : X \hookrightarrow \mathfrak{u}(n)^*$ is the standard moment map (3.39) given by the inclusion and $\pi : \mathfrak{u}(n)^* \to \mathfrak{t}_0^*$ is the projection induced by the natural inclusion $\mathfrak{t}_0 \subset \mathfrak{u}(n)$. Consider the identification $\mathfrak{t}_0^* \simeq \mathbb{R}^{n-1}$ induced by the identification (3.33). Using the description (4.1) of $X := X_\lambda$, one readily checks that

$$\begin{aligned} \mu_0 : X &\longrightarrow \mathbb{R}^{n-1} \\ \Pi_z &\longmapsto (|z_2|^2, |z_3|^2, \cdots, |z_n|^2) . \end{aligned} \tag{4.3}$$

In particular, as $\sum_{j=2}^n |z_j|^2 = 1 - |z_1|^2$ for all $z \in \mathbb{C}^n$ with $|z| = 1$, its image over X as in (4.1) is given by the simplex

$$\Delta := \left\{(\nu_2, \cdots, \nu_n) \in \mathbb{R}^{n-1} \;\middle|\; \sum_{j=2}^{n} \nu_j \leqslant 1 \text{ and } \nu_k \geqslant 0\,,\ \text{for all } 2 \leqslant k \leqslant n\right\}. \tag{4.4}$$

We write

$$\iota_\nu : \Lambda_\nu := \mu^{-1}(\nu) \longrightarrow X\,, \tag{4.5}$$

for the fibre of (4.3) over any $\nu \in \Delta$. As the torus $T_0 \subset U(n)$ acts on X by scalar multiplication on each component of $z \in \mathbb{C}^n$ in the identification (4.1), we see from (4.3) that T_0 acts transitively on the fibres of the moment map $\mu_0 : X \to \Delta$, and freely over the interior $\Delta^0 \subset \Delta$. For any $\nu \in \Delta$, we write

$$T_\nu := \{g \in T_0 \mid g.x = x \text{ for all } x \in \Lambda_\nu\}\,, \tag{4.6}$$

which is a closed subgroup of T_0. By completing a basis of the integral lattice of its Lie subalgebra $\mathfrak{t}_\nu := \mathrm{Lie}(T_\nu) \subset \mathfrak{t}_0$ into an basis of the integral lattice of $\mathfrak{t}_0$, we get a splitting

$$T_0 = T_\nu \times (T_0/T_\nu)\,, \tag{4.7}$$

inducing a freely transitive action of T_0/T_ν on Λ_ν. Write $\mathfrak{t}_0 = \mathfrak{t}_\nu \oplus (\mathfrak{t}_0/\mathfrak{t}_\nu)$ for the induced splitting of its Lie algebra.

Recalling Definition 2.4 of a Bohr-Sommerfeld submanifold in (X, ω), we then have the following basic fundamental result.

Proposition 4.1. *The fibre $\Lambda_\nu \subset X$ over $\nu \in \Delta$ of the moment map (4.3) for the Hamiltonian action of $T_0 \subset U(n)$ on (X, ω) satisfy the Bohr-Sommerfeld condition at level $p \in \mathbb{N}$ if and only if*

$$\nu \in \Delta \cap \left(\frac{1}{p}\,\mathbb{Z}^{n-1}\right). \tag{4.8}$$

Proof. Let $\nu \in \Delta$, and fix a splitting (4.7). For any $p \in \mathbb{N}$, the Kostant formula (3.6) applied to $\mu_0 : X \to \Delta$ over the fibre $\Lambda_\nu \subset X$ shows that a section $\zeta^p \in \mathscr{C}^\infty(\Lambda_\nu, \iota_\nu^* L^p)$ satisfies $\nabla^{\iota_\nu^* L^p} \zeta^p = 0$ if and only if, for all $x \in \Lambda_\nu$ and $\xi \in \mathfrak{t}$, we have

$$e^{\xi}.\zeta^p(e^{-\xi}x) = e^{2\pi\sqrt{-1}p\langle \nu, \xi\rangle}\zeta^p(x)\,. \tag{4.9}$$

Note that if $\xi \in \mathfrak{t}_\nu \subset \mathfrak{t}_0$, so that $e^{\xi}x = x$ by (4.6), equation (4.9) is automatically verified for any section over $x \in \Lambda_\nu$ by definition (3.44) of the line bundle $L := L_\lambda$. As $e^{\xi} = 1$ if and only if $\xi \in \mathbb{Z}^{n-1}$, a non-vanishing section can satisfy (4.9) only if $\langle \nu, \xi\rangle \in \frac{1}{p}\mathbb{Z}$ for all $\xi \in \mathfrak{t}_\nu \cap \mathbb{Z}^{n-1}$.

Choose now $x \in \Lambda_\nu$ and $\eta_x \in L_x$, and define a section $\eta \in \mathscr{C}^\infty(\Lambda_b, \iota_b^* L)$ for all $g \in T_0/T_\nu$ in the splitting (4.7) by the formula

$$\eta(g.x) := g.\eta_x\,, \tag{4.10}$$

which is well-defined since T_0/T_ν acts freely and transitively on Λ_ν. We then have $g\eta = \eta$ for all $g \in T_0/T_\nu$ by definition of the action (3.1), so that $L_\xi\,\eta = 0$ for all $\xi \in \mathfrak{t}_0/\mathfrak{t}_\nu \subset \mathfrak{t}_0$

by definition of the Lie derivative (3.2). We then get from (4.9) that a section non-vanishing $\zeta^p \in \mathscr{C}^\infty(\Lambda_\nu, \iota_\nu^* L^p)$ satisfies $\nabla^{\iota_\nu^* L^p}\zeta^p = 0$ if and only if there exists $c \in \mathbb{C}^*$ such that for all $x \in \Lambda_\nu$ and $\xi \in \mathfrak{t}_0/\mathfrak{t}_\nu$, we have

$$\zeta^p(e^\xi.x) = c\, e^{2\pi\sqrt{-1}p\langle\nu,\xi\rangle}\eta^p(e^\xi.x)\,. \tag{4.11}$$

As $e^\xi.x = x$ if and only if $\xi \in \mathbb{Z}^n$, we thus see that (4.11) defines a non-vanishing section if and only if $\langle\nu,\xi\rangle \in \frac{1}{p}\mathbb{Z}$ for all $\xi \in (\mathfrak{t}_0/\mathfrak{t}_\nu) \cap \mathbb{Z}^{n-1}$. Hence a non-vanishing section over $\Lambda_\nu \subset X$ satisfying (4.9) exists if and only $\nu \in \Delta$ satisfies (4.8), which proves the result by the Definition 2.4 of the Bohr-Sommerfeld condition at level $p \in \mathbb{N}$.

□

Following for instance [37, Ex. 1.25], recall that for any $p \in \mathbb{N}$ and under the identification (4.2), the space of holomorphic sections $H^0(X, L^p)$ naturally identifies with the space of homogoneous polynomials of degree p over $\mathbb{C}^n$, and the induced action of $U(n)$ on polynomials over $\mathbb{C}^n$ corresponds to the action (3.1) of $U(n)$ on $H^0(X, L^p)$. One then readily sees that for all $\nu = (\nu_2, \cdots, \nu_n) \in \mathbb{N}^{n-1}$ such that $|\nu| := \sum_{j=2}^n \nu_j \leqslant p$, the monomial

$$\begin{aligned} z^\nu : \mathbb{C}^n &\longrightarrow \mathbb{C} \\ z &\longmapsto z_1^{p-|\nu|}\prod_{j=2}^n z_j^{\nu_j} \end{aligned} \tag{4.12}$$

generates the weight space (3.28) of weight $\nu \in \mathbb{Z}^{n-1}$ in the decomposition (3.27) of the space of homogeneous polynomials with respect to the $T_0 \subset U(n)$ action in the sense of (3.28), so that the the weight space $\Gamma_p := \Gamma^{T_0}_{H^0(X,L^p)}$ defined in (3.29) is given by

