A RIEMANN-ROCH FORMULA FOR SINGULAR REDUCTIONS BY CIRCLE ACTIONS

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ABSTRACT. We compute a Hirzebruch-Riemann-Roch type formula for the invariant Riemann-Roch number of a quantizable Hamiltonian S^1-manifold $(M, \omega, \mathcal{J})$, allowing 0 to be a singular value of the moment map $\mathcal{J} : M \to \mathbb{R}$. Our formula represents an instance of the Guillemin-Sternberg principle, which states that quantization should commute with reduction. The conceptual novelty of our result is that the involved reduced system only depends on the symplectic data of M. To establish this, we derive a complete singular stationary phase expansion of the Witten integral without appealing to any kind of desingularization. As a consequence, our formula expresses the invariant Riemann-Roch number purely in terms of symplectic invariants of the singular symplectic quotient. In particular, it involves a new explicit symplectic invariant of the singularities.

1. INTRODUCTION

Let $(M, \omega, \mathcal{J})$ be a compact connected symplectic manifold equipped with a Hamiltonian action of a compact connected Lie group G with moment map $\mathcal{J} : M \to \mathfrak{g}^*$. Then the associated *Marsden-Weinstein reduced space*, or simply the *symplectic quotient*, is given by

$$\mathscr{M}_0 := \mathcal{J}^{-1}(\{0\})/G\,. \tag{1.1}$$

If 0 is a regular value of the moment map, the symplectic quotient naturally inherits the structure of a symplectic orbifold $(\mathscr{M}_0, \omega_0)$, but in general, it is only a *stratified symplectic space*, each smooth stratum being naturally equipped with a symplectic structure.

This paper is devoted to the computation of the *invariant Riemann-Roch number* of $(M, \omega, \mathcal{J})$ in terms of the geometry of the associated symplectic quotient (1.1). To introduce it, one has to impose the condition that the cohomology class $[\omega] \in H^2(M, \mathbb{Z})$ is integral, in which case (M, ω) is called *quantizable*. This condition is equivalent to the existence of a Hermitian line bundle (L, h^L) over M equipped with a Hermitian connection ∇^L with curvature R^L satisfying the *prequantization condition*

$$\omega = \frac{i}{2\pi} R^L\,. \tag{1.2}$$

The Hamiltonian G-manifold $(M, \omega, \mathcal{J})$ is *prequantized* if the action of G lifts to an action on (L, h^L, ∇^L) and the moment map $\mathcal{J} : M \to \mathfrak{g}^*$ is given in terms of the lifted action

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by the *Kostant formula* (2.12). Upon choosing a G-invariant almost complex structure $J \in \mathrm{End}(TM)$ over M compatible with ω, one gets a Riemannian metric on M defined by

$$g^{TM} := \omega(\cdot, J\cdot)\,. \tag{1.3}$$

On can then consider the associated *Spin^c-Dirac operators* $D^\pm : \Omega^{0,\pm}(M, L) \to \Omega^{0,\mp}(M, L)$ defined in Section 2.2, which are elliptic first order differential operators, and thus have finite dimensional kernels. The action of G on (L, h^L) induces an action on these kernels, and one defines the associated *invariant Riemann-Roch number* as

$$\mathrm{RR}^G(M, L) := \dim(\mathrm{Ker}\, D^+)^G - \dim(\mathrm{Ker}\, D^-)^G\,, \tag{1.4}$$

where $(\mathrm{Ker}\, D^\pm)^G \subset \mathrm{Ker}\, D^\pm$ denotes the subspace of G-invariant vectors. By the classical invariance of the index of Fredholm operators, the invariant Riemann-Roch number (1.4) does not depend on the choice of the compatible almost complex structure $J \in \mathrm{End}(TM)$ over (M, ω), nor on the choice of h^L and ∇^L satisfying the prequantization condition (1.2). In case that G is trivial, it reduces to the classical Riemann-Roch number $\mathrm{RR}(M, L)$, which is computed by the celebrated *Hirzebruch-Riemann-Roch formula*

$$\mathrm{RR}(M, L) = \int_M e^\omega\ \mathrm{Td}(M)\,, \tag{1.5}$$

where the closed form $\mathrm{Td}(M) \in \Omega^*(M, \mathbb{C})$ of mixed degrees is the *Todd form* of (M, J, g^{TM}), whose cohomology class does not depend on J and hence is a symplectic invariant of (M, ω).

In case of a general G and when 0 is a regular value of the moment map $\mathcal{J}$, so that the symplectic quotient $(\mathscr{M}_0, \omega_0)$ is an orbifold prequantized by a line bundle (L_0, h^{L_0}), by a general result of Meinrenken [28] one has

$$\mathrm{RR}^G(M, L) = \int_{\mathscr{M}_0} e^{\omega_0}\, \mathrm{Td}(\mathscr{M}_0) = \mathrm{RR}(\mathscr{M}_0, L_0)\,. \tag{1.6}$$

In the special case $G = S^1$, this was previously established independently by Meinrenken in [27, Theorem 2.1, (16)] and by Vergne in [38]. As explained in [38], these results rely on localization formulas for certain oscillatory integrals of equivariant differential forms which were first studied by Witten in [39], based on previous work of Duistermaat and Heckman in [9]. To obtain (1.6) within this approach, one expresses $\mathrm{RR}^G(M, L)$ by means of the equivariant Hirzebruch-Riemann-Roch formula in a guise due to Berline and Vergne [3], called the *Kirillov formula*. This formula involves the *equivariant Todd form* $\mathrm{Td}_{\mathfrak{g}}(M) \in \Omega^*_G(M)$ defined in Section 2.3, whose cohomology class in the equivariant cohomology with analytic coefficients $H^\omega_G(M)$ is an invariant of the Hamiltonian G-action on (M, ω). The resulting expression for $\mathrm{RR}^G(M, L)$ is given in terms of a Witten integral, which can then be treated using the stationary phase principle. In doing so, one passes from the equivariant cohomology $H^\omega_G(M)$ to the usual cohomology $H(\mathscr{M}_0)$ of the symplectic quotient through the well-known *Kirwan map* $\kappa : H^\omega_G(M) \to H(\mathscr{M}_0)$ described in Proposition 4.1. Since in the case $G = S^1$ one has $\kappa(\mathrm{Td}_{\mathfrak{g}}(M)) = \mathrm{Td}(\mathscr{M}_0)$, this gives the first equality in (1.6), while the second follows from (1.5).

In this paper, we are mainly interested in the considerably more involved case when 0 is a singular value of the moment map and restrict ourselves to the case $G = S^1$, which already encompasses essential features of the singular case. Denoting by $M^{S^1} \subset M$ the set of fixed points of the S^1-action and setting $\mathcal{J}^{-1}(\{0\})^{\mathrm{reg}} := \mathcal{J}^{-1}(\{0\}) \setminus \big(M^{S^1} \cap \mathcal{J}^{-1}(\{0\})\big)$ we have a stratification of $\mathcal{J}^{-1}(\{0\})$ according to

$$\mathcal{J}^{-1}(\{0\}) = \big(M^{S^1} \cap \mathcal{J}^{-1}(\{0\})\big) \sqcup \mathcal{J}^{-1}(\{0\})^{\mathrm{reg}}\,, \tag{1.7}$$

where the connected components of each stratum are smooth submanifolds of M. To avoid any artificial complications, we will assume for simplicity that S^1 acts freely on $\mathcal{J}^{-1}(\{0\})^{\mathrm{reg}}$, so that the *regular stratum* of the symplectic quotient, defined by

$$\mathscr{M}_0^{\mathrm{reg}} := \mathcal{J}^{-1}(\{0\})^{\mathrm{reg}}/S^1\,,$$

has no orbifold singularities. We still write ω_0 for the symplectic form over the smooth stratum $\mathscr{M}_0^{\mathrm{reg}}$ characterized by the formula $\mathrm{inc}_0^*\,\omega = \pi_0^*\,\omega_0$, where $\mathrm{inc}_0 : \mathcal{J}^{-1}(\{0\})^{\mathrm{reg}} \hookrightarrow M$ is the inclusion map and $\pi_0 : \mathcal{J}^{-1}(\{0\})^{\mathrm{reg}} \to \mathscr{M}_0^{\mathrm{reg}}$ the quotient map. Following Notation 3.1, let us write $\mathcal{F}^0$ for the set of connected components of $M^{S^1} \cap \mathcal{J}^{-1}(\{0\})$. A connected component $F \in \mathcal{F}^0$ is called *definite* if $\mathcal{J}$ attains a local extreme value at F, and *indefinite* otherwise. We write $\mathcal{F}^0_{\mathrm{def}} \subset \mathcal{F}^0$ and $\mathcal{F}^0_{\mathrm{indef}} \subset \mathcal{F}^0$ for the corresponding subsets. Note that because $\mathcal{J}^{-1}(\{0\})$ is connected, all $F \in \mathcal{F}^0$ are either definite or indefinite, and we shall say that $\mathcal{J}^{-1}(\{0\})$ is of *definite or indefinite type*, respectively. For any $F \in \mathcal{F}^0$, let $\omega_F := \mathrm{inc}_F^*\,\omega \in \Omega^2(F,\mathbb{R})$ be the symplectic form on $F \subset M$ induced by the inclusion $\mathrm{inc} : F \hookrightarrow M$, and let $\nu_{\Sigma_F} : \Sigma_F \to F$ be the associated symplectic normal bundle. We write

$$\Sigma_F =: \bigoplus_{k\in W} \Sigma_F^{(k)}$$

for its decomposition into isotypic components with respect to the induced linear S^1-action, where $W \subset \mathbb{Z}$ denotes the finite subset of weights as described in Section 3.1. The compatible almost complex structure $J \in \mathrm{End}(TM)$ over (M,ω) and the associated Riemannian metric g^{TM} given by (1.3) induce for each weight $k \in W \subset \mathbb{Z}$ a complex structure and Hermitian norm $\|\cdot\|_F$ on $\Sigma_F^{(k)}$ by restriction, and we write $R^{\Sigma_F^{(k)}} \in \Omega^2(F, \mathrm{End}(\Sigma_F^{(k)}))$ for the curvature of the connection on $\Sigma_F^{(k)}$ induced by the Levi-Civita connection of g^{TM} for each $k \in W$. Let us finally consider for $F \in \mathcal{F}^0_{\mathrm{indef}}$ the fiberwise product

$$\nu_{S_F} : S_F := S_F^+ \times_F S_F^- \longrightarrow F\,, \tag{1.8}$$

where $S_F^\pm \to F$ are the unit sphere bundles of the subbundles $\Sigma_F^\pm \subset \Sigma_F$ of positive and negative weights, respectively. By the local normal form theorem of Proposition 3.2, the total space of S_F is naturally identified with the boundary of a neighborhood of F inside $\mathcal{J}^{-1}(\{0\})$. We write

$$\nu_{\mathfrak{S}_F} : \mathfrak{S}_F := (S_F^+/S^1) \times_F (S_F^-/S^1) \longrightarrow F \tag{1.9}$$

for the orbifold bundle obtained by taking the fiberwise product of the quotient of each sphere by the induced locally free S^1-action. Its de Rham cohomology ring will be denoted by $H(\mathfrak{S}_F)$.

Our main result is the following Hirzebruch-Riemann-Roch type formula for the invariant Riemann-Roch number (1.4).

Theorem 1.1. *Let $(M,\omega,\mathcal{J})$ be a compact connected prequantized Hamiltonian S^1-manifold such that the S^1-action is free on $\mathcal{J}^{-1}(\{0\})^{reg}$. The S^1-invariant Riemann-Roch number* (1.4) *is given by*

$$\mathrm{RR}^{S^1}(M,L) = \int_{\mathscr{M}_0^{\mathrm{reg}}} e^{\omega_0}\kappa(\mathrm{Td}_{\mathfrak{g}}(M)) + \sum_{F\in\mathcal{F}^0_{\mathrm{indef}}} \int_{\mathfrak{S}_F} e^{\nu^*_{\mathfrak{S}_F}\omega_F}\kappa_F(\mathrm{Td}_{\mathfrak{g}}(M)) + \sum_{F\in\mathcal{F}^0} \mathrm{Res}_F\left(z^{-1}\int_F \frac{e^{\omega_F}\,\mathrm{Td}(F)}{\prod_{k\in W}\det_{\Sigma_F^{(k)}}(1-z^k\exp(R^{\Sigma_F^{(k)}}/2\pi i))}\right), \tag{1.10}$$

where $\kappa:\Omega^*_{S^1}(M)\longrightarrow\Omega^*(\mathscr{M}_0^{\mathrm{reg}},\mathbb{C})$ and $\kappa_F: H^\omega_{S^1}(M)\to H(\mathfrak{S}_F)$ denote the regular Kirwan map (4.3) *and the* exceptional Kirwan map (1.11)*, respectively, and* Res_F *stands for the residue at $z=0$ if $\mathcal{J}$ has a local minimum at F, the residue at $z=\infty$ if $\mathcal{J}$ has a local maximum at F, or the average of the two residues otherwise.*

Furthermore, every term on the right-hand side of Formula (1.10) *is independent of the choice of a compatible almost complex structure $J\in\mathrm{End}(TM)$ over (M,ω) and of the choice of (L,h^L,∇^L) satisfying the prequantization condition* (1.2)*.*

As we explain in more detail below, Theorem 1.1 represents an instance of the Guillemin-Sternberg principle in the singular case and is characterized by the novel conceptual feature that it expresses the invariant Riemann-Roch number purely in terms of the symplectic invariants of the singular symplectic quotient $\mathscr{M}_0$. In particular, replacing L by the tensor power $L^m := L^{\otimes m}$ for any $m\in\mathbb{N}$, so that the symplectic form ω is replaced by $m\omega$, one sees that each term of the right hand side of Formula (1.10) for $RR^{S^1}(M,L^m)$ is polynomial in $m\in\mathbb{N}$, and Theorem 1.1 thus gives explicit formulas to compute the coefficients of $RR^{S^1}(M,L^m)$ as a polynomial in $m\in\mathbb{N}$ in terms of symplectic invariants of $\mathscr{M}_0$. Let us also point out that Theorem 1.1 is already relevant in the complex case, giving an explicit formula for the canonical ring of singular projective varieties $\mathscr{M}_0$ obtained as GIT quotients by a $\mathbb{C}^*$-action of a smooth projective variety M.