$$\Gamma_p = \mathbb{Z}^{n-1} \cap p\Delta\,, \tag{4.13}$$

with all weights of multiplicity 1. Note that $\frac{1}{p}\Gamma_p$ becomes dense in Δ as $p \to +\infty$. Now by the consequence (3.53) of the Borel-Weil Theorem 3.9, the action (3.1) of $U(n)$ identifies $H^0(X, L^p)$ with the irreducible represention $V(p\lambda)$ of $U(n)$ with highest weight $p\lambda = (p, 0, \cdots 0) \in \overline{\Gamma}^*_+$ via (3.35), so that we have a weight decomposition (3.27) into 1-dimensional subspaces

$$V(p\lambda) = \bigoplus_{\nu_p \in \Gamma_p} \mathbb{C}_{\nu_p}\,, \tag{4.14}$$

such that for any $\xi \in \mathfrak{t}_0$ and $v \in \mathbb{C}_{\nu_p}$, we have $e^\xi.v = e^{2\pi\sqrt{-1}\langle\nu_p,\xi\rangle}v$. Comparing (4.13) with Proposition 4.1, we then see that for any $p \in \mathbb{N}$, the weight basis of $V(p\lambda)$ is in bijection with the fibres of the moment map (4.3) satisfying the Bohr-Sommerfeld condition at level $p \in \mathbb{N}$. The following result shows that this is not a coincidence.

Proposition 4.2. *For any $p, k \in \mathbb{N}$ and $v_p \in \frac{1}{p}\Gamma_{p-k} \subset \Delta$, let $f \in \mathscr{C}^\infty(\Lambda_{v_p}, \iota_{v_p}^* L^{-k})$ be invariant by the action of T_0, and let $\zeta^p \in \mathscr{C}^\infty(\Lambda_{v_p}, \iota_{v_p}^* L^p)$ satisfy $\nabla^{\iota_{v_p}^* L^p}\zeta^p \equiv 0$. Then the section $s_{\Lambda_{v_p}} \in H^0(X, L^{p-k})$ defined by*

$$s_{\Lambda_{v_p}}(x) = \int_{\Lambda_{v_p}} P_{p-k}(x,y).\zeta^p f(y)\, dv_{\Lambda_{v_p}}(y)\,, \tag{4.15}$$

belongs to $\mathbb{C}_{pv_p} \subset V((p-k)\lambda)$ *in the decomposition* (4.14)*. Furthermore, if* $\zeta^p \neq 0$ *and* $f \neq 0$*, then* $s_{\Lambda_{v_p}} \neq 0$*.*

Proof. Recall first that the action (3.1) of G on $\mathscr{C}^\infty(X, L^{p-k})$ preserves the orthogonal projection $P_{p-k} : \mathscr{C}^\infty(X, L^{p-k}) \to H^0(X, L^{p-k})$ with respect to the L^2-product (2.6), which translates through Definition 2.3 to the property that for any $g \in G$, any x, $y \in X$ and $\eta_y \in L^{p-k}_y$, we have

$$g.P_{p-k}(g^{-1}x, y).\eta_y = P_{p-k}(x, gy).(g.\eta_y) \in L^{p-k}_x \,. \tag{4.16}$$

Then by formula (4.15), since $f \in \mathscr{C}^\infty(\Lambda_{v_p}, \iota^*_{v_p} L^{-k})$ is T_0-invariant and by formula (4.9) for the flat section $\zeta^p \in \mathscr{C}^\infty(\Lambda_{v_p}, \iota^*_{v_p} L^p)$, for any $\xi \in \mathfrak{t}$ and $x \in X$, we have

$$\begin{aligned}
(e^\xi.s_{\Lambda_{v_p}})(x) &= \int_{\Lambda_{v_p}} e^\xi.P_{p-k}(e^{-\xi}x, y).\zeta^p f(y)\, dv_{\Lambda_{v_p}}(y) \\
&= e^{2\pi\sqrt{-1}p\langle v_p, \xi\rangle} \int_{\Lambda_{v_p}} P_{p-k}(x, e^\xi y).\zeta^p f(e^\xi y)\, dv_{\Lambda_{v_p}}(y) \\
&= e^{2\pi\sqrt{-1}p\langle v_p, \xi\rangle} \int_{\Lambda_{v_p}} P_{p-k}(x, y).\zeta^p f(y)\, dv_{\Lambda_{v_p}}(y) = e^{2\pi\sqrt{-1}p\langle v_p, \xi\rangle} s_{\Lambda_{v_p}}(x) \,,
\end{aligned} \tag{4.17}$$

so that $s_{\Lambda_{v_p}} \in \mathbb{C}_{pv_p}$ in the decomposition (4.14).

Let us now assume that $\zeta^p \neq 0$ and $f \neq 0$. Writing $z^{pv_p} \in H^0(X, L^{p-k})$ for the monomial (4.12) associated with $pv_p \in \Gamma_{p-k}$, one readily sees from (4.3) that z^{pv_p} does not vanish along $\Lambda_{v_p} \subset \mathbb{CP}^{n-1}$. From formula (4.9) and picking $x_0 \in \Lambda_{v_p}$, since $z^{pv_p} \in \mathbb{C}_{pv_p}$ and using that h^L is G-invariant, for any $\xi \in \mathfrak{t}_0$ we get

$$h^{L^{p-k}}\Big(z^{pv_p}(e^\xi.x_0), \zeta^p f(e^\xi.x_0)\Big) = h^{L^{p-k}}\Big(z^{pv_p}(x_0), \zeta^p f(x_0)\Big) \,, \tag{4.18}$$

so that $h^{L^{p-k}}(z^{pv_p}, \zeta^p f)$ is constant along Λ_{v_p} since T_0 acts transitively. Using the reproducing property of Proposition 2.6, this implies

$$\begin{aligned}
\langle z^{pv_p}, s_{\Lambda_{v_p}}\rangle_{L^2} &= \int_{\Lambda_{v_p}} h^{L^{p-k}}(z^{pv_p}(x), \zeta^p f(x))\, dv_{\Lambda_{v_p}}(x) \\
&= \mathrm{Vol}(\Lambda_{v_p}, dv_{\Lambda_{v_p}})\, h^{L^{p-k}}(z^{pv_p}(x_0), \zeta^p f(x_0)) \neq 0 \,,
\end{aligned} \tag{4.19}$$

which concludes the proof ☐

For $k = 0$, Proposition 4.2 gives an explicit bijection between the Bohr-Sommerfeld fibres of (4.3) at level $p \in \mathbb{N}$ as in Proposition 4.1 and a weight basis of $V(p\lambda)$ as in (4.14) via the Borel-Weil Theorem 3.9. In the proof of Theorem 1.2 given in the next section, we will need Proposition 4.2 for $k = n$.

4.2 Proof of Theorem 1.2

First note that by Proposition 4.2, for any sequence $\{v_p \in \frac{1}{p}\Gamma_p\}_{p\in\mathbb{N}}$, an associated sequence of unit vectors $\{e_{pv_p} \in \mathbb{C}_{pv_p}\}_{p\in\mathbb{N}}$ satisfies

$$e_{pv_p} = z_p \frac{s_{\Lambda_{v_p}}}{\|s_{\Lambda_{v_p}}\|} \,, \tag{4.20}$$

with $z_p \in \mathbb{C}$ such that $|z_p| = 1$ for all $p \in \mathbb{N}$, where $\{s_{\Lambda_{v_p}} \in H^0(X, L^p)\}_{p\in\mathbb{N}}$ is the isotropic state of Definition 2.5 associated with the sequence of Bohr-Sommerfeld submanifolds $\{(\Lambda_{v_p}, \zeta^p)\}_{p\in\mathbb{N}}$. On the other hand, since $\mu_0 : X \to \Delta$ is a fibration over the interior of any face of its image $\Delta \subset \mathbb{R}^{n-1}$ by (4.3), we readily get that for any sequence $\{v_p \in \Delta\}_{p\in\mathbb{N}}$ such that $v_p \to v$ as $p \to +\infty$ and belonging to the face of $v \in \Delta$ after some rank, the sequence $\{\Lambda_{v_p} \subset X\}_{p\in\mathbb{N}}$ converges smoothly towards $\Lambda_v \subset X$ in the sense of Definition 2.7. Then the first statement of Theorem 1.2 is a straightforward consequence of Theorems 2.9 and 2.10.

To get the asymptotic expansion (1.15) from the asymptotic expansion (2.21) for the scalar product of isotropic states, we need to show that for any $x \in \Lambda_p^{(1)} \cap \Lambda_p^{(2)}$, the factor $\lambda_p^{(q)}(x) := h^{L^p}(\zeta_1^p(x), \zeta_2^p(x))$ appearing in (2.21) satisfies

$$\lambda_p^{(q)}(x) = h^{L^p}(\zeta_1^p(x_p), \zeta_2^p(x_p))\, e^{2\pi\sqrt{-1}p\eta_p^{(q)}} \tag{4.21}$$

where $\eta_p^{(q)} > 0$ is the symplectic area of a disk $D \subset X$ bounded by a path $\gamma_1 \subset \Lambda_p^{(1)}$ joining any point $x_p \in \Lambda_p^{(1)} \cap \Lambda_p^{(2)}$ to any point $y_p \in \Lambda_p^{(1)} \cap \Lambda_p^{(2)}$, followed by a path $\gamma_2 \subset \Lambda_p^{(2)}$ returning to $x_p \in \Lambda_p^{(1)} \cap \Lambda_p^{(2)}$, which exists under the hypotheses of Theorem 1.2 since $\Lambda_p^{(1)} = g\Lambda_{v_p}$ and $\Lambda_p^{(2)} = \Lambda_{w_p}$ are connected and isotopic to each other. Let us trivialize L^p over D in such a way that $\zeta_2^p \equiv 1$ along γ_2. Writing $\nabla^{L^p}|_D = d - 2\pi\sqrt{-1}p\,\alpha$ in this trivialisation, we have $\alpha|_{\gamma_2} \equiv 0$ by the flatness condition (2.13) for ζ_2^p, and $\omega = d\alpha$ by the prequantization formula (1.17). Then Stokes theorem gives

$$\int_D \omega = \int_{\gamma_1} \alpha\,. \tag{4.22}$$

On the other hand, by the flatness condition (2.13) for ζ_1^p and the fact that $d\zeta_2^p \equiv 0$ in the chosen trivialization of L^p over D, we get

$$\begin{aligned} \frac{d}{dt} h^{L^p}(\zeta_1(\gamma_1(t)), \zeta_2(\gamma_1(t))) &= h^{L^p}\left(\zeta_1(\gamma_1(t)), \nabla^{L^p}\zeta_2(\gamma_1(t))\right) \\ &= 2\pi\sqrt{-1}\,p\,\alpha_{\gamma_1(t)}\, h^{L^p}(\zeta_1(\gamma_1(t)), \zeta_2(\gamma_1(t)))\,, \end{aligned} \tag{4.23}$$

so that for all $t \in [0,1]$, we can solve the corresponding ODE to get

$$h^{L^p}(\zeta_1(\gamma_1(t)), \zeta_2(\gamma_1(t))) = \exp\left(2\pi\sqrt{-1}p\int_0^t \gamma^*\alpha\right) h^{L^p}(\zeta_1(x_p), \zeta_2(x_p))\,. \tag{4.24}$$

Setting $t = 1$, we get (4.21) from (4.22). Together with Theorems 2.9 and 2.10, this gives the asymptotic expansion (1.15).

Let us now consider the assumptions of formula (1.16), and recall for instance from [37, Ex. 1.27] that the canonical line bundle of $X \simeq \mathbb{CP}^n$ is given by $K_X = L^{-n}$, so that when $n \in \mathbb{N}^*$ is even, we get a natural choice of square root given by $K_X^{1/2} = L^{-n/2}$, with induced Hermitian metric and connection. In particular, for all $p \in \mathbb{N}$ we get

$$L^p \otimes K_X^{1/2} = L^{p-\frac{n}{2}}\,. \tag{4.25}$$

The action of $T_0 \subset U(n)$ on L induces an action on $K_X^{1/2}$ compatible with its natural action on K_X, and since it acts by isometries on (X, g^{TX}), for any $v_p \in \Delta^0 \subset \Delta$, the Riemannian volume form $dv_{\Lambda_{v_p}} \in \mathscr{C}^\infty(\Lambda_{v_p}, \det(T\Lambda_{v_p}))$ is invariant with respect to the action of T_0. Letting $dv_{\Lambda_{v_p}}^{1/2} \in \mathscr{C}^\infty(X, \iota_{v_p}^* K_X^{1/2})$ be a square root in the identification (2.23) of $\iota_{v_p}^* K_X$ with $\det(T_{\mathbb{C}}^*\Lambda_{v_p})$, Definition 2.4 and Proposition 4.2 show that the Lagrangian state $\{s_{\Lambda_{v_p}} \in H^0(X, L^p \otimes K_X^{1/2})\}_{p\in\mathbb{N}}$ associated with the sequence of Bohr-Sommerfeld Lagrangian submanifolds $\{(\Lambda_{v_p}, dv_{\Lambda_{v_p}}^{1/2}, \zeta^p)\}_{p\in\mathbb{N}}$ converging to $\{(\Lambda_v, dv_{\Lambda_v}^{1/2})\}_{p\in\mathbb{N}}$ belong to $\mathbb{C}_{pv_p} \subset V((p-n/2)\lambda)$ and is non-vanishing for all $p \in \mathbb{N}$.

Now since v_p, $w_p \in \Delta^0 \subset \Delta$ for all $p \in \mathbb{N}$ big enough under the assumptions of formula (1.16), the $(n-1)$-dimensional torus $T_0 \subset U(n)$ acts freely and transitively on Λ_{v_p} and on Λ_{w_p}, so that, letting $\{\xi_j\}_{j=1}^{n-1}$ be a basis of the integral lattice of $\mathfrak{t}_0 \subset \mathfrak{u}(n)$, we get that $\{\xi_j^X\}_{j=1}^n$ induces a basis of $T_x\Lambda_{w_p}$ at all $x \in \Lambda_{w_p}$ and that $\{\mathrm{Ad}_g\,\xi_j^X\}_{i=1}^n$ induces a basis of $T_x(g\Lambda_{v_p})$ at all $x \in g\Lambda_{w_p}$. Proposition 3.7 then gives

$$\omega_\alpha(\mathrm{Ad}_g\,\xi_j^X, \xi_k^X) = \alpha([\mathrm{Ad}_g\,\xi_j, \xi_k])\,, \tag{4.26}$$

for any $\alpha \in g\Lambda_{v_p} \cap \Lambda_{w_p} \subset \mathfrak{u}(n)^*$ and $1 \leqslant j, k \leqslant n-1$. Furthermore, by T_0-invariance and the fact that $\mathrm{Vol}(T_0) = 1$, we get the following identity of functions over Λ,

$$dv_{\Lambda_{w_p}}(\xi_1^X, \cdots, \xi_n^X) \equiv \mathrm{Vol}(\Lambda_{w_p}, dv_{\Lambda_{w_p}})\,. \tag{4.27}$$

On the other hand, Theorem 2.9 applied to $\{s_{\Lambda_{w_p}} \in H^0(X, L^p \otimes K_X^{1/2})\}_{p\in\mathbb{N}}$ gives the following asymptotic expansion as $p \to +\infty$,

$$\left\| s_{\Lambda_{w_p}} \right\|_{L^2} = 2^{\frac{n}{4}} p^{\frac{n}{4}} \left(\mathrm{Vol}(\Lambda_{w_p}, dv_{\Lambda_{w_p}})^{1/2} + O(p^{-1}) \right)\,, \tag{4.28}$$

Applying Theorem 2.11 to (4.20) with the vectors $\xi_j := \mathrm{Vol}(\Lambda_{w_p}, dv_{\Lambda_{w_p}})^{-\frac{1}{n-1}} \xi_j^X$ for all $1 \leqslant j \leqslant n-1$ as well as the analogous formula for $\{g s_{\Lambda_{v_p}} = s_{g\Lambda_{v_p}} \in H^0(X, L^p \otimes K_X^{1/2})\}_{p\in\mathbb{N}}$ since $U(n)$ acts by biholomorphic isometries, this gives formula (1.16) and concludes the proof.