In case that 0 is a regular value of the moment map $\mathcal{J}$, so that $M^{S^1}\cap\mathcal{J}^{-1}(\{0\})=\emptyset$, the last two terms of Formula (1.10) vanish and Theorem 1.1 reduces to the invariant Riemann-Roch formula (1.6), as already explained there. At the other extreme end, in case that $M^{S^1}\cap\mathcal{J}^{-1}(\{0\})=\mathcal{J}^{-1}(\{0\})$, so that the action of S^1 on $\mathcal{J}^{-1}(\{0\})$ is trivial and $\mathscr{M}_0=\mathcal{J}^{-1}(\{0\})$ inherits a natural prequantizing line bundle (L_0,h^{L_0}) by restriction, the first two terms of Formula (1.10) vanish and Theorem 1.1 reduces again to (1.6) by taking into account the fact that in this case all $F\in\mathcal{F}^0$ are definite, so that all weights $k\in W$ of the S^1-action on Σ_F are of the same sign, and that in this case the sum over all residues precisely equals the middle term in (1.6). This was first established by Duistermaat, Guillemin, Meinrenken and Wu in [8, § 2]. Theorem 1.1 can thus be seen as an interpolation between these two extreme cases which covers also the genuinely singular case. In particular, the middle term in Formula (1.10) appears to be a previously unknown object involving a new explicit symplectic invariant of the singularities.

To describe this term more closely, let $\Theta^{\pm}_{S_F}\in\Omega^1(S_F,\mathbb{R})$ be the pullbacks to (1.8) of connections for the S^1-actions on $S^{\pm}_F$ in the sense of (2.8) for any $F\in\mathcal{F}^0_{\mathrm{indef}}$. At the level of S^1-equivariant differential forms as in Definition (2.7) and using the same notation as for the Kirwan map (4.1), the image of an equivariantly closed form $\varrho \in \Omega^*_{S^1}(M)$ by the *exceptional Kirwan map* is the element in $\Omega^*(\mathfrak{S}_F, \mathbb{C})$ defined by

$$\kappa_F(\varrho) := \frac{\frac{1}{2}\big(\varrho_{S_F}(\frac{i}{2\pi}d\Theta^+_{S_F}) + \varrho_{S_F}(\frac{i}{2\pi}d\Theta^-_{S_F})\big) - \varrho_{S_F}\big(\frac{i}{4\pi}(d\Theta^+_{S_F} + d\Theta^-_{S_F})\big)}{d\Theta^+_{S_F} - d\Theta^-_{S_F}}, \tag{1.11}$$

where we wrote $\varrho_{S_F} := \nu^*_{S_F}\mathrm{inc}^*_F\varrho \in \Omega^*_{S^1}(S_F)$ for the pullback to S_F of the restriction of ϱ to F. The numerator on the right-hand side of (1.11) is a multiple of $d\Theta^+_{S_F} - d\Theta^-_{S_F}$ inside the ring $\Omega^*(S_F, \mathbb{C})$, which gives an obvious sense to the fraction. Hence, as it is also closed and basic for the action of S^1 on both sphere bundles $S^\pm_F$, by Proposition 2.4, Formula (1.11) induces a well-defined map $\kappa_F : H^\omega_{S^1}(M) \to H(\mathfrak{S}_F)$ in cohomology. In fact, the numerator in (1.11) is even a multiple of $(d\Theta^+_{S_F} - d\Theta^-_{S_F})^2$, so that the right-hand side of (1.11) is itself a multiple of $d\Theta^+_{S_F} - d\Theta^-_{S_F}$. This implies in particular that κ_F, and hence the middle term in Formula (1.10), vanishes when $\dim M < 6$.

The invariant Riemann-Roch formula (1.6) is the content of the celebrated *Quantization commutes with Reduction* principle of Guillemin and Sternberg, which they formulated in [11] in the case of Kähler manifolds, always under the assumption that 0 a regular value of the moment map. In that case, the kernel of the Spin^c-Dirac operator $D^\pm$ reduces to the kernel of the Dolbeault $\overline{\partial}$-operator acting on $\Omega^{0,\pm}(M, L)$. In particular, the kernel of the $\overline{\partial}$-operator restricted to $\Omega^{0,0}(M, L)$ coincides with the space $H^0(M, L)$ of holomorphic sections of L over M, which is interpreted in [11] as the quantization of the classical phase space (M, ω). Furthermore, 0 being a regular value, the symplectic reduction $(\mathscr{M}_0, \omega_0)$ inherits a natural structure of a Kähler orbifold, and it was shown in [11] that there is a natural isomorphism

$$H^0(M, L)^G \simeq H^0(\mathscr{M}_0, L_0)\,. \tag{1.12}$$

The identity (1.12) was generalized to the kernel of the $\overline{\partial}$-operator restricted to $\Omega^{0,j}(M, L)$ for each $j > 0$ by Teleman [35] and Zhang [40]. Taking the alternating sum of dimensions of these spaces, this precisely leads to the invariant Riemann-Roch formula (1.6) in the context of Kähler manifolds. As we explain in Section 2.2, Formula (1.6) naturally extends to general symplectic manifolds, so that it constitutes the appropriate extension of Quantization commutes with Reduction in the general symplectic case, since the isomorphism (1.12) does not make sense in general. In case that L is replaced by the tensor power $L^m := L^{\otimes m}$ for $m \in \mathbb{N}$ large enough, Formula (1.6) was established by Meinrenken in [27] by the approach already described above, then by Jeffrey-Kirwan in [19] relying on the so-called *Witten non-abelian localization formula* stated in [39], which was established rigorously by Jeffrey and Kirwan in [18]. In general, Formula (1.6) was first established by Meinrenken in [28] using the symplectic cutting techniques of Lerman [23], then by Tian and Zhang [36] using the analytic localisation techniques of Bismut-Lebeau [4]. Formula (1.6) was generalized to the case of manifolds with boundary by Tian and Zhang in [37], then to the case of non-compact (M, ω) by Ma and Zhang in [26] and Paradan in [32]. A generalization to compact CR-manifolds was recently established by Ma, Marinescu, and Hsiao in [15].

In case that 0 is a singular value of the moment map, there is an immediate difficulty coming from the fact that there is no natural definition of the Riemann-Roch number $\mathrm{RR}(\mathscr{M}_0, L_0) \in \mathbb{Z}$ of a singular symplectic quotient $\mathscr{M}_0$, and one is tempted to define it directly as $\mathrm{RR}(\mathscr{M}_0, L_0) := \mathrm{RR}^G(M, L)$, making the result tautological. More substantially, one can consider the Riemann-Roch number of various notions of symplectic desingularization of stratified symplectic spaces, such as Kirwan's *partial desingularization* or the *shift desingularization*. The invariant Riemann-Roch number $\mathrm{RR}^G(M, L)$ was shown to coincide with the Riemann-Roch number of Kirwan's partial desingularization by Meinrenken and Sjamaar in [29], and of the shift desingularization by Meinrenken and Sjamaar in [29], Zhang [40] and Paradan in [31]. In the case $G = S^1$, Tian and Zhang established in [37, Theorem 6.4] an identity in terms of a Riemann-Roch number for shift desingularizations containing a term similar to the last term in Formula (1.10).

A main drawback of the approaches described above is that the desingularization depends on a number of choices and is in no way unique, while the symplectic structure also depends on the choice of an auxiliary parameter $\varepsilon > 0$. In particular, these results do not allow to compute $\mathrm{RR}^G(M, L)$ in terms of the symplectic invariants of the symplectic quotient $\mathscr{M}_0$ itself and do not provide a canonical Hirzebruch-Riemann-Roch type formula such as (1.6). In contrast, Theorem 1.1 does not rely on any desingularization process and provides an explicit formula for $\mathrm{RR}^G(M, L)$ in which every term is a well-defined invariant of the Hamiltonian action of S^1 on (M, ω). Thus, our formula satisfies the key requirement of the Guillemin-Sternberg principle that the involved reduced system should only depend on the symplectic data of M, answering an almost 30 years old question of Sjamaar in [34, pp. 124-126].

In order to compare Theorem 1.1 more closely with the mentioned previous results, we provide in Section 4.2 natural topological conditions on the S^1-action under which the first term in (1.10) can be interpreted topologically as

$$\int_{\mathscr{M}_0^{\mathrm{reg}}} e^{\omega_0} \kappa(\mathrm{Td}_{\mathfrak{g}}(M)) = \int_{\widetilde{\mathscr{M}_0}} e^{\widetilde{\omega}_0} \widetilde{\kappa}(\mathrm{Td}_{\mathfrak{g}}(M)) \,, \tag{1.13}$$

where $\pi : \widetilde{\mathscr{M}_0} \to \mathscr{M}_0$ denotes the partial resolution of $\mathscr{M}_0$ and $\widetilde{\omega}_0$ denotes the degenerate 2-form over $\widetilde{\mathscr{M}_0}$ coinciding with ω_0 on the dense open set $\mathscr{M}_0^{\mathrm{reg}}$, while $\widetilde{\kappa} : H^{\omega}_{S^1}(M) \longrightarrow H(\widetilde{\mathscr{M}_0})$ is the usual Kirwan map of the resolution. By contrast, the Riemann-Roch formula obtained via the method of Meinrenken and Sjamaar in [29] reads

$$\mathrm{RR}^G(M, L) = \int_{\widetilde{\mathscr{M}_0}} e^{\widetilde{\omega}_\varepsilon} \, \mathrm{Td}_\varepsilon(\widetilde{\mathscr{M}_0}) \,, \tag{1.14}$$

where the symplectic form $\widetilde{\omega}_\varepsilon \in \Omega^2(\widetilde{\mathscr{M}_0}, \mathbb{R})$ depends on the choice of a parameter $\varepsilon > 0$ such that $\widetilde{\omega}_\varepsilon \xrightarrow{\varepsilon \to 0} \widetilde{\omega}_0$, while $\mathrm{Td}_\varepsilon(\widetilde{\mathscr{M}_0}) \in \Omega^*(\widetilde{\mathscr{M}_0}, \mathbb{R})$ denotes the induced Todd form. In particular, as $\widetilde{\omega}_0$ is degenerate, the Todd form $\mathrm{Td}_\varepsilon(\widetilde{\mathscr{M}_0})$ does not admit a limit as $\varepsilon \to 0$, so that the right hand side is not well defined a priori for $\varepsilon = 0$. Let us also point out that Jeffrey, Kiem, Kirwan, and Woolf in [17], and Lerman and Tolman in [24] in the case of $G = S^1$, studied Kirwan maps to the intersection cohomology of the singular quotient. The relation of their Kirwan maps to our resolution Kirwan map is explained in Remark 4.8.

Our main tool for the proof of Theorem 1.1 is the already mentioned *Witten integral*, which we introduce in Definition 5.1. In fact, Theorem 1.1 is established as a consequence of our second main result, Theorem 5.7, where we compute an asymptotic expansion of the Witten integral in terms of geometric invariants associated with the singular symplectic quotient. This allows us to directly compute the invariant Riemann-Roch number $\mathrm{RR}^{S^1}(M, L)$ by adapting the method of Meinrenken in [27] to the singular case. The asymptotic parameter in the Witten integral is given by the power $m \to \infty$ of the tensor power $L^m := L^{\otimes m}$ of the prequantizing line bundle. It can therefore be regarded as a semiclassical limit, so that from this viewpoint Theorem 1.1 becomes an instance of the correspondence principle of quantum mechanics. Asymptotics of the Witten integral for general Hamiltonian G-manifolds and when 0 is a singular value of the moment map were first obtained by Paradan in [30, Theorem 5.1] relying on partitions of unity in equivariant cohomology with generalized coefficients. There, the coefficients in the expansion are given by integrals over the shifted Marsden-Weinstein reduced spaces $\mathscr{M}_\varepsilon = \mathcal{J}^{-1}(\{\varepsilon\})/G$ for some $\varepsilon \in \mathfrak{g}^*$ close to but different from 0. In certain algebraic settings, an asymptotic expansion of the Witten integral was also derived by Jeffrey, Kiem, Kirwan, and Woolf in [17, Section 9] using the localization principle. Accordingly, the coefficients in the asymptotics are given in terms of residues. In contrast, the coefficients in our asymptotic expansion of the Witten integral are given in terms of integrals on the symplectic strata of $\mathscr{M}_0$, which is crucial in proving Theorem 1.1. Our approach was preceded by work of Küster (n.k.a. Delarue) and Ramacher in [6] within a purely analytic context, motivated by the original attempt of Ramacher [33] of proving residue formulae via singular equivariant asymptotics.

Let us finally note that the assumption of S^1 acting freely on $\mathcal{J}^{-1}(\{0\})^{\mathrm{reg}}$, meaning that there are no orbifold singularities over $\mathscr{M}_0^{\mathrm{reg}}$, is not essential. In fact, the result of Meinrenken in [27], which we extend in this paper to case when 0 is a singular value of the moment map, holds for orbifolds as well, and our contribution mainly focuses on the most singular stratum $M^{S^1} \cap \mathcal{J}^{-1}(\{0\})$. Following the general method of Meinrenken, one can certainly extend our method to orbifolds, obtaining a Kawasaki-Riemann-Roch type formula which extends (1.10). In a future work, we intend to generalize our results to general compact group actions.

We begin our exposition in Section 2 by giving a detailed account on the background and setup of our paper. In Section 3 we study the geometry of the zero level set of the moment map around its singularities, which will be crucial for the ensuing analysis, and introduce the relevant Kirwan maps in Section 4. Based on these results, we derive in Section 5 a complete asymptotic expansion of the Witten integral. Finally, we establish in Section 6 the proof of our main result Theorem 1.1.