5 Quantization of Gelfand-Zetlin systems

In this Section, we fix $n \in \mathbb{N}^*$ and consider the case of irreducible representations of $G = U(n)$ with highest weight

$$\lambda \in \overline{\Gamma}_+^* = \{(\lambda_1, \cdots \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 > \lambda_2 > \cdots > \lambda_n\}\,, \tag{5.1}$$

in the sense of Definition 3.5, by definition (3.51) of the regular dominant weights under the identifications (3.25) and (3.35).

5.1 Gelfand-Zetlin bases

Recall from Theorem 3.6 that the unitary irreducible representations of $U(n)$ are in bijective correspondence with the discrete set (5.1), sending $\lambda \in \overline{\Gamma}_+^*$ to the associated *highest weight representation* $V(\lambda)$. Recalling also the sequence of inclusions (1.1), we have the following result of Weyl, whose statement can be found in [23, Prop. 6.2].

Theorem 5.1. *[21, Th. 8.1.1] For any $\lambda = (\lambda_1, \cdots, \lambda_n) \in \Gamma_+^*$, the associated highest weight representation $V(\lambda)$ admits the following decomposition into irreducible representations of $U(n-1) \subset U(n)$,*

$$V(\lambda) = \bigoplus_{\mu \in \Gamma_1(\lambda)} V(\mu)\,, \tag{5.2}$$

where

$$\Gamma_1(\lambda) = \{\mu \in \mathbb{Z}^{n-1} \mid \lambda_1 \geqslant \mu_1 \geqslant \lambda_2 \geqslant \mu_2 \geqslant \cdots \geqslant \lambda_{n-1} \geqslant \mu_{n-1} \geqslant \lambda_n\} \tag{5.3}$$

and $V(\mu)$ is the irreducible representation of $U(n-1)$ with highest weight $\mu \in \mathbb{Z}^{n-1}$.

For any $k \in \mathbb{N}^*$, let us write $T_k \subset U(k)$ for the maximal torus of the unitary group $U(k)$ given by diagonal matrices, and $\mathfrak{t}_k \subset \mathfrak{u}(k)$ for the associated Lie subalgebra of the Lie algebra $\mathfrak{u}(k) := \mathrm{Lie}(U(k))$. Under the identification $\mathfrak{t}_k^* \simeq \mathbb{R}^k$ given by (3.33), we get an identification

$$\bigoplus_{k=1}^{n} \mathfrak{t}_k^* \simeq \bigoplus_{k=1}^{n} \mathbb{R}^k\,, \tag{5.4}$$

which we will consider as the space of upper-triangular $n \times n$ matrices with real coefficients. For each $1 \leqslant i \leqslant n$, we write

$$\begin{aligned} \pi_i : \bigoplus_{k=1}^{n} \mathfrak{t}_k^* &\longrightarrow \mathfrak{t}_i^* \\ \left(\nu_j^{(k)}\right)_{1 \leqslant j \leqslant k \leqslant n} &\longmapsto (\nu_j^{(i)})_{1 \leqslant j \leqslant i}\,. \end{aligned} \tag{5.5}$$

for the associated canonical projection.

The following result follows immediately from Theorem 5.1 by decreasing induction on $n \in \mathbb{N}^*$ along the sequence of inclusions (1.1).

Corollary 5.2. *For any $\lambda = (\lambda_1, \cdots, \lambda_n) \in \Gamma_+^*$, there exists a unique decomposition*

$$V(\lambda) = \bigoplus_{\nu \in \Gamma(\lambda)} \mathbb{C}_\nu\,, \tag{5.6}$$

with

$$\begin{aligned} \Gamma(\lambda) = \{(\nu_j^{(k)})_{1 \leqslant j \leqslant k \leqslant n} \in \bigoplus_{k=1}^{n} \mathbb{Z}^k \mid \nu_j^{(n)} = \lambda_j \textit{ for all } 1 \leqslant j \leqslant n\,, \\ \nu_j^{(k)} \geqslant \nu_j^{(k-1)} \geqslant \nu_{j+1}^{(k)} \text{ for all } 1 \leqslant j \leqslant k \leqslant n\}\,. \end{aligned} \tag{5.7}$$

where for any $\nu \in \Gamma(\lambda)$, the subspace $\mathbb{C}_\nu \subset V(\lambda)$ is the 1-dimensional space characterized by the fact that $\mathbb{C}_\nu \subset V(\pi_k(\nu))$ at each step $1 \leqslant k \leqslant n-1$ in the decomposition (5.2) by decreasing induction from n to $k < n$ along the sequence of inclusions (1.1).

For any $\lambda \in \overline{\Gamma}_{+}^{*}$, the decomposition (5.6) is called the *Gelfand-Zetlin decomposition* of $V(\lambda)$. Note that Corollary 5.2 immediately implies that

$$\dim V(\lambda) = \#\Gamma(\lambda)\,, \tag{5.8}$$

as stated in [23, Prop. 6.5]. This also leads to the following definition, introducing the main notion used in this paper.

Definition 5.3. A *Gelfand-Zetlin basis* of the unitary irreducible representation $V(\lambda)$ with highest weight $\lambda \in \overline{\Gamma}_{+}^{*}$ is an orthonormal basis $\{e_\nu\}_{\nu\in\Gamma(\lambda)}$ such that $e_\nu \in \mathbb{C}_\nu$ for all $\nu \in \Gamma(\lambda)$ in the decomposition (5.6).

As the Gelfand-Zetlin decomposition (5.6) of Corollary 5.2 is unique, the Gelfand-Zetlin basis of Definition 5.3 is also unique up to multiplication of each basis vector by a unit scalar.

Recall now the sequence of inclusions (1.1), where for any $k \in \mathbb{N}$ with $k \leqslant n$, the subgroup $U(k) \subset U(n)$ consists of the two-blocks diagonal matrices with lower-right $(n-k)\times(n-k)$ block given by the diagonal identity matrix of $\mathbb{C}^{n-k}$. For any such $k \in \mathbb{N}$, we write $\mathfrak{u}(k) \subset \mathfrak{u}(n)$ for the Lie subalgebra of $U(k) \subset U(n)$. We then get a natural sequence of inclusions

$$U(\mathfrak{u}(1)) \subset U(\mathfrak{u}(2)) \subset \cdots \subset U(\mathfrak{u}(n))\,, \tag{5.9}$$

where for all $1 \leqslant k \leqslant n$, the subalgebra $U(\mathfrak{u}(k)) \subset U(\mathfrak{u}(n))$ of the envelopping algebra $U(\mathfrak{u}(n))$ is the subalgebra generated by $j(\mathfrak{u}(k)) \subset U(\mathfrak{u}(n))$ via the canonical inclusion (3.54). The subalgebra $U(\mathfrak{u}(k)) \subset U(\mathfrak{u}(n))$ is naturally isomorphic to the envelopping algebra of $\mathfrak{u}(k)$, for all $1 \leqslant k \leqslant n$.

Definition 5.4. The *Gelfand-Zetlin subalgebra* $\mathscr{A}_n \subset U(\mathfrak{u}(n))$ is the subalgebra generated by the centers $Z[U(\mathfrak{u}(k))] \subset U(\mathfrak{u}(n))$ via the inclusion (5.9), for all $1 \leqslant k \leqslant n$.

The following well-known property, which can be found for instance in [18], is a straightforward consequence of Definition 5.4.