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2. Background and setup

Before we introduce our setup, let us first fix some global notation. For any vector bundle E over a smooth manifold M and for all $k \in \mathbb{N}$, we will write $\Omega^k(M, E)$ for the space of differential forms of degree k with values in E, and $\Omega^*(M, E) := \bigoplus_{k=0}^{\dim M} \Omega^k(M, E)$ for the space of differential forms of mixed degrees with value in E. We will use the same notation for any vector space V, which we regard as a trivial vector bundle over M. We will denote the inclusion $A \hookrightarrow B$ of a subset $A \subset B$ into a set B by inc_A, the target set B being clear from the context. For a finite Cartesian product $X = X_1 \times \cdots X_N$, we will write $\mathrm{pr}_{X_j} : X \to X_j$ for the canonical projection onto the factor X_j.

2.1. Equivariant cohomology.

Let M be a smooth manifold equiped with a smooth action of a compact Lie group G. Write $S^\omega(\mathfrak{g}^*)$ for the complex vector space of analytic series on $\mathfrak{g}$ with complex coefficients converging in a neighborhood of $0 \in \mathfrak{g}$. The coadjoint action of G on $\mathfrak{g}^*$ induces a natural action on $S^\omega(\mathfrak{g}^*)$.

Definition 2.1. The *complex of analytic G-equivariant differential forms* is the complex vector space of G-invariants

$$\Omega^*_G(M) := \left(\Omega^*(M, \mathbb{C}) \otimes S^\omega(\mathfrak{g}^*)\right)^G$$

for the natural G-action on the tensor product, equipped with the *equivariant differential*

$$\begin{aligned} d_{\mathfrak{g}} : \Omega^*_G(M) &\longrightarrow \Omega^*_G(M), \\ \sigma(X) &\longmapsto d\sigma(X) + 2\pi i\, \widetilde{X} \lrcorner\, \sigma(X)\,. \end{aligned} \tag{2.1}$$

The cohomology of the complex $(\Omega^*_G(M), d_{\mathfrak{g}})$ is denoted by $H^\omega_G(M)$ and called *equivariant cohomology* of M with analytic coefficients.

In the above definition and in the sequel, we will often write equivariant differential forms $\sigma(X) \in \Omega^*_G(M)$ with explicit dependence on the variable $X \in \mathfrak{g}$. We point out that there are several competing conventions for the definition of the equivariant differential in the literature, with the factor $2\pi i$ in (2.1) replaced by other constants. Similarly, the sign in (2.10) is a convention. We follow here Meinrenken's conventions in [27]. The following version of the Stokes Lemma for equivariant differential forms can be found for instance in [5, §4].

Lemma 2.2. (Equivariant Stokes' lemma) *Let $U \subset M$ be an open set such that its closure $\overline{U} \subset M$ is a compact submanifold with boundary. Then, for any $\sigma \in \Omega^*_G(M)$ we* *have*

$$\int_U d_{\mathfrak{g}}\sigma(X) = \int_{\partial\overline{U}} \sigma(X)\,.$$

For a submanifold $N \subset M$ we write $\nu_{\Sigma_N} : \Sigma_N \to N$ for its normal bundle inside TM and identify N canonically with the zero section in Σ_N. The following version of the classical homotopy lemma for equivariant differential forms follows for instance from [10, Theorem 6.1].

Lemma 2.3. (Equivariant homotopy lemma) *Let $N \subset M$ be a submanifold preserved by the action of G, and let $\Phi_N : V_N \to U_N$ be a G-equivariant diffeomorphism between a tubular neighborhood $V_N \subset \Sigma_N$ of the zero section of Σ_N and a tubular neighborhood $U_N \subset M$ of N. Then, for any equivariant cohomology class $[\varrho] \in H^{\omega}_G(M)$ there exists an equivariant form $\beta_N \in \Omega^*_G(V_N)$ such that*

$$\Phi^*_N(\varrho|_{U_N}) = \nu^*_{\Sigma_N}\mathrm{inc}^*_N\varrho|_{V_N} + d_{\mathfrak{g}}\beta_N. \tag{2.2}$$

Furthermore, the equivariant form $\beta_N \in \Omega^*_G(V_N)$ can be chosen such that $\mathrm{inc}^*_N\,\beta_N \equiv 0$ *and such that for any open set $S \subset V_N$ on which we already have $\Phi^*_N(\varrho|_{U_N})|_S = \nu^*_{\Sigma_N}\mathrm{inc}^*_N\varrho|_S$, one has $\beta_N|_S \equiv 0$.*

The second part of Lemma 2.3, which will be used in Section 5 to establish Lemma 5.3, follows from the fact that N is a strong deformation retract of U_N and the explicit form of the homotopy operator used for instance in [10, Theorem 6.1] to construct the equivariant form $\beta_N \in \Omega^*_G(V_N)$ of Formula (2.2).

Consider now the important special case when the G-action is locally free, so that the quotient map

$$\pi : M \longrightarrow M/G \tag{2.3}$$

is a G-principal bundle over the orbifold M/G. We then have the following basic result.

Proposition 2.4. *The pullback by the quotient map* (2.3) *induces an isomorphism of complexes*

$$\pi^* : (\Omega(M/G,\mathbb{R}), d) \xrightarrow{\sim} (\Omega(M)^G_{\mathrm{bas}}, d)\,, \tag{2.4}$$

where the subcomplex of basic differential forms $\Omega(M)^G_{\mathrm{bas}} \subset \Omega(M,\mathbb{R})$ *is defined by*

$$\Omega(M)^G_{\mathrm{bas}} := \{\sigma \in \Omega^*(M,\mathbb{R})^G \mid \widetilde{X} \lrcorner\, \sigma = 0\}\,, \tag{2.5}$$

The cohomology of the complex (2.5) is called the *basic cohomology* of M, and Proposition (2.4) shows that it is isomorphic to the de Rham cohomology $H(M/G)$ of the orbifold M/G. On the other hand, one has the following fundamental notion in Chern-Weil theory, which will be of crucial importance in this paper.

Definition 2.5. A *connection* for a locally free G-action on M is a $\mathfrak{g}$-valued 1-form $\Theta \in \Omega^1(M,\mathfrak{g})$ satisfying for all $X \in \mathfrak{g}$

$$\tilde{X} \lrcorner\, \Theta = X\,.$$

In what follows, we will mainly be interested in the case where $G = S^1$ is the circle group, which we realize as the subgroup of complex numbers of modulus 1, inducing an identification

$$\begin{aligned} \mathfrak{g} &\xrightarrow{\sim} \mathbb{R} \\ X &\longmapsto x\,, \end{aligned} \tag{2.6}$$

in such a way that $X \in \mathfrak{g}$ exponentiates to $e^{2\pi i x} \in S^1 \subset \mathbb{C}$. This induces in turn an identification $\mathfrak{g}^* \simeq \mathbb{R}$ of the dual of the Lie algebra with $\mathbb{R}$ and of the Lebesgue measure on the interval $[0,1]$ with the normalized Haar measure on S^1. Under this identification, Definition 2.1 becomes

$$\Omega^*_{S^1}(M) = \Omega(M, S^\omega(\mathbb{R}))^{S^1}\,, \tag{2.7}$$

the complex vector space of S^1-invariant differential forms with values in entire analytic series of the variable $x \in \mathbb{R}$. We will write S^1-equivariant differential forms $\sigma(x) \in \Omega^*_{S^1}(M)$ with explicit dependence in the variable $x \in \mathbb{R}$, when they are understood in the identification (2.7), so that they can actually be seen as functions of $x \in \mathbb{R}$ with values in S^1-invariant differential forms. Furthermore, Definition 2.5 of a connection for the S^1-action on M becomes under this identification a 1-form $\Theta \in \Omega^1(M, \mathbb{R})$ such that for any $X \in \mathfrak{g}$ with image $x \in \mathbb{R}$ by (2.6), we have

$$\tilde{X} \lrcorner\, \Theta = x\,. \tag{2.8}$$

The following basic lemma is then a straightforward consequence of Proposition 2.4 and Definition 2.5 via the identification (2.6).

Lemma 2.6. *Assume that the S^1-action on M is locally free, let $\Theta \in \Omega^1(M, \mathbb{R})$ be an associated connection in the sense of* (2.8)*, and let $\sigma \in \Omega(M/S^1)$ be closed. Then we have*

$$\int_M \pi^*\sigma \wedge \Theta = \int_{M/S^1} \sigma \tag{2.9}$$

in terms of the isomorphism of complexes (2.4)*. In particular, the integral on the left-hand side of* (2.9) *only depends on the basic cohomology class of σ.*

2.2. **Dirac operators and invariant Riemann-Roch numbers.** To introduce our proper setup, let us now focus on the case of a Hamiltonian G-action on a compact connected symplectic manifold (M, ω) of dimension $2n$. Recall that a G-equivariant map $\mathcal{J} : M \to \mathfrak{g}^*$ is called a *moment map* for such an action if for all $X \in \mathfrak{g}$, the function $\mathcal{J}(X) \in \mathrm{C}^\infty(M, \mathbb{R})$ satisfies

$$d\mathcal{J}(X) = -\widetilde{X} \lrcorner\, \omega\,, \tag{2.10}$$

where d is the de Rham differential and $\lrcorner$ denotes the contraction. By definition, a Hamiltonian G-action on a symplectic manifold (M, ω) induces a locally free action on the level set $\mathcal{J}^{-1}(\{0\})$ if and only if 0 is a regular value of $\mathcal{J} : M \to \mathfrak{g}^*$. For $G = S^1$, under the identification (2.6) the moment map $\mathcal{J} : M \to \mathfrak{g}^*$ corresponds to a function $\mathcal{J} \in \mathrm{C}^\infty(M, \mathbb{R})$

which we still call moment map, such that for any $X \in \mathfrak{g}$ with image $x \in \mathbb{R}$ by (2.6), we have

$$\mathcal{J}(X) = x\mathcal{J} \in \mathrm{C}^\infty(M,\mathbb{R})\,. \tag{2.11}$$

Next, let (M,ω) be endowed with a so-called *prequantizing* line bundle (L,h^L,∇^L), so that (L,h^L) is a Hermitian line bundle over M with a Hermitian connection ∇^L whose curvature $R^L \in \Omega^2(M,\mathbb{C})$ satisfies the prequantization condition (1.2). We denote by $\mathrm{C}^\infty(M,L)$ the space of smooth sections of L. Let further G be a connected compact Lie group such that G acts on L over M, preserving the Hermitian metric h^L and the connection ∇^L. Such an action is called *prequantized*, and the induced action of G on (M,ω) then preserves the symplectic form ω. Following for instance [27, (25)] adapted to our conventions, there is a canonical choice of moment map $\mathcal{J} : M \to \mathfrak{g}^*$ for a prequantized action of G on (M,ω) defined by the *Kostant formula*

$$\mathcal{J}(X)s := \frac{i}{2\pi}\left(L_X s - \nabla^L_{\widetilde{X}} s\right) \tag{2.12}$$

for all $s \in \mathrm{C}^\infty(M,L)$ and $X \in \mathfrak{g}$, where $\widetilde{X} \in \mathrm{C}^\infty(M,TM)$ denotes the fundamental vector field on M associated with X and L_X denotes the Lie derivative with respect to X induced by the G-action. This is the moment map which will underly all our considerations from now on.

Next, choose a G-invariant compatible almost complex structure $J \in \mathrm{End}(TM)$ over (M,ω), inducing a splitting

$$TM \otimes \mathbb{C} = T^{(1,0)}M \oplus T^{(0,1)}M \tag{2.13}$$

on the complexification $TM \otimes \mathbb{C}$ of TM into the eigenspaces of J corresponding to the eigenvalues i and $-i$, respectively. Consider the total exterior product

$$\Lambda(T^{*(0,1)}M) := \bigoplus_{j=0}^{n} \Lambda^j(T^{*(0,1)}M)\,, \tag{2.14}$$

where $T^{*(0,1)}M$ denotes the dual bundle of $T^{(0,1)}M$. For any $v \in TM$ with decomposition $v = v^{1,0} + v^{0,1}$ according to (2.13), we define its Clifford action on $\alpha \in \Lambda(T^{*(0,1)}M)$ by

$$c(v)\alpha := \sqrt{2}\,(v^{1,0})^* \wedge \alpha - v^{0,1} \lrcorner\, \alpha\,, \tag{2.15}$$

where $(v^{1,0})^*$ denotes the metric dual of $v^{1,0}$ in $T^{*(0,1)}M$ with repect to the induced Hermitian metric g^{TM} defined by (1.3). As explained for instance in [22, Appendix D], the Clifford action (2.15) on $\Lambda(T^{*(0,1)}M)$ is associated with the canonical Spin^c *structure* of the almost Hermitian manifold (M,J,g^{TM}), and there is an induced connection $\nabla^{\Lambda(T^{*(0,1)}M)}$ on $\Lambda(T^{*(0,1)}M)$, which we call the *Clifford connection*. Following [25, §1.3.1], the Clifford connection $\nabla^{\Lambda(T^{*(0,1)}M)}$ is given in any local orthonormal frame $\{e_j\}_{j=1}^n$ of (TM,g^{TM}) by the formula

$$\nabla^{\Lambda(T^{*(0,1)}M)} = d + \frac{1}{4}\sum_{j,k=1}^{2n} \langle \Gamma^{TM} e_j, e_k\rangle_{g^{TM}} c(e_j)c(e_k) + \frac{1}{2}\Gamma^{\det}\,,$$

where Γ^{TM} is the local connection form of the Levi-Civita connection ∇^{TM} associated with g^{TM} and $\Gamma^{\det}$ is the local connection form associated with the connection on $\det(T^{(1,0)}M) := \Lambda^n(T^{(1,0)}M)$ induced by ∇^{TM}. Finally, for any $m \in \mathbb{N}$, we write $\Omega^{0,*}(M, L^m)$ for the space of smooth sections of $\Lambda(T^{*(0,1)}M) \otimes L^m$, where L^m denotes the m-th tensor power of L. We then have the following

Definition 2.7. For any $m \in \mathbb{N}$, the *Spin^c-Dirac operator* D_m is defined in any local orthonormal frame $\{e_j\}_{j=1}^n$ of (TM, g^{TM}) by the formula

$$D_m = \sum_{j=1}^{2n} c(e_j) \nabla_{e_j}^{\Lambda(T^{*(0,1)}M)\otimes L^m} : \Omega^{0,*}(M, L^m) \longrightarrow \Omega^{0,*}(M, L^m),$$

where $\nabla^{\Lambda(T^{*(0,1)}M)\otimes L^m}$ is the connection on $\Lambda(T^{*(0,1)}M) \otimes L^m$ induced by $\nabla^{\Lambda(T^{*(0,1)}M)}$ and ∇^{L^m}.

By definition of the Clifford action (2.15), the Spin^c-Dirac operator D_m interchanges the space $\Omega^{0,+}(M, L^m) \subset \Omega^{0,*}(M, L^m)$ of even-degree forms with the space $\Omega^{0,-}(M, L^m) \subset \Omega^{0,*}(M, L^m)$ of odd-degree forms in the decomposition (2.14) of $\Lambda(T^{*(0,1)}M)$. We write $D_m^{\pm}$ for the restriction of D_m to $\Omega^{0,\pm}(M, L^m)$. On the other hand, as shown for instance in [25, Lemma 1.3.4], D_m is a formally self-adjoint elliptic operator on $\Omega^{0,*}(M, L^m)$ with respect to the L^2-Hermitian product induced by g^{TM} and h^L. In particular, it has finite-dimensional kernel inside $\Omega^{0,*}(M, L^m)$, which allows us to state the following definition.

Definition 2.8. For any $m \in \mathbb{N}$, the *Riemann-Roch number* $\mathrm{RR}(M, L^m) \in \mathbb{N}$ of the bundle L^m over M is defined by the formula

$$\mathrm{RR}(M, L^m) := \dim \operatorname{Ker} D_m^+ - \dim \operatorname{Ker} D_m^- .$$

The standard invariance property of indices of elliptic operators with respect to deformations shows that these Riemann-Roch numbers do not depend on the choice of an almost complex structure J nor on h^L and ∇^L satisfying the prequantization condition (1.2).

On the other hand, recall that the action of G preserves all additional data on M and L by construction, so that there is an induced action of G on $\Omega^{0,*}(M, L^m)$ commuting with D_m. In particular, this induces unitary representations of G on the finite dimensional Hermitian vector spaces $\operatorname{Ker} D_m^+$ and $\operatorname{Ker} D_m^-$. For any $g \in G$, we write

$$\chi^{(m)}(g) := \operatorname{Tr}_{\operatorname{Ker} D_m^+}[g] - \operatorname{Tr}_{\operatorname{Ker} D_m^-}[g] .$$

We write $(\operatorname{Ker} D_m^+)^G$ and $(\operatorname{Ker} D_m^-)^G$ for the subspaces of G-invariant vectors inside $\operatorname{Ker} D_m^+$ and $\operatorname{Ker} D_m^-$ respectively.

Definition 2.9. For any $m \in \mathbb{N}$, the *G-invariant Riemann-Roch number* $\mathrm{RR}^G(M, L^m) \in \mathbb{N}$ of the bundle L^m over M is defined by the formula

$$\mathrm{RR}^G(M, L^m) := \dim(\operatorname{Ker} D_m^+)^G - \dim(\operatorname{Ker} D_m^-)^G .$$

Again by invariance of the index of elliptic operators, the G-invariant Riemann-Roch number do not depend on the choice of a G-invariant almost complex structure J nor on the choice of G-invariant h^L and ∇^L satisfying the prequantization condition (1.2). Note also that we have

$$\mathrm{RR}^G(M,L^m) = \int_G \mathrm{Tr}_{\mathrm{Ker}\,D_m^+}[g]\,dg - \int_G \mathrm{Tr}_{\mathrm{Ker}\,D_m^-}[g]\,dg = \int_G \chi^{(m)}(g)\,dg\,, \tag{2.16}$$

where dg is the normalized Haar volume form on G.

2.3. **Equivariant characteristic forms and Riemann-Roch formulas.** In the setting of Section 2.2, and following [2, Definition 7.5], we define for any Hermitian vector bundle $(E, h^E) \to M$ to which the G-action lifts and which is equipped with a G-equivariant Hermitian connection ∇^E the associated *moment map* $\mathcal{J}^E : M \to \mathrm{End}(E) \otimes \mathfrak{g}^*$ via the Kostant formula

$$\mathcal{J}^E(X) := L_X - \nabla^E_{\widetilde{X}}\,, \qquad X \in \mathfrak{g},$$

where L_X denotes the Lie derivative with respect to X induced by the action of G on E. Its *equivariant curvature* $R^E_{\mathfrak{g}}(X) \in \Omega^*(M, \mathrm{End}(E))$ evaluated at $X \in \mathfrak{g}$ is then given by the formula

$$R^E_{\mathfrak{g}}(X) := R^E + 2\pi i \mathcal{J}^E(X)\,,$$

where R^E denotes the curvature of ∇^E. The associated *equivariant Todd form* is the equivariantly closed form $\mathrm{Td}_{\mathfrak{g}}(E) \in \Omega^*_G(M)$ defined for all $X \in \mathfrak{g}$ by the formula

$$\mathrm{Td}_{\mathfrak{g}}(E,X) := \det{}_E \left(\frac{R^E_{\mathfrak{g}}(X)/2\pi i}{\exp(R^E_{\mathfrak{g}}(X)/2\pi i) - \mathrm{Id}_E} \right) . \tag{2.17}$$

One readily checks that its cohomology class in $H^\omega_G(M)$ does not depend on the choice of a Hermitian metric h^E and connection ∇^E. In the important case of the tangent bundle $TM \to M$ equipped with the chosen G-invariant compatible almost complex structure $J \in \mathrm{End}(TX)$ over (M,ω), one can consider the induced Hermitian metric g^{TM} given by Formula (1.3) and the associated Levi-Civita connection ∇^{TM}, which are both G-equivariant. This induces a Hermitian metric and connection on $T^{(1,0)}M$ via the splitting (2.13), and the associated equivariant Todd form $\mathrm{Td}_{\mathfrak{g}}(M) \in \Omega^*_G(M)$ given by $\mathrm{Td}_{\mathfrak{g}}(M,X) := \mathrm{Td}_{\mathfrak{g}}(T^{(1,0)}M, X)$ does not depend on the choice of the compatible almost complex structure, and hence is an invariant of the Hamiltonian action of G on (M,ω). The closed form $\mathrm{Td}(M) := \mathrm{Td}_{\mathfrak{g}}(M,0) \in \Omega^*(M)$ is called the *Todd form* of M, and its cohomology class is a symplectic invariant of (M,ω).

Recalling the prequantization condition (1.2), the *equivariant Chern character* of L^m is the equivariantly closed form $\mathrm{ch}_{\mathfrak{g}}(L^m) \in \Omega^*_G(M)$ defined for all $m \in \mathbb{N}$ and $X \in \mathfrak{g}$ by the formula

$$\mathrm{ch}_{\mathfrak{g}}(L^m, X) := \exp(m\omega + 2\pi i m \mathcal{J}(X))\,. \tag{2.18}$$

The closed form $\mathrm{ch}(L^m) := \mathrm{ch}_{\mathfrak{g}}(L^m, 0) = e^{m\omega} \in \Omega^*(M)$ is called the *Chern character* of L^m. We now have the following fundamental *Kirillov formula*, which we present in a guise that can be found in [27, Theorem 3.1 and Remark (4) after Theorem 2.1].

Theorem 2.10. [2, Theorem 8.2] *For any $m \in \mathbb{N}$ and $X \in \mathfrak{g}$ sufficiently close to 0, we have*

$$\chi^{(m)}(\exp(X)) = \int_M \mathrm{Td}_{\mathfrak{g}}(M, X)\ \mathrm{ch}_{\mathfrak{g}}(L^m, X)\,.$$

This result can be seen as a consequence of the *Berline-Vergne localization formula* [3] applied to the *equivariant Atiyah-Segal-Singer index theorem.* To state this last theorem, consider for any $g \in G$ the fixed point set

$$M^g := \{p \in M \mid g \cdot p = p\} \subset M\,,$$

which is a smooth submanifold of M, and write $\Sigma_{M^g} \to M^g$ for its normal bundle inside TM. Let $R^{\Sigma_{M^g}} \in \Omega^2(F, \mathrm{End}(\Sigma_{M^g}))$ be the curvature of the connection on Σ_{M^g} induced by the Levi-Civita connection of g^{TM}. The following form of the equivariant index theorem can also be found in [27, Theorem 3.1 and Remark (4) after Theorem 2.1].

Theorem 2.11. [1] *For any $m \in \mathbb{N}$ and $g \in G$, we have*

$$\chi^{(m)}(g) = \int_{M^g} \frac{\mathrm{Td}(M^g)\ \mathrm{Tr}_{L^m}[g^{-1}]\exp(-mR^L/2\pi i)}{\det{}_{\Sigma_{M^g}}(1 - g\exp(R^{\Sigma_{M^g}}/2\pi i))}.$$

Note that either Theorems (2.10) and (2.11) immediately imply the classical Hirzebruch-Riemann-Roch formula (1.5) for the usual Riemann-Roch numbers of Definition 2.8.

3. Geometry of the zero level set of the moment map

Let (M, ω) be a compact symplectic manifold of dimension $2n$ on which $G = S^1$ acts in a Hamiltonian fashion with moment map $\mathcal{J} : M \to \mathfrak{g}^* \simeq \mathbb{R}$. In this section, we will study the geometry of $\mathcal{J}^{-1}(\{0\})$ using a local normal form theorem for the S^1-action, and introduce corresponding retractions that will be needed later for the implementation of homotopy arguments. We begin by introducing some general notation that will be frequently used.

Notation 3.1. We denote by $\mathcal{F}$ the set of all connected components of the fixed point set $M^{S^1} \subset M$. Given $F \in \mathcal{F}$, we write $\mathcal{J}(F) \in \mathbb{R}$ for the constant value of the moment map $\mathcal{J} : M \to \mathbb{R}$ on F. Furthermore, we introduce the following subsets of $\mathcal{F}$:

$$\mathcal{F}_+ := \{F \in \mathcal{F} \,|\, \mathcal{J} \geq \mathcal{J}(F) \text{ near } F\}, \qquad \mathcal{F}_- := \{F \in \mathcal{F} \,|\, \mathcal{J} \leq \mathcal{J}(F) \text{ near } F\},$$
$$\mathcal{F}_{\mathrm{def}} := \mathcal{F}_+ \cup \mathcal{F}_-, \qquad \mathcal{F}_{\mathrm{indef}} := \mathcal{F} \setminus \mathcal{F}_{\mathrm{def}}.$$

We say that $F \in \mathcal{F}$ is *positive definite* if $F \in \mathcal{F}_+$, *negative definite* if $F \in \mathcal{F}_-$, *definite* if $F \in \mathcal{F}_{\mathrm{def}}$ and *indefinite* if $F \in \mathcal{F}_{\mathrm{indef}}$. *For the connected components $F \in \mathcal{F}$ in $\mathcal{J}^{-1}(\{0\})$ satisfying $\mathcal{J}(F) = 0$ we set*

$$\mathcal{F}^0 := \{F \in \mathcal{F} \,|\, F \subset \mathcal{J}^{-1}(\{0\})\}, \qquad \mathcal{F}^0_\pm := \mathcal{F}^0 \cap \mathcal{F}_\pm, \qquad \mathcal{F}^0_{\mathrm{def/indef}} := \mathcal{F}^0 \cap \mathcal{F}_{\mathrm{def/indef}}\,.$$

3.1. **Local normal form theorem.** Let $F \in \mathcal{F}$ and note that F is a symplectic submanifold of M. When considering a fiber bundle over F with total space E, we will use the notation $\nu_E : E \to F$ for its bundle projection. Let $\nu_{\Sigma_F} : \Sigma_F \to F$ be the symplectic normal bundle of F inside TM, so that we have a decomposition

$$(TM, \omega)|_F = (TF, \omega_F) \oplus (\Sigma_F, \omega^\perp) \tag{3.1}$$

into symplectic vector bundles, where $\omega_F := \mathrm{inc}_F^* \omega \in \Omega^2(F)$ and $\omega^\perp$ is the fibrewise restriction of ω to Σ_F. Note that the action of S^1 on (M, ω) induces a fiberwise action on Σ_F, and choose an S^1-invariant compatible complex structure $J_{\Sigma_F} \in \mathrm{End}(\Sigma_F)$ on $(\Sigma_F, \omega^\perp)$. The Formula (1.3) then induces a Hermitian metric $g_{\Sigma_F} := \omega^\perp(\cdot, J_{\Sigma_F} \cdot)$ on the complex bundle (Σ_F, J_{Σ_F}), making Σ_F into a Hermitian vector bundle over F. For any $k \in \mathbb{N}$ and under the identification (2.6) we write

$$\Sigma_F^{(k)} := \left\{ v \in \Sigma_F \;\middle|\; \exp(X) \cdot v = e^{2\pi i k x} v \text{ for all } X \in \mathfrak{g} \right\}$$

for the associated isotypic component, as well as $W := \{k \in \mathbb{Z} \mid \Sigma_F^{(k)} \neq 0\} \subset \mathbb{Z}$ for the set of weights, which is finite and does not contain 0 by definition of Σ_F, and let $\ell_F^{(k)} := \mathrm{rk}_{\mathbb{C}} \Sigma_F^{(k)}$ be the complex rank of the vector bundle $\Sigma_F^{(k)}$. We thus have a finite decomposition

$$\Sigma_F =: \bigoplus_{k \in W} \Sigma_F^{(k)} \tag{3.2}$$

into hermitian subbundles, with associated structure group

$$K_F := \prod_{k \in W} \mathrm{U}\big(\ell_F^{(k)}\big)\,,$$

so that there is a principal K_F-bundle $\nu_{P_F} : P_F \to F$ satisfying

$$\Sigma_F \cong P_F \times_{K_F} \bigoplus_{k \in W} \mathbb{C}^{\ell_F^{(k)}}\,, \tag{3.3}$$

and inducing the decomposition (3.2), where K_F acts linearly on $\bigoplus_{k \in W} \mathbb{C}^{\ell_F^{(k)}}$. The diagonal action of S^1 on $\bigoplus_{k \in W} \mathbb{C}^{\ell_F^{(k)}}$ of weight $k \in W$ on each summand $\mathbb{C}^{\ell_F^{(k)}}$ commutes with the action of K_F, inducing an S^1-action on the right-hand side of (3.3), which makes it an S^1-equivariant identification. Note also that the symplectic structure on Σ_F is induced by the standard complex structure on each $\mathbb{C}^{\ell_F^{(k)}}$ via this decomposition. In what follows, we will write $W_\pm := \{w \in W \mid \pm w \in \mathbb{N}\}$ for the sets of positive and negative weights, respectively, so that by (3.2) there is a decomposition

$$\Sigma_F = \Sigma_F^+ \oplus \Sigma_F^-, \tag{3.4}$$

where $\Sigma_F^\pm := \bigoplus_{k \in W_\pm} \Sigma_F^{(k)}$. Setting $\ell_F^\pm := \mathrm{rk}_{\mathbb{C}} \Sigma_F^\pm = \sum_{k \in W_\pm} \ell_F^{(k)}$, the decomposition (3.4) is induced by the natural embedding $K_F \subset U(\ell_F^+) \times U(\ell_F^-)$. We write $\mathbb{C}^{\ell_F^\pm} = \bigoplus_{k \in W_\pm} \mathbb{C}^{\ell_F^{(k)}}$ and set $\ell_F := \mathrm{rk}_{\mathbb{C}} \Sigma_F$.