Proposition 5.5. *The Gelfand-Zetlin subalgebra $\mathscr{A}_n \subset U(\mathfrak{u}(n))$ of Definition 5.4 is commutative.*

Proof. By Definition 5.4, it is enough to check the statement for $z_1, z_2 \in \mathscr{A}_n$ satisfying $z_j \in Z[U(\mathfrak{u}(k_j))]$ with $j = 1, 2$ and $k_1 \geqslant k_2$ respectively. Now $z_1 \in Z[U(\mathfrak{u}(k_1))]$ is naturally included in $U(\mathfrak{u}(k_2))$ via the sequence of inclusions (5.9), and as $z_2 \in Z[U(\mathfrak{u}(k_2))]$ belongs to the center of $U(\mathfrak{u}(k_2))$, we get that z_1 and z_2 commute. This concludes the proof of the Proposition. □

Recall that for any $1 \leqslant k \leqslant n$, the Weyl group of $\mathfrak{u}(k)$ coincides with the k^{th} symmetric group $\mathfrak{S}_k$, acting on $\mathfrak{t}_k^* \simeq \mathbb{R}^k$ by permutation of coordinates under the identification (3.33). We consider the cartesian product

$$\widetilde{W} = \prod_{k=1}^{n} \mathfrak{S}_k\,, \tag{5.10}$$

with induced action on $\bigoplus_{k=1}^n \mathfrak{t}_k^*$. Recall also formula (3.36) for the half-sum of positive roots $\rho_k \in \mathfrak{t}_k^*$, and let us set

$$\widetilde{\rho} = \left(\frac{k-2j+1}{2}\right)_{1\leqslant j\leqslant k\leqslant n} \in \bigoplus_{k=1}^n \mathfrak{t}_k^*\,, \tag{5.11}$$

so that $\pi_k(\widetilde{\rho}) = \rho_k$, for all $1 \leqslant k \leqslant n$. Recall the action (3.55) of the enveloping algebra $U(\mathfrak{u}(n))$ on any representation V of $U(n)$. We then have the following result, which will be crucial in the sequel.

Proposition 5.6. *The Gelfand-Zetlin subalgebra $\mathscr{A}_n \subset U(\mathfrak{u}(n))$ of Definition 5.4 acts diagonally in the Gelfand-Zetlin decomposition (5.6) of Corollary 5.2, and there is an injective algebra morphism*

$$\widetilde{\gamma} : \mathscr{A}_n \hookrightarrow \mathbb{R}\Big[\bigoplus_{k=1}^n \mathfrak{t}_k^*\Big]^{\widetilde{W}}\,, \tag{5.12}$$

characterised by the fact that for any $\lambda \in \Gamma_+^*$, the action of $z \in \mathscr{A}_n$ on the vector $e_\nu \in V(\lambda)$ parametrized by $\nu \in \Gamma(\lambda)$ in a Gelfand-Zetlin basis of Definition 5.3 is given by

$$\Phi(z).e_\nu = \widetilde{\gamma}(z)(2\pi\sqrt{-1}(\nu+\widetilde{\rho}))\,e_\nu\,. \tag{5.13}$$

Furthermore, for any $1 \leqslant j \leqslant n$, the restriction of $\widetilde{\gamma}$ to the subalgebra $Z[U(\mathfrak{u}(j))] \subset \mathscr{A}_n$ satisfies

$$\widetilde{\gamma}\,\Big|_{Z[U(\mathfrak{u}(j))]} = \pi_j^* \circ \gamma\,. \tag{5.14}$$

Proof. Let $\lambda \in \overline{\Gamma}_+^*$, and consider a Gelfand-Zetlin basis $\{e_\nu\}_{\nu\in\Gamma(\lambda)}$ of the associated highest weight representation $V(\lambda)$ as in Definition 5.3. Then for any $\nu \in \Gamma(\lambda)$, any $1 \leqslant k \leqslant n$ and any $z \in Z[U(\mathfrak{u}(k))] \subset \mathscr{A}_n$, Theorem 3.11 together with Corollary 5.2 readily imply that

$$\Phi(z).e_\nu = \gamma(z)(2\pi\sqrt{-1}\pi_k(\nu+\widetilde{\rho}))\,e_\nu\,. \tag{5.15}$$

Since by Definition 5.4, the Gelfand-Zetlin subalgebra $\mathscr{A}_n \subset U(\mathfrak{u}(n))$ is generated by $Z[U(\mathfrak{u}(k))] \subset U(\mathfrak{u}(n))$ for all $1 \leqslant k \leqslant n$, this readily implies that $\mathscr{A}_n$ acts diagonally on any Gelfand-Zetlin basis, with eigenvalues determined by formula (5.15), and that if a morphism $\widetilde{\gamma}$ as in (5.12) satisfying (5.14) exists, then it satisfies (5.13) and is uniquely determined by this property.

To show the existence of the morphism (5.12), pick $z \in \mathscr{A}_n$ and choose a presentation of it as a linear combination of products of elements of $Z[U(\mathfrak{u}(k))] \subset U(\mathfrak{u}(n))$ for all $1 \leqslant k \leqslant n$. Through the injective morphism

$$\pi_j^* : \mathbb{R}[\mathfrak{t}_j^*]^{\mathfrak{S}_k} \hookrightarrow \mathbb{R}\Big[\bigoplus_{k=1}^n \mathfrak{t}_k^*\Big]^{\widetilde{W}}\,, \tag{5.16}$$

we define $\widetilde{\gamma}(z)$ to be the unique extension of (5.14) preserving sums and products in this presentation. By formula (5.15), we see that formula (5.13) clearly holds for all $\lambda \in \overline{\Gamma}_+^*$ and $\nu \in \Gamma(\lambda)$, so that by (5.7), we see that the values of $\widetilde{\gamma}(z)$ as a polynomial over $\mathbb{R}[\bigoplus_{k=1}^n \mathfrak{t}_k^*]$ are prescribed on the integral elements in an open cone of $\bigoplus_{k=1}^n \mathbb{R}^k$, which implies as in Theorem 3.11 that formula (5.13) uniquely determines $\widetilde{\gamma}(z)$ as a polynomial over $\mathbb{R}[\bigoplus_{k=1}^n \mathfrak{t}_k^*]$. This shows in particular that $\widetilde{\gamma}(z)$ does not depend on the chosen presentation of $z \in \mathscr{A}_n$ and that it is a morphism of algebras. This establishes the existence of the morphism (5.12).

To see that the morphism (5.12) is injective, assume by contradiction that there exists a non-zero $z \in \mathscr{A}_n$ whose action via (3.55) on every irreducible representation vanishes identically, and let $P \in \mathbb{R}[\mathfrak{u}(n)^*]$ be such that

$$z = \operatorname{sym} P \in U(\mathfrak{u}(n)) \tag{5.17}$$

by the Poincaré-Birkhoff-Witt Theorem 3.10. In particular, there exists $\lambda \in \mathfrak{g}^*$ such that $P(\lambda) \neq 0$, and since any element in G is conjugate to an element in $T \subset G$, up to replacing $z \in \mathscr{A}_n$ by $\operatorname{Ad}_g z \in \mathscr{A}_n$ for $g \in G$ such that $\operatorname{Ad}_g^* \lambda \in \mathfrak{t}^*$, which also vanishes on every irreducible representation, one can assume that $\lambda \in \mathfrak{t}^*$. Hence $P \in \mathbb{R}[\mathfrak{u}(n)^*]$ does not vanish identically on $\mathfrak{t}^*$, and since Γ_+^* consists of the integral elements in the open cone $\overline{\mathfrak{t}_+^*} \subset \mathfrak{t}^*$ by (3.20) and (3.31), it cannot vanish identically on Γ_+^*, so that there exists $\lambda \in \Gamma_+^*$ such that $P(\lambda) \neq 0$.

Now by (3.53) and (3.57), for any $\lambda \in \Gamma_+^*$, definition (5.17) implies that the operator $Q_p(P) \in \operatorname{End}(H^0(X_\lambda, L_\lambda^p))$ vanishes for all $p \in \mathbb{N}$. On the other hand, letting $\lambda \in \Gamma_+^*$ such that $P(\lambda) \neq 0$, since the moment map $\mu : X_\lambda \hookrightarrow \mathfrak{u}(n)^*$ of Proposition 3.7 is just the inclusion, the function $\mu^* P \in \mathscr{C}^\infty(X_\lambda, \mathbb{R})$ does not vanish identically. Now if $Q_p(P) = 0$ for all $p \in \mathbb{N}$, Proposition 3.4 implies that $T_p(\mu^* P) = O(p^{-1})$ as $p \to +\infty$ for the operator norm, which then contradicts the norm correspondence (2.9) of Theorem 2.2. This shows that (5.17) cannot vanish on the irreducible representation $V(p\lambda)$ of highest weight $p\lambda$ for all $p \in \mathbb{N}$, showing that the morphism (5.12) is injective and concluding the proof.