Let ω_{Σ_F} be a symplectic form on a neighborhood V_F of the zero section in Σ_F such that its restriction to the fibers of Σ_F coincides with the standard symplectic form induced by $\omega^\perp$, while the restriction $(T\Sigma_F, \omega_{\Sigma_F})|_F$ to the zero section $F \subset \Sigma_F$ coincides with $(TM, \omega)|_F$ via the natural identification of TM with $T\Sigma_F$ over F. The next proposition gives a canonical description of the moment map $\mathcal{J} : M \to \mathbb{R}$ in a neighborhood of each $F \in \mathcal{F}$, and constitutes the basis for the computations of the next sections. It is a straightforward consequence of the equivariant Darboux lemma, as explained for instance in [13, Section 2.2].

Proposition 3.2 (**Local normal form theorem**). *For each $F \in \mathcal{F}$ there is an S^1-equivariant symplectomorphism $\Phi_F : V_F \to U_F$ from the open tubular neighborhood (V_F, ω_{Σ_F}) of the zero section in Σ_F onto an open neighborhood $U_F \subset (M, \omega)$ of F, such that*

$$\mathcal{J} \circ \Phi_F([\wp, w]) = \frac{1}{2} Q_F(w) + \mathcal{J}(F), \quad \textit{for all} \quad [\wp, w] \in \Sigma_F \cong P_F \times_{K_F} \bigoplus_{k \in W} \mathbb{C}^{\ell_F^{(k)}},$$

where Q_F is the K_F-invariant quadratic form on $\bigoplus_{k\in W} \mathbb{C}^{\ell_F^{(k)}}$ defined by

$$Q_F(w) := \sum_{k \in W} k \, \|\pi_k(w)\|^2 . \tag{3.5}$$

Following Notation 3.1, a connected component $F \in \mathcal{F}$ is positive definite, negative definite or indefinite if and only if the quadratic form Q_F from Proposition 3.2 is positive definite, negative definite or indefinite, respectively. We write

$$Z_F := \{[\wp, w] \in \Sigma_F \,|\, w \neq 0 \text{ and } Q_F(w) = 0\} \subset \Sigma_F \tag{3.6}$$

for the submanifold formed fiberwise by the smooth points of the 0-level set of the quadratic form Q_F. Recalling the stratification (1.7), the set Z_F then corresponds to the regular stratum of $J^{-1}(\{0\})$ in the local normal coordinates of Proposition 3.2, since

$$\Phi_F(V_F \cap Z_F) = U_F \cap \mathcal{J}^{-1}(\{0\})^{\mathrm{reg}}. \tag{3.7}$$

3.2. **Local model.** In this section, we introduce for each $F \in \mathcal{F}$ coordinates on $\bigoplus_{k\in W} \mathbb{C}^{\ell_F^{(k)}}$ with respect to the zero level set of the quadric Q_F, which will be extended in Section 3.3 to the symplectic normal bundle using the framework of the previous section. We will consider $\bigoplus_{k\in W} \mathbb{C}^{\ell_F^{(k)}}$ as equipped with the diagonal S^1-action of weight $k \in W$ on each summand $\mathbb{C}^{\ell_F^{(k)}}$. To begin, we define the sets

$$\begin{aligned} \mathbb{C}_{\bullet}^{\ell_F^{\pm}} &:= \mathbb{C}^{\ell_F^{\pm}} \setminus \{0\}, && \text{in case} \quad \ell_F^{\pm} > 0, \ell_F^{\mp} = 0, \\ \mathbb{C}_{\bullet\bullet}^{\ell_F^{+}, \ell_F^{-}} &:= (\mathbb{C}^{\ell_F^{+}} \setminus \{0\}) \times (\mathbb{C}^{\ell_F^{-}} \setminus \{0\}), && \text{in case} \quad \ell_F^{+}, \ell_F^{-} > 0. \end{aligned} \tag{3.8}$$

Further, for any $k_0 \in W$ we write

$$\pi_{k_0} : \bigoplus_{k \in W} \mathbb{C}^{\ell_F^{(k)}} \to \mathbb{C}^{\ell_F^{(k_0)}}$$

for the canonical projection, and introduce the weighted Euclidean norms

$$\|w\|_{F\pm}^2 := \pm \sum_{k \in W^{\pm}} k \, \|\pi_k(w)\|^2 , \quad \text{for all} \quad w \in \mathbb{C}^{\ell_F^{\pm}} = \bigoplus_{k \in W^{\pm}} \mathbb{C}^{\ell_F^{(k)}}. \tag{3.9}$$

We then introduce the ellipsoids

$$S_{\pm}^{2\ell_F^{\pm}-1} := \{ w \in \mathbb{C}^{\ell_F^{\pm}} \mid \|w\|_{F\pm} = 1 \}, \quad \text{when} \quad \ell_F^{\pm} > 0\,, \tag{3.10}$$

Note that the norms $\|\cdot\|_{F\pm}$ are both S^1- and K_F-invariant. The quadratic form Q_F from (3.5) can be written in norm notation as

$$Q_F(w) = \|w^+\|_{F+}^2 - \|w^-\|_{F-}^2\,, \quad \text{for all} \quad w = (w^+, w^-) \in \mathbb{C}^{\ell_F^+ + \ell_F^-}.$$

Later, it will also be convenient to consider the weighted Hermitian norm

$$\|w\|_F^2 := \|w^+\|_{F+}^2 + \|w^-\|_{F-}^2\,, \quad \text{for all} \quad w = (w^+, w^-) \in \mathbb{C}^{\ell_F^+ + \ell_F^-}. \tag{3.11}$$

Next, for each weight $k \in W$, we introduce the following standard symplectic and angular forms over $\mathbb{C}^{\ell_F^{(k)}}$ for any $w = (w_1' + i w_1'', \cdots, w_{\ell_F^{(k)}}' + i w_{\ell_F^{(k)}}'') \in \mathbb{C}^{\ell_F^{(k)}}$ by setting

$$\omega_{(k)}|_w := \sum_{j=1}^{\ell_F^{(k)}} dw_j' \wedge dw_j'', \qquad \alpha_{(k)}|_w := \sum_{j=1}^{\ell_F^{(k)}} w_j'\, dw_j'' - w_j''\, dw_j'\,. \tag{3.12}$$

They are related by $\omega_{(k)} = \frac{1}{2} d\alpha_{(k)}$. For any $X \in \mathfrak{g}$ inducing the fundamental vector field $\tilde{X} \in C^\infty(\mathbb{C}^{\ell_F^{(k)}}, T\mathbb{C}^{\ell_F^{(k)}})$ associated with the S^1-action on $\mathbb{C}^{\ell_F^{(k)}}$ of weight $k \in W$ we then get

$$(\tilde{X} \lrcorner\, \alpha_{(k)})|_w = k\, x\, \|w\|^2\,, \quad \text{for all} \quad w \in \mathbb{C}^{\ell_F^{(k)}}\,, \tag{3.13}$$

where $x \in \mathbb{R}$ is the image of $X \in \mathfrak{g}$ via the identification (2.6). If we now introduce the 1-forms

$$\alpha^{\pm} := \pm \sum_{k \in W^{\pm}} \pi_k^* \alpha_{(k)} \in \Omega^1(\mathbb{C}^{\ell_F^{\pm}}, \mathbb{R}))\,, \tag{3.14}$$

then (3.13) and (3.9) imply that the 1-forms defined by

$$\theta^{\pm}|_w := \frac{\alpha^{\pm}|_w}{\|w\|_{F\pm}^2}, \qquad \text{if } w \in \mathbb{C}_\bullet^{\ell_F^{\pm}}, \tag{3.15}$$

$$\theta|_w := \frac{1}{2} \left(\frac{\alpha^+|_w}{\|w\|_{F+}^2} + \frac{\alpha^-|_w}{\|w\|_{F-}^2} \right), \qquad \text{if } w \in \mathbb{C}_{\bullet\bullet}^{\ell_F^+, \ell_F^-}, \tag{3.16}$$

are connection forms in the sense of Formula (2.8) for the diagonal S^1-action of weight $k \in W$ on the k-th summand on $\mathbb{C}_\bullet^{\ell_F^{\pm}}$ and $\mathbb{C}_{\bullet\bullet}^{\ell_F^+, \ell_F^-}$ respectively defined in (3.8).

The following simple lemma will be crucial for all our local and global considerations. Recall the general notation for inclusions and projections introduced at the beginning of Section 2.

Lemma 3.3. *Let $S \subset \bigoplus_{k\in W} \mathbb{C}^{\ell_F^{(k)}}$ be an embedded smooth submanifold, $f : (0,\infty) \to (0,\infty)$ a smooth function, and consider the smooth map $\Psi_{S,f} : S \times (0,\infty) \to \bigoplus_{k\in W} \mathbb{C}^{\ell_F^{(k)}}$, $(w,r) \mapsto f(r)w$. Then one has for each $k \in W$*

$$\Psi_{S,f}^*(\pi_k^*\alpha_{(k)}) = f(r)^2\, \mathrm{pr}_S^*\mathrm{inc}_S^*(\pi_k^*\alpha_{(k)})\,.$$

Proof. Let $v \in T_y(S\times(0,\infty))$ be a tangent vector at $y := (w,r) \in S\times(0,\infty)$ and $\gamma_v(t) = (w(t), r(t))$ a smooth curve with $\dot\gamma_v(0) = v$, $\gamma_v(0) = y$. Identifying $T_y(S\times(0,\infty))$ with a subspace of $\bigoplus_{k\in W}\mathbb{C}^{\ell_F^{(k)}} \times \mathbb{R}$ we get

$$(\Psi_{S,f})_*(v) = \frac{d}{dt}(f(r(t))w(t))|_{t=0} = \frac{d}{dt}f(r(t))|_{t=0}w + f(r)\dot w(0),$$

so that with the identification $w_j \equiv w_j'\,\partial_{w_j'} + w_j''\,\partial_{w_j''}$ in the notation of (3.12) we obtain

$$\begin{aligned}\Psi_{S,f}^*(\pi_k^*\alpha_{(k)})|_y(v) &= \frac{d}{dt}f(r(t))|_{t=0}\underbrace{(w_j'dw_j'' - w_j''dw_j')|_{\Psi_{S,f}(y)}(w)}_{=0} + f(r)(w_j'dw_j'' - w_j''dw_j')|_{\Psi_{S,f}(y)}(\dot w(0))\\ &= f(r)^2\,\mathrm{pr}_S^*\mathrm{inc}_S^*\pi_k^*\alpha_{(k)}|_y(v).\end{aligned}$$

□

We will now proceed with separate treatments of the cases $\ell_F^+, \ell_F^- > 0$ and $\ell_F^+\ell_F^- = 0$, in which Q_F is indefinite and definite, respectively.

3.2.1. *Definite case.* Let us begin with the easier definite case $F \in \mathcal{F}_{\mathrm{def}}$, so that either $\ell_F^- = 0$ or $\ell_F^+ = 0$, and write $\ell_F := \ell_F^+ > 0$ if $F \in \mathcal{F}_+$ and $\ell_F := \ell_F^- > 0$ if $F \in \mathcal{F}_-$. The diagonal action of S^1 on $\mathbb{C}^{\ell_F} = \bigoplus_{k\in W}\mathbb{C}^{\ell_F^{(k)}}$ considered in Section 3.1 restricts to a locally free action on $S^{2\ell_F-1}$, and we introduce the connection form

$$\Theta := \mathrm{inc}_{S^{2\ell_F-1}}^*\theta \in \Omega^1(S^{2\ell_F-1},\mathbb{R}) \tag{3.17}$$

as the restriction of (3.15) to $S^{2\ell_F-1}$.

We now introduce spherical polar coordinates in $\mathbb{C}_\bullet^{\ell_F}$ via the diffeomorphism

$$\begin{aligned}\Psi : S^{2\ell_F-1}\times(0,\infty) &\longrightarrow \mathbb{C}_\bullet^{\ell_F}\\ (w,r) &\longmapsto rw\,,\end{aligned} \tag{3.18}$$

which is equivariant for the action of S^1 introduced above on the first factor and the trivial action on the second factor of $S^{2\ell_F-1}\times(0,\infty)$. It defines on $\mathbb{C}_\bullet^{\ell_F}$ the radial coordinate

$$r = \|w\|_F = \sqrt{|Q_F(w)|}, \quad \text{for all} \quad w \in \mathbb{C}_\bullet^{\ell_F}, \tag{3.19}$$

which is related to the quadratic form Q_F by

$$Q_F|_{\mathbb{C}_\bullet^{\ell_F}} = \pm r^2 \quad \text{if } F \in \mathcal{F}_\pm. \tag{3.20}$$

Applying Lemma 3.3 to $S = S^{2\ell_F-1}$ and $f = \mathrm{id}$ one immediately sees that

$$\Psi^*(\alpha^\pm) = r^2\,\mathrm{pr}_{S^{2\ell_F-1}}^*\mathrm{inc}_{S^{2\ell_F-1}}^*\alpha^\pm\,.$$

Since $\omega_{\mathrm{std}} = \pm\frac{1}{2}d\theta$ and $\Theta = \mathrm{inc}^*_{S^{2\ell_F-1}}\theta$, we deduce from this the useful identity

(3.21)
$$\Psi^*(\omega_{\mathrm{std}}|_{\mathbb{C}^{\ell_F}_\bullet}) = \pm\frac{1}{2}d(r^2\,\mathrm{pr}^*_{S^{2\ell_F-1}}\Theta) = \pm r\,dr\wedge \mathrm{pr}^*_{S^{2\ell_F-1}}\Theta \pm \frac{1}{2}r^2\,\mathrm{pr}^*_{S^{2\ell_F-1}}d\Theta \quad \text{if } F\in\mathcal{F}_\pm.$$

3.2.2. *Indefinite case.* Let us now turn to the case $F\in\mathcal{F}_{\mathrm{indef}}$, so that $\ell_F^+,\ell_F^->0$. Departing from the quadric formed by the zero level set of Q_F we define the slit quadric

$$\mathcal{Q}_\times := Q_F^{-1}(\{0\})\setminus\{0\}\subset \mathbb{C}^{\ell_F^+,\ell_F^-}_{\bullet\bullet}, \tag{3.22}$$

which is a smooth submanifold of the set $\mathbb{C}^{\ell_F^+,\ell_F^-}_{\bullet\bullet}$ defined in (3.8). The diagonal action of S^1 on $\mathbb{C}^{\ell_F} = \bigoplus_{k\in W}\mathbb{C}^{\ell_F^{(k)}}$ considered in Section 3.1 restricts to locally free S^1-actions on $\mathcal{Q}_\times\subset\mathbb{C}^{\ell_F^+,\ell_F^-}_{\bullet\bullet}$ and the product of ellipsoids

$$S_+^{2\ell_F^+-1}\times S_-^{2\ell_F^--1}\subset\mathbb{C}^{\ell_F^++\ell_F^-} = \mathbb{C}^{\ell_F^+}\oplus\mathbb{C}^{\ell_F^-} \tag{3.23}$$

of the ellipsoids (3.10). Besides the connection form (3.16) we introduce now also the 1-form

$$\bar\theta|_w := \frac{1}{2}\left(\frac{\alpha^+|_w}{\|w\|^2_{F+}} - \frac{\alpha^-|_w}{\|w\|^2_{F-}}\right) \tag{3.24}$$

on $\mathbb{C}^{\ell_F^+,\ell_F^-}_{\bullet\bullet}$, and introduce the corresponding restrictions

$$\Theta := \mathrm{inc}^*_{S_+^{2\ell_F^+-1}\times S_-^{2\ell_F^--1}}\theta, \qquad \overline{\Theta} := \mathrm{inc}^*_{S_+^{2\ell_F^+-1}\times S_-^{2\ell_F^--1}}\bar\theta \tag{3.25}$$

to $S_+^{2\ell_F^+-1}\times S_-^{2\ell_F^--1}$. Clearly, $\Theta\in\Omega^1(S_+^{2\ell_F^+-1}\times S_-^{2\ell_F^--1},\mathbb{R})$ is a connection for the S^1-action on $S_+^{2\ell_F^+-1}\times S_-^{2\ell_F^--1}$. On the other hand, the 1-form $\overline{\Theta}\in\Omega^1(S_+^{2\ell_F^+-1}\times S_-^{2\ell_F^--1},\mathbb{R})$ is basic in the sense of Proposition 2.4.

Let us now introduce polar moment coordinates in $\mathbb{C}^{\ell_F^+,\ell_F^-}_{\bullet\bullet}$ via the diffeomorphism

(3.26)
$$\begin{aligned}\Psi : S_+^{2\ell_F^+-1}\times S_-^{2\ell_F^--1}\times(0,\infty)\times\mathbb{R} &\longrightarrow \mathbb{C}^{\ell_F^+,\ell_F^-}_{\bullet\bullet},\\ (w^+,w^-,r,q)&\longmapsto\left(\sqrt{\sqrt{r^4+q^2}+q}\;w^+,\sqrt{\sqrt{r^4+q^2}-q}\;w^-\right).\end{aligned}$$

The map Ψ is at the basis of our local model. It is equivariant under the action of S^1 introduced above on $S_+^{2\ell_F^+-1}\times S_-^{2\ell_F^--1}$ and the trivial action on $(0,\infty)\times\mathbb{R}$, and defines for $w\in\mathbb{C}^{\ell_F^+,\ell_F^-}_{\bullet\bullet}$ the coordinates

$$r = \sqrt{\|w^+\|_{F+}\,\|w^-\|_{F-}}, \qquad q = \frac{1}{2}Q_F(w). \tag{3.27}$$

The slit quadric $\mathcal{Q}_\times \subset \mathbb{C}^{\ell_F^+,\ell_F^-}_{\bullet\bullet}$ defined in (3.22) corresponds to the set $\{q = 0\}$, while the coordinate r is radial on $\mathcal{Q}_\times$. Moreover, composing the retraction

$$\begin{aligned} &\mathrm{ret}_{\mathcal{Q}_\times} : \mathbb{C}^{\ell_F^+,\ell_F^-}_{\bullet\bullet} \longrightarrow \mathcal{Q}_\times, \\ &w = (w^+, w^-) \longmapsto \sqrt{\|w^+\|_{F+}\,\|w^-\|_{F-}}\Big(\frac{w^+}{\|w^+\|_{F+}}, \frac{w^-}{\|w^-\|_{F-}}\Big) \end{aligned} \tag{3.28}$$

with Ψ gives us an S^1-equivariant surjection

$$\begin{aligned} \pi_{\mathcal{Q}_\times} : S_+^{2\ell_F^+-1} \times S_-^{2\ell_F^--1} \times (0,\infty) \times \mathbb{R} &\longrightarrow \mathcal{Q}_\times, \\ (w^+, w^-, r, q) &\longmapsto \Psi(w^+, w^-, r, 0) = (rw^+, rw^-). \end{aligned} \tag{3.29}$$

Let $\mathrm{pr}_{S_+^{2\ell_F^+-1}\times S_-^{2\ell_F^--1}} : S_+^{2\ell_F^+-1} \times S_-^{2\ell_F^--1} \times (0,\infty) \times \mathbb{R} \longrightarrow S_+^{2\ell_F^+-1} \times S_-^{2\ell_F^--1}$ be the canonical projection. We then have

Lemma 3.4. *Write* $\alpha := \frac{1}{2}(\alpha^+ + \alpha^-)$ *and* $\overline{\alpha} := \frac{1}{2}(\alpha^+ - \alpha^-)$. *In the coordinates* (3.27)*, one has*

$$\begin{aligned} \Psi^*(\alpha|_{\mathbb{C}^{\ell_F^+,\ell_F^-}_{\bullet\bullet}}) &= \sqrt{r^4+q^2}\ \mathrm{pr}^*_{S_+^{2\ell_F^+-1}\times S_-^{2\ell_F^--1}}\Theta + q\,\mathrm{pr}^*_{S_+^{2\ell_F^+-1}\times S_-^{2\ell_F^--1}}\overline{\Theta}, \\ \Psi^*(\overline{\alpha}|_{\mathbb{C}^{\ell_F^+,\ell_F^-}_{\bullet\bullet}}) &= \sqrt{r^4+q^2}\ \mathrm{pr}^*_{S_+^{2\ell_F^+-1}\times S_-^{2\ell_F^--1}}\overline{\Theta} + q\,\mathrm{pr}^*_{S_+^{2\ell_F^+-1}\times S_-^{2\ell_F^--1}}\Theta\,. \end{aligned}$$

Proof. Consider Ψ and $\mathrm{pr}_{S_+^{2\ell_F^+-1}\times S_-^{2\ell_F^--1}}$ as families of maps defined on $S_+^{2\ell_F^+-1} \times S_-^{2\ell_F^--1} \times (0,\infty)$ and parametrized by $q \in \mathbb{R}$. Defining $f_q^\pm : (0,\infty) \to (0,\infty)$ by $f_q^\pm(r) := \sqrt{\sqrt{r^4+q^2} \pm q}$, we can apply Lemma 3.3 with $S = S_+^{2\ell_F^+-1} \times S_-^{2\ell_F^--1}$ and $f = f_q^\pm$ to get

$$\Psi^*(\alpha^\pm|_{\mathbb{C}^{\ell_F^+,\ell_F^-}_{\bullet\bullet}}) = f_q^\pm(r)^2 \mathrm{pr}^*_{S_+^{2\ell_F^+-1}\times S_-^{2\ell_F^--1}} \mathrm{inc}^*_{S_+^{2\ell_F^+-1}\times S_-^{2\ell_F^--1}} \alpha^\pm\,,$$

concluding the proof. □

A direct consequence of Lemma 3.4 is the following indefinite version of (3.21).

Corollary 3.5. *In the coordinates* (3.26) *one has*

$$\Psi^*(\omega_{\mathrm{std}}|_{\mathbb{C}^{\ell_F^+,\ell_F^-}_{\bullet\bullet}}) = \pi^*_{\mathcal{Q}_\times}\mathrm{inc}^*_{\mathcal{Q}_\times}\omega_{\mathrm{std}} + d\Big(q\,\mathrm{pr}^*_{S_+^{2\ell_F^+-1}\times S_-^{2\ell_F^--1}}\Theta + \big(\sqrt{r^4+q^2} - r^2\big)\mathrm{pr}^*_{S_+^{2\ell_F^+-1}\times S_-^{2\ell_F^--1}}\overline{\Theta}\Big).$$

Proof. As $\omega_{\mathrm{std}} = d\bar{\alpha}$, we apply d on both sides of the equation in Lemma 3.4 involving $\bar{\alpha}$, yielding

$$\Psi^*(\omega_{\mathrm{std}}|_{\mathbb{C}^{\ell_F^+,\ell_F^-}_{\bullet\bullet}}) = d\big(q\,\mathrm{pr}^*_{S_+^{2\ell_F^+-1}\times S_-^{2\ell_F^--1}}\Theta + \sqrt{r^4+q^2}\mathrm{pr}^*_{S_+^{2\ell_F^+-1}\times S_-^{2\ell_F^--1}}\overline{\Theta}\big).$$

We now assert that

$$\pi^*_{\mathcal{Q}_\times}\mathrm{inc}^*_{\mathcal{Q}_\times}\omega_{\mathrm{std}} = d\big(r^2\mathrm{pr}^*_{S_+^{2\ell_F^+-1}\times S_-^{2\ell_F^--1}}\overline{\Theta}\big). \tag{3.30}$$

Indeed, since $\overline{\Theta} = \mathrm{inc}^*_{S_+^{2\ell_F^+-1}\times S_-^{2\ell_F^--1}}\bar{\alpha}$, this is a consequence of the relation

$$\pi_{\mathcal{Q}_\times}^* \mathrm{inc}_{\mathcal{Q}_\times}^* \bar{\alpha} = r^2 \,\mathrm{pr}^*_{S_+^{2\ell_F^+-1}\times S_-^{2\ell_F^--1}} \mathrm{inc}^*_{S_+^{2\ell_F^+-1}\times S_-^{2\ell_F^--1}} \bar{\alpha}$$

which follows from Lemma 3.3 applied with $S = S_+^{2\ell_F^+-1} \times S_-^{2\ell_F^--1}$, $f = \mathrm{id}$, and $q \in \mathbb{R}$ as an additional parameter. ☐

3.3. **Application to the symplectic normal bundle.** We now translate the fiberwise considerations from Section 3.2 into global statements on the symplectic normal bundle Σ_F, identified with the associated bundle $P_F \times_{K_F} \bigoplus_{k\in W} \mathbb{C}^{\ell_F^{(k)}}$ as in (3.3).

Choose a K_F-connection on the principal bundle $\nu_{P_F} : P_F \to F$, inducing a Hermitian connection ∇^{Σ_F} on Σ_F that preserves the decomposition (3.2). This provides us with a splitting of the short exact sequence

$$0 \longrightarrow \nu_{\Sigma_F}^* \Sigma_F \longrightarrow T\Sigma_F \longrightarrow \nu_{\Sigma_F}^* TF \longrightarrow 0$$

of vector bundles over Σ_F, yielding the global decomposition

$$T\Sigma_F \cong T^{\mathrm{hor}}F \oplus \nu_{\Sigma_F}^* \Sigma_F \,, \tag{3.31}$$

where $T^{\mathrm{hor}}F \simeq \nu_{\Sigma_F}^* TF$ is the horizontal distribution defined by ∇^{Σ_F} and $\nu_{\Sigma_F}^*\Sigma_F \subset T\Sigma_F$ is the vertical tangent bundle to the fibers. Choose a compatible almost complex structure $J_F \in \mathrm{End}(TF)$ over (F, ω_F). Together with the natural complex structure $J_{\Sigma_F} \in \mathrm{End}(\Sigma_F)$, this induces via the decomposition (3.31) an S^1-invariant almost complex structure $J \in \mathrm{End}(T\Sigma_F)$ over the total space of Σ_F.