□

5.2 Gelfand-Zetlin systems

Recall from Section 3.3 that for any $\lambda \in \overline{\Gamma}_+^*$, the action of $U(n)$ on the assocated coadjoint orbit $X_\lambda \subset \mathfrak{u}(n)^*$ is Hamiltonian with moment map $\mu : X_\lambda \hookrightarrow \mathfrak{u}(n)^*$ given by the inclusion. For any $1 \leqslant k \leqslant n$, write $p_k : \mathfrak{u}(n)^* \to \mathfrak{u}(k)^*$ for the projection associated with the the inclusion $U(k) \subset U(n)$ induced by the sequence of inclusions (1.1). Then the map

$$\mu_k := p_k \circ \mu : X_\lambda \longrightarrow \mathfrak{u}(k)^* \,, \tag{5.18}$$

satisfies the Kostant formula Definition 3.1, hence is a moment map for the action of the subgroup $U(k) \subset U(n)$. The following definition is adapted from [23, p. 119].

Definition 5.7. For any $\lambda \in \overline{\Gamma}_+^*$, the *Gelfand-Zetlin system* over the coadjoint orbit X_λ is the collection of functions

$$H_j^{(k)} = \mu_k^* s_j^{(k)} \in \mathscr{C}^\infty(X_\lambda, \mathbb{R}) \quad \text{for all } 1 \leqslant j \leqslant k \leqslant n-1 \,, \tag{5.19}$$

where for all $1 \leqslant k \leqslant n$ and all $1 \leqslant j \leqslant k$, the polynomial $s_j^{(k)} \in \mathbb{R}[\mathfrak{u}(k)^*]^{U(k)}$ is given by the j^{th} coefficient of the characteristic polynomial over the set of $k \times k$ Hermitian matrices under the identification (3.32).

We then have the following basic result from [23, Prop. 3.1, p. 119], mirroring the analogous Proposition 5.5 on the side of representation theory.

Proposition 5.8. *For any $\lambda \in \overline{\Gamma}^*_+$, the Gelfand-Zetlin system of Definition 5.7 is a family of Poisson-commuting functions inside $\mathscr{C}^\infty(X_\lambda, \mathbb{R})$,*

Proof. Following the proof of Proposition 5.5, let $1 \leqslant k_1, k_2 \leqslant n-1$ be such that $k_1 \leqslant k_2$, and note that for any $1 \leqslant j \leqslant k_1$ and $1 \leqslant i \leqslant k_2$, we have

$$\{H_j^{(k_1)}, H_i^{(k_2)}\} = \mu_{k_2}^* \{p_{k_2,k_1}^* s_j^{(k_1)}, s_i^{(k_2)}\}\,, \tag{5.20}$$

where $p_{k_2,k_1} : \mathfrak{u}(k_2)^* \to \mathfrak{u}(k_1)^*$ denotes the canonical projection induced by the inclusion $\mathfrak{u}(k_1) \subset \mathfrak{u}(k_2)$, and where the Poisson bracket on $\mathbb{R}[\mathfrak{u}(k_2)^*]$ is defined by the natural extension of the Lie bracket of $\mathfrak{u}(k_2)$ through the identification (3.15). Now as $s_i^{(k_2)} \in \mathbb{R}[\mathfrak{u}(k_2)^*]^{U(k_2)}$ vanishes under the adjoint action of $\mathfrak{u}(k_2)$, the bracket (5.20) necessarily vanishes, which concludes the proof. □

Recall now by (3.41) and (3.42) that if $\lambda \in \Gamma^*_+$ belongs to the set of dominant regular weights (3.51), then we have

$$\dim X_\lambda = \dim U(n) - \dim T = n(n-1)\,. \tag{5.21}$$

By the celebrated *Arnold-Liouville theorem* [2, Chap.10, § 50], Proposition 5.8 then implies the existence of *action-angle coordinates* over the open set $U \subset \mathbb{R}^{\frac{n(n-1)}{2}}$ of regular points inside the image of the map

$$\begin{aligned} H : X_\lambda &\longrightarrow \mathbb{R}^{\frac{n(n-1)}{2}} \\ x &\longrightarrow (H_j^{(k)}(x))_{1 \leqslant j \leqslant k \leqslant n-1}\,, \end{aligned} \tag{5.22}$$

which means that there exists a diffeomorphism $\Psi : \Delta^0 \to U$ with $\Delta^0 \subset \mathbb{R}^{\frac{n(n-1)}{2}}$ open and a free Hamiltonian action of the torus $\mathbb{T}^{\frac{n(n-1)}{2}}$ on $H^{-1}(U)$ with moment map

$$M := \Psi^{-1} \circ H : H^{-1}(U) \longrightarrow \Delta^0 \subset \mathbb{R}^{\frac{n(n-1)}{2}}\,. \tag{5.23}$$

To make such action-angle coordinates explicit, let $1 \leqslant k \leqslant n$, and for any $\alpha \in X_\lambda$, consider the coadjoint orbit $X_{\mu_k(\alpha)} \subset \mathfrak{u}(k)^*$ passing through its image $\mu_k(\alpha) \in \mathfrak{u}(k)^*$ under the moment map (5.18) for the action of $U(k) \subset U(n)$. By definition (3.19), we know that the intersection $X_{\mu_k(\alpha)} \cap \mathfrak{t}_k^*$ is an orbit of the Weyl group $W = \mathfrak{S}_k$, and since it acts simply and transitively on Weyl chambers, the intersection of $X_{\mu_k(\alpha)}$ with the closure $(\overline{\mathfrak{t}_k})^*_+ \subset \mathfrak{t}_k^*$ of the positive Weyl chamber (3.25) consists of only one point

$$M^{(k)}(\alpha) = (M_1^{(k)}(\alpha), M_2^{(k)}(\alpha), \cdots, M_k^{(k)}(\alpha)) \in \mathfrak{t}_k^* \simeq \mathbb{R}^k\,. \tag{5.24}$$

Following for instance [19, § A.1], recall on the other hand that for any $1 \leqslant j \leqslant k \leqslant n$, the j^{th} coefficient of the characteristic polynomial $s_j^{(k)} \subset \mathbb{R}[\mathfrak{u}(k)^*]^{U(k)}$ used in Definition 5.7 restricts over $\mathfrak{t}_k^* \simeq \mathbb{R}^n$ to the j^{th} elementary polynomial, which we still write $s_j^{(k)} \subset \mathbb{R}[\mathfrak{t}_k^*]^{\mathfrak{S}_k}$. We then have the following fundamental result of Guillemin and Sternberg in [24], which we present in the form given in [23, § 5].

Theorem 5.9. *[24, §4] The map*

$$\begin{aligned} M : X_\lambda &\longrightarrow \mathbb{R}^{\frac{n(n-1)}{2}} \\ \alpha &\mapsto (M_j^{(k)}(\alpha))_{1\leqslant j\leqslant k\leqslant n-1}\,, \end{aligned} \tag{5.25}$$

is a continuous map, whose image is the convex polytope

$$\Delta_\lambda = \Big\{(\nu_j^{(k)})_{1\leqslant j\leqslant k\leqslant n-1} \in \mathbb{R}^{\frac{n(n-1)}{2}} \;\Big|\; \lambda_j \geqslant \nu_j^{(n-1)} \geqslant \lambda_{j+1}\,, \\ \nu_j^{(k)} \geqslant \nu_j^{(k-1)} \geqslant \nu_{j+1}^{(k)}\,,\ \text{for all } 1\leqslant j<k\leqslant n\Big\}\,. \tag{5.26}$$

Furthermore, the open subset $M^{-1}(\Delta_\lambda^0)\subset X_\lambda$ over the interior $\Delta_\lambda^0\subset\Delta_\lambda$ is a dense open set over which the map (5.25) is smooth and regular, and the diffeomorphism

$$\begin{aligned} \Psi : \Delta_\lambda^0 &\longrightarrow U\subset \mathbb{R}^{\frac{n(n-1)}{2}} \\ \nu &\longmapsto (s_j^{(k)}(\pi_k(\nu)))_{1\leqslant j\leqslant k\leqslant n-1}\,, \end{aligned} \tag{5.27}$$

induces action-angle coordinates for the map (5.22) over the open set $U := \Psi(\Delta_\lambda^0)$ of its regular values, with moment map for the Hamiltonian $\mathbb{T}^{\frac{n(n-1)}{2}}$-action over $H^{-1}(U)\subset X_\lambda$ given by (5.25).