For any $k_0 \in W$, we define $f_{k_0} \in \mathrm{C}^\infty(\Sigma_F, \mathbb{R})$ by setting

$$f_{k_0}([\wp, w]) := \frac{1}{2}\,\|\pi_{k_0}(w)\|^2\,, \quad \text{for all} \quad [\wp, w] \in P_F \times_{K_F} \bigoplus_{k\in W} \mathbb{C}^{\ell_F^{(k)}}\,, \tag{3.32}$$

where $\|\cdot\|$ denotes the standard Hermitian norm. Note that f_{k_0} is well-defined since for each $k \in W$ the structure group K_F acts by $\mathrm{U}(\ell_F^{(k)})$ on $\mathbb{C}^{\ell_F^{(k)}}$ and thus preserves the Hermitian norm. Using this, we define the 1-form $\alpha_{k_0} \in \Omega^1(\Sigma_F, \mathbb{R})$ by

$$\alpha_{k_0}(v) := -df_{k_0}(Jv), \quad \text{for all} \quad v \in T\Sigma_F\,. \tag{3.33}$$

Comparing with (3.12), one readily checks that for any $p \in F$ the restriction of $\alpha_{k_0} \in \Omega^1(\Sigma_F, \mathbb{R})$ to the fiber $(\Sigma_F)_p \subset \Sigma_F$, seen as a submanifold in the total space of Σ_F, satisfies

$$\alpha_{k_0}|_{(\Sigma_F)_p} = \pi_{k_0}^* \alpha_{(k_0)}\,, \tag{3.34}$$

in any trivialization $(\Sigma_F)_p \simeq \bigoplus_{k\in W} \mathbb{C}^{\ell_F^{(k)}}$ compatible with the K_F-principal bundle structure (3.3). Following (3.14), (3.15) and (3.16), we write

$$\alpha_F^\pm := \pm \sum_{k\in W^\pm} \alpha_k\,, \tag{3.35}$$

so that the 1-forms given by

$$\theta_F^{\pm}|_{[\wp,w]} := \frac{\alpha_F^{\pm}|_{[\wp,w]}}{\|w\|_{F\pm}^2} \quad \text{for} \quad [\wp,w] \in P_F \times_{K_F} \mathbb{C}_{\bullet}^{\ell_F^{\pm}}, \tag{3.36}$$

$$\theta_F|_{[\wp,w]} := \frac{1}{2}\left(\frac{\alpha_F^{+}|_{[\wp,w]}}{\|w\|_{F+}^2} + \frac{\alpha_F^{-}|_{[\wp,w]}}{\|w\|_{F-}^2}\right) \quad \text{for} \quad [\wp,w] \in P_F \times_{K_F} \mathbb{C}_{\bullet\bullet}^{\ell_F^{+},\ell_F^{-}}, \tag{3.37}$$

define connections for the S^1-action on $P_F \times_{K_F} \mathbb{C}_{\bullet}^{\ell_F^{\pm}}$ and $P_F \times_{K_F} \mathbb{C}_{\bullet\bullet}^{\ell_F^{+},\ell_F^{-}}$, respectively, such that for all $p \in F$ the restriction of $\theta_F^{\pm}$, θ_F to the fiber over p coincides with $\theta^{\pm}$, θ in any trivialization $(\Sigma_F)_p \simeq \bigoplus_{k\in W} \mathbb{C}^{\ell_F^{(k)}}$ compatible with the K_F-principal bundle structure (3.3). We also define the vertical form $\omega_{\mathrm{vert}} \in \Omega^2(\Sigma_F,\mathbb{R})$ by

$$\omega_{\mathrm{vert}} := \frac{1}{2}\sum_{k\in W} d\alpha_k \,. \tag{3.38}$$

Comparing with (3.12) and (3.34), for all $p \in F$ the restriction of $\omega_{\mathrm{vert}} \in \Omega^2(\Sigma_F,\mathbb{R})$ to the fiber $(\Sigma_F)_p \subset \Sigma_F$ satisfies

$$\omega_{\mathrm{vert}}|_{(\Sigma_F)_p} = \omega_{\mathrm{std}} \,, \tag{3.39}$$

in any trivialization $(\Sigma_F)_p \simeq \bigoplus_{k\in W} \mathbb{C}^{\ell_F^{(k)}}$ compatible with the K_F-principal bundle structure (3.3), where ω_{std} is the standard symplectic form on $\bigoplus_{k\in W} \mathbb{C}^{\ell_F^{(k)}}$ induced by (3.12) for each $k \in W$. Furthermore, note that the restriction $T\Sigma_F|_F$ over the zero section $F \subset \Sigma_F$ naturally identifies with the restriction $TM|_F$ over $F \subset M$, and the restriction of ω_{vert} to $T\Sigma_F|_F$ coincides with $\omega^{\perp}$ in the decomposition (3.1). Consequently, we can explicitly choose the symplectic form ω_{Σ_F} from Proposition 3.2 to be given by

$$\omega_{\Sigma_F} := \omega_{\mathrm{vert}} + \nu_{\Sigma_F}^* \omega_F \in \Omega^2(\Sigma_F,\mathbb{R}). \tag{3.40}$$

By construction, the almost complex structure $J \in \mathrm{End}(T\Sigma_F)$ over Σ_F is compatible with ω_{Σ_F}, and we write $g^{T\Sigma_F} := \omega_{\Sigma_F}(\cdot, J\cdot)$ for the induced S^1-invariant Riemannian metric over V_F.

3.3.1. *Definite case.* Let $F \in \mathcal{F}_{\mathrm{def}}$. Recall the definition of ℓ_F from the beginning of Section 3.2.1. As the linear action of K_F on $\mathbb{C}^{\ell_F}$ preserves $S^{2\ell_F-1}$, we can consider the associated sphere bundle $S_F := P_F \times_{K_F} S^{2\ell_F-1}$ inside $\Sigma_F = P_F \times_{K_F} \mathbb{C}^{\ell_F}$, with bundle projection $\nu_{S_F} : S_F \to F$. The S^1-action on Σ_F restricts to a locally free S^1-action on S_F for which we have the connection

$$\Theta_{S_F} := \mathrm{inc}_{S_F}^* \theta_F \tag{3.41}$$

given by restriction of the connection θ_F from (3.36). Using (3.8) to define the subbundle

$$\Sigma_{F\bullet} := P_F \times_{K_F} \mathbb{C}_{\bullet}^{\ell_F} = \Sigma_F \setminus F,$$

the S^1-equivariant diffeomorphism Ψ from (3.18) globalizes to an S^1-equivariant diffeomorphism

$$\begin{aligned} \Psi_F : S_F \times (0,\infty) &\longrightarrow \Sigma_{F\bullet} \\ ([\wp, w], r) &\longmapsto [\wp, \Psi(w,r)]\,, \end{aligned} \tag{3.42}$$

where S^1 acts on $S_F \times (0,\infty)$ by the product of the S^1-action on S_F and the trivial action on $(0,\infty)$. Ψ_F promotes the coordinate r from (3.19) to a global fiber-radial coordinate r on the bundle $\Sigma_{F\bullet}$. Its relation to Q_F is given by (3.20) with $\mathbb{C}^{\ell_F}_{\bullet}$ replaced by $\Sigma_{F\bullet}$. Note that $S_F \times (0,\infty)$ is a fiber bundle over F with the projection $\nu_{S_F\times(0,\infty)} := \nu_{S_F} \circ \mathrm{pr}_{S_F}$. We state the global version of (3.21) in

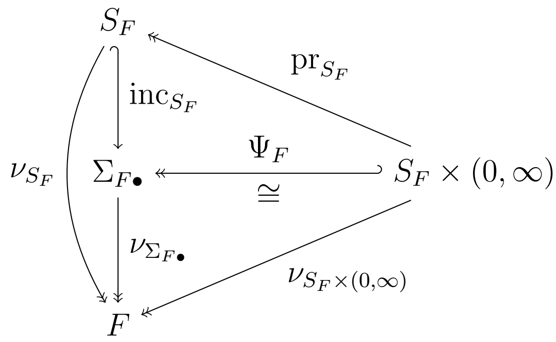
Corollary 3.6. *In the coordinate* (3.19) *provided by the diffeomorphism* (3.42) *we have*

$$\Psi_F^*(\omega_{\Sigma_F}|_{\Sigma_{F\bullet}}) = \nu^*_{S_F\times(0,\infty)}\omega_F \pm d\Big(\frac{r^2}{2}\mathrm{pr}^*_{S_F}\Theta_{S_F}\Big) \quad \textit{if } F \in \mathcal{F}_\pm.$$

Proof. In view of the definition (3.41) of Θ_{S_F} and the definition (3.42) of Ψ_F the assertion follows from (3.40), (3.21), and (3.39). □

For a better overview, we illustrate the maps appearing in Corollary 3.6 in the diagram

(3.43)



in which everything commutes except the triangle formed by pr_{S_F}, inc_{S_F}, and Ψ_F, which commutes only up to S^1-equivariant homotopy.

3.3.2. *Indefinite case.* Let $F \in \mathcal{F}_{\mathrm{indef}}$. As the linear action of K_F on $\mathbb{C}^{\ell_F^+ + \ell_F^-}$ preserves the bi-ellipsoid (3.23), we can consider the associated fiber bundle

$$S_F := P_F \times_{K_F} (S_+^{2\ell_F^+ - 1} \times S_-^{2\ell_F^- - 1}) \tag{3.44}$$

inside $\Sigma_F \cong P_F \times_{K_F} \mathbb{C}^{\ell_F^+ + \ell_F^-}$. Furthermore, the 0-level set $Z_F \subset \Sigma_F \setminus F$ defined in (3.6) has the fiber bundle structure

$$Z_F \cong P_F \times_{K_F} \mathcal{Q}_\times \tag{3.45}$$

associated with the slit quadric (3.22). Recall that we write $\nu_{S_F} : S_F \to F$ and $\nu_{Z_F} : Z_F \to F$ for the fiber bundle projections. The action of S^1 on Σ_F restricts to locally free

S^1-actions on S_F and Z_F, respectively, and we set

$$\Theta_{S_F} := \mathrm{inc}^*_{S_F}\theta_F = \frac{1}{2}(\Theta^+_{S_F} + \Theta^-_{S_F}), \quad \overline{\Theta}_{S_F} := \frac{1}{2}(\Theta^+_{S_F} - \Theta^-_{S_F}), \quad \text{with} \quad \Theta^{\pm}_{S_F} := \mathrm{inc}^*_{S_F}\theta^{\pm}_F, \tag{3.46}$$

$$\Theta_{Z_F} := \mathrm{inc}^*_{Z_F}\theta_F = \frac{1}{2}(\Theta^+_{Z_F} + \Theta^-_{Z_F}), \quad \overline{\Theta}_{Z_F} := \frac{1}{2}(\Theta^+_{Z_F} - \Theta^-_{Z_F}), \quad \text{with} \quad \Theta^{\pm}_{Z_F} := \mathrm{inc}^*_{Z_F}\theta^{\pm}_F\,, \tag{3.47}$$

where θ_F is the connection form on $\Sigma_F \setminus F$ from (3.37) and the 1-forms α_k were defined in (3.33). Thus, Θ_{S_F} and Θ_{Z_F} are connections for the S^1-actions on S_F and Z_F, respectively. On the other hand, $\overline{\Theta}_{S_F}$ and $\overline{\Theta}_{Z_F}$ are basic differential forms in the sense of Proposition 2.4. Using (3.8) to define the subbundle

$$\Sigma_{F\bullet\bullet} := P_F \times_{K_F} \mathbb{C}^{\ell^+_F,\ell^-_F}_{\bullet\bullet},$$

the S^1-equivariant diffeomorphism Ψ from (3.26) globalizes to an S^1-equivariant diffeomorphism

$$\begin{aligned} \Psi_F : S_F \times (0,\infty) \times \mathbb{R} &\longrightarrow \Sigma_{F\bullet\bullet} \\ ([\wp, w], r, q) &\longmapsto [\wp, \Psi(w, r, q)], \end{aligned} \tag{3.48}$$

where S^1 acts on $S_F \times (0,\infty) \times \mathbb{R}$ by the product of the S^1-action on S_F and the trivial action on $(0,\infty)\times\mathbb{R}$. The map Ψ_F defines coordinates (3.27) on $\Sigma_{F\bullet\bullet}$, in which Z_F coincides with $\{q = 0\}$. The retraction $\mathrm{ret}_{\mathcal{Q}_\times}$ from (3.28) and the surjection $\pi_{\mathcal{Q}_\times}$ from (3.29) globalize to an S^1-equivariant retraction and an S^1-equivariant surjection onto Z_F, respectively, by defining

$$\begin{aligned} \mathrm{ret}_{Z_F} : \Sigma_{F\bullet\bullet} &\longrightarrow Z_F \\ [\wp, w] &\longmapsto [\wp, \mathrm{ret}_{\mathcal{Q}_\times}(w)], \end{aligned} \tag{3.49}$$

as well as

$$\begin{aligned} \pi_{Z_F} := \mathrm{ret}_{Z_F} \circ \Psi_F : S_F \times (0,\infty) \times \mathbb{R} &\longrightarrow Z_F, \\ ([\wp, w], r, q) &\longmapsto [\wp, \pi_{\mathcal{Q}_\times}(w, r, q)] = \Psi_F([\wp, w], r, 0) = [\wp, rw]. \end{aligned}$$

The restricted diffeomorphism

$$\Psi_F : S_F \times (0,\infty) \times \{0\} \longrightarrow Z_F, \tag{3.50}$$

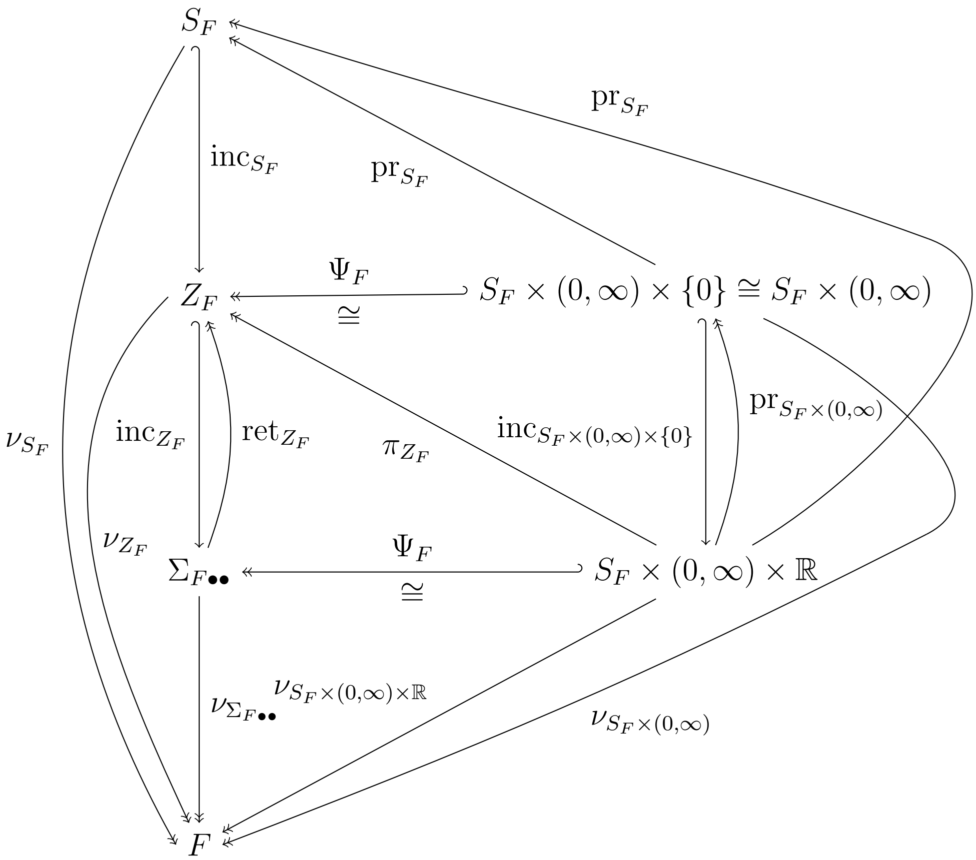
denoted again just by Ψ_F for simplicity, provides the radial coordinate r on Z_F. Note that the space $S_F \times (0,\infty)$, which we identify canonically with $S_F \times (0,\infty) \times \{0\}$, as well as the space $S_F \times (0,\infty) \times \mathbb{R}$, are fiber bundles over F with the corresponding projections $\nu_{S_F \times (0,\infty)}$, $\nu_{S_F \times (0,\infty) \times \mathbb{R}}$ given by the composition of ν_{S_F} with the canonical projections onto S_F, respectively. These constructions lead us to the following