Finally, the Bohr-Sommerfeld fibres of the map (5.25) over the interior $\Delta_\lambda^0\subset\Delta_\lambda$ are the fibres over $\Delta_\lambda^0\cap\mathbb{Z}^n$.

Note that the map (5.27) sends each decreasing sequence $\nu_1^{(k)} > \nu_2^{(k)} > \cdots > \nu_k^{(k)}$ to the coefficients of the polynomial $\prod_{j=1}^k (X-\nu_j^{(k)})$, which is clearly a diffeomorphism on its image. As explained in [23, p. 121], under the natural identification (3.32) for all $1\leqslant k\leqslant n$, the Hermitian matrices $\mu_k(\alpha)\in \mathrm{Herm}(\mathbb{C}^k)$ are the $k\times k$ upper-left minors of $\alpha\in\mathrm{Herm}(\mathbb{C}^n)$, with non-increasing sequence of eigenvalues $M_1^{(k)}(\alpha)\geqslant\cdots\geqslant M_k^{(k)}(\alpha)$ given by (5.24). We then have the natural bijection

$$\begin{aligned} \Gamma(\lambda) &\xrightarrow{\ \sim\ } \Delta_\lambda\cap\mathbb{Z}^{\frac{n(n-1)}{2}} \\ (\nu_j^{(k)})_{1\leqslant j\leqslant k\leqslant n} &\mapsto (\nu_j^{(k)})_{1\leqslant j\leqslant k\leqslant n-1}\,, \end{aligned} \tag{5.28}$$

which is interpreted in [23, Prop 6.5] as the correspondence between Bohr-Sommerfeld quantization and holomorphic quantization in the case of coadjoint orbits of $U(n)$. Recalling $\overline{\rho}\in\Gamma_+^*$ from (1.6), we then get that for any $p\in\mathbb{N}$, the set

$$\Gamma_p := \Delta_{p\lambda+\overline{\rho}}\cap\mathbb{Z}^{\frac{n(n-1)}{2}}\,, \tag{5.29}$$

parametrizes the elements of a Gelfand-Zetlin basis of the irreducible representation $V(p\lambda+\overline{\rho})$ of $U(n)$ with highest weight $p\lambda+\overline{\rho}$, and is such that $\frac{1}{p}\Gamma_p\subset\mathbb{R}^{\frac{n(n-1)}{2}}$ becomes dense in Δ_λ as $p\to+\infty$, in the sense that for any open set $U\subset\mathbb{R}^{\frac{n(n-1)}{2}}$ containing Δ_λ, there is $p_0\in\mathbb{N}$ such that $\frac{1}{p}\Gamma_p\subset U$ for all $p\geqslant p_0$, and for any $v\in\Delta_\lambda$ there is a sequence $(v_p\in\frac{1}{p}\Gamma_p)_{p\in\mathbb{N}}$ such that $v_p\to v$ as $p\to+\infty$.

5.3 Proof of Theorem 1.1

Let $\lambda \in \Gamma_+^*$ be a regular weight as in (5.1), and consider the associated coadjoint orbit $X_\lambda \subset \mathfrak{g}^*$, endowed with its natural symplectic structure ω defined by (3.38) and the compatible complex structure defined by (3.47). Recall also the holomorphic Hermitian line bundle $(L_\lambda, h^{L_\lambda}, \nabla^{L_\lambda})$ defined by (3.44), which prequantizes (X_λ, ω) in the sense of (1.17) by Proposition 3.8. For any $v \in \Delta_\lambda^0 \subset \Delta_\lambda$, we write

$$\iota_\nu : \Lambda_\nu := M^{-1}(\nu) \longrightarrow X \tag{5.30}$$

for the fibre of the action-angle coordinates (5.25) over v.

The proof of Theorem 1.1 is based on a result of Charles in [13], where Theorem 2.11 is established for a more general notion of Lagrangian states than Definition 2.5. In particular, this relies on the existence of a metaplectic structure over (X_λ, ω), that is a square root $K_{X_\lambda}^{1/2}$ of the canonical bundle (2.5) of X_λ. To describe it in our setting, recall formula (3.36) for the half-sum of positive roots $\rho_n \in \mathfrak{t}^*$ of $\mathfrak{u}(n)$, and write $\overline{\rho} \in \mathbb{Z}^n$ for its integral part as in formula (1.6). In the case $n \in \mathbb{N}^*$ is odd, we see that $\rho = \overline{\rho} \in \Gamma_*^+$, so that by (3.49), the canonical line bundle K_{X_λ} over the coadjoint orbit (X_λ, ω) endowed with the complex structure induced by (3.47) admits a natural square root given by $K_{X_\lambda}^{1/2} := L_{-\overline{\rho}}$. In the case when $n \in \mathbb{N}^*$ is even, we get by (3.48) that

$$K_{X_\lambda} := L_{-2\rho} = L_{-2\overline{\rho}} \otimes L_\theta\,, \tag{5.31}$$

with $\theta = (1, 1, \cdots, 1) \in \Gamma^* \simeq \mathbb{Z}^n$. From its definition (3.44), we see that the holomorphic Hermitian line bundle (L_θ, h^{L_θ}) is trivial over X_λ, albeit endowed with a non-trivial lift of the $U(n)$ action on X_λ. This means that for any $n \in \mathbb{N}^*$, the holomorphic Hermitian line bundle

$$(K_{X_\lambda}^{1/2}, h^{K_{X_\lambda}^{1/2}}) := (L_{-\overline{\rho}}, h_{-\overline{\rho}}) \tag{5.32}$$

defines a square root of K_{X_λ} over X_λ. By the Borel-Weil Theorem 3.9, for any $p \in \mathbb{N}$, the unitary irreducible representation $V(p\lambda + \overline{\rho})$ of $U(n)$ with highest weight $p\lambda + \overline{\rho}$ satisfies the following identity of unitary representations of $U(n)$,

$$V(p\lambda + \overline{\rho}) = (H^0(X_\lambda, L_\lambda^p \otimes K_{X_\lambda}^{1/2}), \langle \cdot, \cdot \rangle_{L^2})\,, \tag{5.33}$$

where the lift of the action of $U(n)$ on $K_{X_\lambda}^{1/2}$ is determined by (5.32).

For any $k \in \mathbb{N}$ and any $1 \leqslant j \leqslant k$, recall that $s_j^{(k)} \in \mathbb{R}[\mathfrak{u}(k)^*]^{U(k)}$ stands for the j^{th} coefficient of the characteristic polynomial over the set of $k \times k$ Hermitian matrices as in Definition 5.7 and $p_k : \mathfrak{u}(n)^* \hookrightarrow \mathfrak{u}(k)^*$ for the projection induced by the sequence of inclusion (1.1), so that $p_k^* s_j^{(k)} \in \mathbb{R}[\mathfrak{u}(n)^*]$ is a homogeneous polynomial of degree $j \in \mathbb{N}$. Since on the other hand the symmetrization map (3.56) is equivariant with respect to the action of $U(n)$, we know that

$$\mathrm{sym}(p_k^* s_j^{(k)}) \in Z[U(\mathfrak{u}(k)] \subset \mathscr{A}_n\,, \tag{5.34}$$

where $\mathscr{A}_n \subset U(\mathfrak{u}(n))$ is the Gelfand-Zetlin subalgebra of Definition 5.4. Since the coefficients of the characteristic polynomial generate $\mathbb{R}[\mathfrak{u}(k)^*]^{U(k)}$ and via the bijection

(3.60) with $Z[U(\mathfrak{u}(k)]$, elements of the form (5.34) for all $1 \leqslant j \leqslant k \leqslant n-1$ generate the Gelfand-Zetlin subalgebra of Definition 5.4. Furthermore, for any $p \in \mathbb{N}$, the action of $\mathrm{sym}(p_k^* s_j^{(k)})$ on the irreducible representation $V(p\lambda + \overline{\rho})$ of $U(n)$ with highest weight $p\lambda + \overline{\rho}$ is given via (3.57) and (5.33) by the operator $Q_p(\mathrm{sym}(p_k^* s_k^{(j)})) \in \mathrm{End}(V(p\lambda + \overline{\rho}))$, which are pairwise commuting for all $1 \leqslant j \leqslant k \leqslant n-1$ by Proposition 5.5.

Let now $w \in \Delta_\lambda^0$, and consider a sequence $(w_p \in \frac{1}{p}\Gamma_p)_{p\in\mathbb{N}}$ such that $w_p \to w$ as $p \to +\infty$. For any $p \in \mathbb{N}$, let $e_{pw_p} \in V(p\lambda + \overline{\rho})$ be the element of a Gelfand-Zetlin basis parametrized by $p\, w_p \in \Gamma_p$ as in (5.29). By Proposition 5.6, these elements are common eigenvectors of the operators $Q_p(\mathrm{sym}(p_k^* s_k^{(j)})) \in \mathrm{End}(V(p\lambda + \overline{\rho}))$ for all $1 \leqslant j \leqslant k \leqslant n-1$, which by Corollary 3.12 satisfy

$$\begin{aligned} \frac{1}{(2\pi\sqrt{-1}p)^j} Q_p(\mathrm{sym}(p_k^* s_k^{(j)}))\, e_{pw_p} &= \frac{1}{(2\pi\sqrt{-1}p)^j}\, \widetilde{\gamma}(z)(2\pi\sqrt{-1}(\nu + \widetilde{\rho}))\, e_{pw_p} \\ &= (H_j^{(k)}(w_p) + O(p^{-1}))\, e_{pw_p}\,, \end{aligned} \tag{5.35}$$

where the functions $H_j^{(k)} \in \mathscr{C}^\infty(X, \mathbb{R})$ for all $1 \leqslant j \leqslant k \leqslant n-1$ form the Gelfand-Zetlin system introduced in Definition 5.7, by definition (5.18) of the application $\mu_k : X \to \mathfrak{u}(k)^*$. On the other hand, Proposition 3.4 gives the following asymptotic estimate in the sense of the operator norm of (5.33) as $p \to +\infty$,

$$\frac{1}{(2\pi\sqrt{-1}p)^j} Q_p(p_k^* s_j^{(k)}) = T_p(H_j^{(k)}) + O(p^{-1})\,. \tag{5.36}$$

By Proposition 5.8, the functions $H_j^{(k)} \in \mathscr{C}^\infty(X, \mathbb{R})$ are pairwise Poisson-commuting for all $1 \leqslant j \leqslant k \leqslant n$. The result of Charles in [14, Th. 5.2] applied to the family (5.36) of pairwise commutating operators for all $1 \leqslant j \leqslant k \leqslant n$ then implies that the sequence $\{e_{pw_p} \in H^0(X_\lambda, L_\lambda \otimes K_{X_\lambda}^{1/2})\}_{p\in\mathbb{N}}$ is a Lagrangian state associated with the sequence of Lagrangian submanifolds $\{(\Lambda_{w_p}, \zeta^p, f_{w_p})\}_{p\in\mathbb{N}}$ in the sense of [13, § 3], with sections $f_{w_p} \in \mathscr{C}^\infty(\Lambda_{w_p}, \iota_p^* K_{X_\lambda}^{1/2})$ satisfying

$$f_{w_p}^2 = dv_{w_p} \in \mathscr{C}^\infty(\Lambda_{w_p}, \det(T_\mathbb{C}^* \Lambda_{w_p})) \tag{5.37}$$

in the identification (2.23), where dv_{w_p} is the volume form of volume 1 over Λ_{w_p} invariant by the free torus action of $\mathbb{T}^{\frac{n(n-1)}{2}}$ with moment map (5.23) induced by the action-angle coordinates of Theorem 5.9. It is determined by that fact that

$$dv_{w_p}\Big(\xi_1^{X_\lambda}, \cdots \xi_{\frac{n(n-1)}{2}}^{X_\lambda}\Big) \equiv 1\,, \tag{5.38}$$

over Λ_{w_p}, where $\{\xi_j\}_{j=1}^{\frac{n(n-1)}{2}}$ is the canonical basis of the torus $\mathbb{T}^{\frac{n(n-1)}{2}}$ acting on $M^{-1}(\Delta_\lambda^0)$ from the definition (5.23) of action-angle coordinates.

Let now $v \in \Delta_\lambda^0$ and consider a sequence $(v_p \in \frac{1}{p}\Gamma_p)_{p\in\mathbb{N}}$ such that $v_p \to v$ as $p \to +\infty$. As any $g \in U(n)$ acts by a biholomorphic isometry, applying [14, Th. 5.2] as before, the sequence $\{g e_{pv_p} \in V(p\lambda + \overline{\rho})\}_{p\in\mathbb{N}}$ is also a Lagrangian state associated with the sequence

$\{(g\Lambda_{v_p}, g\zeta^p, gf_{v_p})\}_{p\in\mathbb{N}}$ of Bohr-Sommerfeld submanifolds in the sense of [13, § 3], so that $gf_{v_p} = g.dv_{v_p}$. Then if $g\Lambda_v \cap \Lambda_w = \emptyset$, the rapid vanishing (1.9) readily follows from [14, Th. 5.1], as in Theorem 2.9.

To establish the asymptotics (1.10), let us first point out that Definition 2.5 of a Lagrangian state is only a particular case of the general notion of a Lagrangian state in the sense of [13, § 3]. The asymptotics (1.10) then rather follow from the formula given in [14, Th. 6.1] for the Hermitian product of two Lagrangian states with $K = K_{X_\lambda}^{1/2}$, in strict analogy with Theorem 2.11, so that the asymptotic expansion (1.10) coincides with the general form of the formula [14, Th. 6.1], as computed for instance in [16, Th. 11.1], along with the fact that $\mathrm{Vol}(\Lambda_{v_p}, dv_{v_p}) = \mathrm{Vol}(\Lambda_{w_p}, dv_{w_p}) = 1$. In particular, a description of $\kappa(x) \in \mathbb{Z}/4\mathbb{Z}$ as a Maslov index is given in [16, § 5.4]. As explained in [12, § 2.3] and made explicit in [14, Th. 6.1], first order asymptotics of Lagrangian states have a universal behavior, depending only on the linear symplectic data at each intersection point of the Lagrangian submanifolds. Since the local model in flat space for the Definition 2.5 of a Lagrangian state is a Lagrangian state in the sense of [13, § 3], and since the first order asymptotics in Theorem 2.10, (2.21), and Theorem 2.11, (2.24), are computed using this local model in [27], one can thus also compute (1.10) using Definition 2.5 of a Lagrangian state. We then get formula (1.10) from formulas (2.24), (4.21) and (5.38), as well as the standard fact that for all $1 \leqslant j, k \leqslant \frac{n(n-1)}{2}$, we have

$$\omega_x(g.\xi_j^{X_\lambda}, \xi_k^{X_\lambda}) = \{g_* M_j, M_k\}(x)\,, \tag{5.39}$$

for any $x \in g\Lambda_{v_p} \cap \Lambda_{w_p}$, since $g \in U(n)$ acts as a Hamiltonian diffeomorphism and since $\xi_j^{X_\lambda}$ is the Hamiltonian vector field over $M^{-1}(\Delta_\lambda^0)$ associated with the j^{th} component $M_j \in \mathscr{C}^\infty(X, \mathbb{R})$ of the map (5.25) in $\mathbb{R}^{\frac{n(n-1)}{2}}$ for each $1 \leqslant j \leqslant \frac{n(n-1)}{2}$, by its description (5.23) as a moment map for the $\mathbb{T}^{\frac{n(n-1)}{2}}$-action on $M^{-1}(\Delta_\lambda^0)$. This concludes the proof.

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