Corollary 3.7. *In the coordinates* (3.27) *defined by the diffeomorphism* (3.48) *one has*

$$\Psi_F^*(\omega_{\Sigma_F}|_{\Sigma_{F\bullet\bullet}}) = \pi_{Z_F}^* \mathrm{inc}_{Z_F}^* \omega_{\Sigma_F} + d\Big(q\, \mathrm{pr}_{S_F}^* \Theta_{S_F} + \big(\sqrt{r^4+q^2} - r^2\big) \mathrm{pr}_{S_F}^* \overline{\Theta}_{S_F} \Big).$$

Proof. By the definition of ω_{Σ_F} in (3.40) and (3.39) the proof is a straightforward consequence of Corollary 3.5. ☐

For an overview, the maps introduced above, and in particular those involved in Corollary 3.7, are illustrated in the diagram

(3.51)


This is the indefinite version of (3.43). In the diagram, everything commutes except the triangles involving an inclusion followed by the retraction ret_{Z_F}, a projection, or the surjection π_{Z_F}, which commute only up to S^1-equivariant homotopy. Note that the two canonical projections onto S_F are both denoted by pr_{S_F}, whereas each fiber bundle projection onto F has its own name.

4. Kirwan maps

In the setting of Section 2.1, let $(M, \omega, \mathcal{J})$ be a Hamiltonian S^1-manifold such that the restriction of the S^1-action to $\mathcal{J}^{-1}(\{0\})$ is locally free, and write $\mathrm{inc}_0 : \mathcal{J}^{-1}(\{0\}) \to M$

for the inclusion map. The following classical notion goes back to Kirwan [20], which we introduce here only for $G = S^1$.

Proposition 4.1. [16, (18)] *Assume that S^1 acts locally freely on the level set $\mathcal{J}^{-1}(\{0\}) \subset M$, and let $\Theta \in \Omega^1(\mathcal{J}^{-1}(\{0\}), \mathbb{R})$ be a connection for the S^1-action. Then, for any $\sigma(x) \in \Omega^*_{S^1}(M)$*

$$\kappa(\sigma(X)) := \mathrm{inc}_0^* \, \sigma\big(\frac{i}{2\pi} d\Theta\big) - \frac{i}{2\pi}\Theta \wedge \left(\widetilde{X} \lrcorner \, \mathrm{inc}_0^* \, \sigma\big(\frac{i}{2\pi} d\Theta\big)\right) \tag{4.1}$$

defines a map of complexes $\kappa : (\Omega^*_{S^1}(M), d_{\mathfrak{g}}) \to (\Omega(\mathcal{J}^{-1}(\{0\}))^{S^1}_{\mathrm{bas}}, d)$, inducing via the isomorphism of Proposition 2.4 a morphism

$$\kappa : H^{\omega}_{S^1}(M) \longrightarrow H(\mathscr{M}_0)\,, \tag{4.2}$$

which does not depend on the choice of the connection $\Theta \in \Omega^1(\mathcal{J}^{-1}(\{0\}), \mathbb{R})$. The morphism κ is called the Kirwan map.

The notation $\mathrm{inc}_0^* \, \sigma(\frac{i}{2\pi} d\Theta) \in \Omega^*(\mathcal{J}^{-1}(\{0\}), \mathbb{R})$ means that we substitute $\frac{i}{2\pi} d\Theta$ for X in $\mathrm{inc}_0^* \, \sigma(X)$, taking the wedge product with the coefficients of the power series $\mathrm{inc}_0^* \, \sigma(X)$.

4.1. **Regular Kirwan map of a singular symplectic quotient.** Let now $(M, \omega, \mathcal{J})$ be a general Hamiltonian S^1-manifold, so that the symplectic quotient (1.1) is only a stratified space. To introduce the regular Kirwan maps appearing in Theorem 1.1, we will need the following definition.

Definition 4.2. A connection $\Theta \in \Omega^1(\mathcal{J}^{-1}(\{0\})^{\mathrm{reg}}, \mathbb{R})$ for the S^1-action on $\mathcal{J}^{-1}(\{0\})^{\mathrm{reg}}$ is said to have *normal form near the singularities* if for each $F \in \mathcal{F}^0_{\mathrm{indef}}$ one has

$$\Phi_F^*(\Theta|_{U_F \cap \mathcal{J}^{-1}(\{0\})^{\mathrm{reg}}}) = \Theta_{Z_F}|_{V_F \cap Z_F}\,,$$

where $U_F \subset M$ is a neighborhood of F as in Proposition 3.2 and $\Theta_{Z_F} \in \Omega^1(Z_F, \mathbb{R})$ denotes the connection form (3.47) over $Z_F \subset \Sigma_F$ via (3.7).

Using a partition of unity, one readily constructs a connection with normal form near all $F \in \mathcal{F}^0_{\mathrm{indef}}$.

Definition 4.3. The *regular Kirwan map* is the map of complexes

$$\kappa : (\Omega^{\omega}_{S^1}(M), d_{\mathfrak{g}}) \longrightarrow (\Omega(\mathscr{M}_0^{\mathrm{reg}}), d) \tag{4.3}$$

defined by the restriction of the formula (4.1) to the regular stratum $\mathcal{J}^{-1}(\{0\})^{\mathrm{reg}}$ of the stratification (1.7) and using a connection $\Theta \in \Omega^1(\mathcal{J}^{-1}(\{0\})^{\mathrm{reg}}, \mathbb{R})$ with normal form near the singularities in the sense of Definition 4.2.

The following result will be used in a crucial way in Section 5.4 to establish Theorem 5.7 on the topological interpretation of the asymptotics of the Witten integral, which forms the core of the proof of Theorem 1.1.

Lemma 4.4. *Let $\sigma \in \Omega^*_{S^1}(M)$ be such that for every $F \in \mathcal{F}^0_{\mathrm{indef}}$ we have $\mathrm{inc}_F^* \sigma \equiv 0$. Then for all $m \in \mathbb{N}$, we have*

$$\int_{\mathscr{M}_0^{\mathrm{reg}}} e^{m\omega_0} \kappa(d_{\mathfrak{g}}\sigma) = 0\,, \tag{4.4}$$

where $\kappa : \Omega^*_{S^1}(M) \to \Omega(\mathscr{M}_0^{\mathrm{reg}})$ denotes the Kirwan map (4.3).

Proof. Let $F \in \mathcal{F}^0_{\mathrm{indef}}$. Recall from (3.7) that the diffeomorphism Φ_F of the local normal form of Proposition 3.2 sends $Z_F \cap V_F$ to $\mathcal{J}^{-1}(\{0\})^{\mathrm{reg}} \cap U_F$, where $Z_F = Q_F^{-1}(\{0\}) \setminus F$ has been introduced in Equation (3.6). Keeping in mind the coordinates (3.27), for any small enough $\varepsilon > 0$ let $B_{F,\varepsilon} \subset \mathcal{J}^{-1}(\{0\}) \cap U_F$ be the neighborhood of F defined by

$$B_{F,\varepsilon} := \left\{ x \in \mathcal{J}^{-1}(\{0\}) \;\middle|\; \Phi_F^{-1}(x) = [\wp, w] \text{ with } \sqrt{\|w^+\|_{F+}\,\|w^-\|_{F-}} < \varepsilon \right\}.$$

Let $\Theta \in \Omega^1(\mathcal{J}^{-1}(\{0\})^{\mathrm{reg}}, \mathbb{R})$ be a connection for the S^1-action whose restriction to $\mathcal{J}^{-1}(\{0\})^{\mathrm{reg}} \cap U_F$ pulls back along Φ_F to the connection $\Theta_{Z_F} \in \Omega^1(Z_F, \mathbb{R})$ introduced in (3.47), and write $\sigma_F := \mathrm{inc}_F^* \sigma \in \Omega^*_{S^1}(F)$, as well as $\mathrm{inc}_0 : \mathcal{J}^{-1}(\{0\}) \to M$ for the inclusion map.

Using Proposition 2.6 and the fact that the Kirwan map (4.3) is a map of complexes we can now apply the usual Stokes' theorem to get

$$\begin{aligned}
\int_{\mathscr{M}_0^{\mathrm{reg}}} e^{m\omega_0} \kappa(d_{\mathfrak{g}}\sigma) &= \int_{\mathcal{J}^{-1}(\{0\})^{\mathrm{reg}}} e^{m\,\mathrm{inc}_0^*\omega} d\kappa(\sigma) \wedge \Theta \\
&= \lim_{\varepsilon \to 0} \int_{\mathcal{J}^{-1}(\{0\}) \setminus \bigcup_{F \in \mathcal{F}^0_{\mathrm{indef}}} B_{F,\varepsilon}} e^{m\,\mathrm{inc}_0^*\omega} d\kappa(\sigma) \wedge \Theta \\
&= \sum_{F \in \mathcal{F}^0_{\mathrm{indef}}} \lim_{\varepsilon \to 0} \int_{\partial B_{F,\varepsilon}} e^{m\,\mathrm{inc}_0^*\omega} \kappa(\sigma) \wedge \Theta \\
&\qquad - \lim_{\varepsilon \to 0} \int_{\mathcal{J}^{-1}(\{0\}) \setminus \bigcup_{F \in \mathcal{F}^0_{\mathrm{indef}}} B_{F,\varepsilon}} e^{m\,\mathrm{inc}_0^*\omega} \kappa(\sigma) \wedge d\Theta \\
&= \sum_{F \in \mathcal{F}^0_{\mathrm{indef}}} \lim_{\varepsilon \to 0} \int_{\partial B_{F,\varepsilon}} e^{m\,\mathrm{inc}_0^*\omega_0}\, \mathrm{inc}_0^* \sigma\big(\frac{i}{2\pi} d\Theta\big) \wedge \Theta,
\end{aligned}$$

where the second term of the third line vanishes since the integrand is basic for the S^1-action and therefore its top form component is zero. Next, we define for all small enough $\varepsilon > 0$ the diffeomorphism

$$\begin{aligned}
\tilde{b}_{F,\varepsilon} : S_F &\longrightarrow \Phi_F^{-1}(\partial B_{F,\varepsilon}) = \Psi_F(S_F \times \{\varepsilon\} \times \{0\}) \subset Z_F \subset \Sigma_F, \\
p = [\wp, w] &\longmapsto \Psi_F(p, \varepsilon, 0) = [\wp, \varepsilon w]
\end{aligned}$$

analogous to (5.31) using the diffeomorphism Ψ_F from (3.48), and we pull back along $\Phi_F \circ \tilde{b}_{F,\varepsilon}$ to get

$$\int_{\mathscr{M}_0^{\mathrm{reg}}} e^{m\omega_0}\kappa(d_{\mathfrak{g}}\sigma) = \sum_{F\in\mathcal{F}^0_{\mathrm{indef}}} \lim_{\varepsilon\to 0}\int_{S_F} e^{m\tilde{b}^*_{F,\varepsilon}\omega_{\Sigma_F}}\tilde{b}^*_{F,\varepsilon}\Phi_F^*\sigma_{|U_F}\big(\frac{i}{2\pi}d\tilde{b}^*_{F,\varepsilon}\Theta_{Z_F}\big)\wedge\tilde{b}^*_{F,\varepsilon}\Theta_{Z_F}. \tag{4.5}$$

We now investigate each of the three pullbacks along $\tilde{b}_{F,\varepsilon}$ on the right-hand side. From (3.38), (3.33) and (3.32) we get as in (5.33)

$$\tilde{b}^*_{F,\varepsilon}\omega_{\Sigma_F} = \varepsilon^2\mathrm{inc}^*_{S_F}\omega_{\mathrm{vert}} + \nu^*_{S_F}\,\omega_F. \tag{4.6}$$

Similarly, (3.47) implies

$$\tilde{b}^*_{F,\varepsilon}\Theta_{Z_F} = \Theta_{S_F}. \tag{4.7}$$

Furthermore, we claim that for any differential form $\alpha\in\Omega^*(M)$ we have

$$\lim_{\varepsilon\to 0}\tilde{b}^*_{F,\varepsilon}\Phi_F^*\alpha_{|U_F} = \nu^*_{S_F}\alpha_F \quad \text{in } \Omega^*(S_F), \tag{4.8}$$

where we write $\alpha_F := \mathrm{inc}_F^*\,\alpha\in\Omega^*(F)$ and we use on $\Omega^*(S_F)$ the standard Fréchet topology of uniform convergence of all derivatives, recalling that S_F is compact. To prove the claim, we first note that it is enough to prove (4.8) when $\alpha\in\Omega^0(M) = \mathrm{C}^\infty(M)$ is a smooth function and the convergence takes place in the Fréchet subspace $\mathrm{C}^\infty(S_F)\subset\Omega^*(S_F)$, since (4.8) is a local formula involving pullbacks which commute with wedge products and the exterior differential d, which is a continuous operator $\Omega^*(S_F)\to\Omega^*(S_F)$, and any differential form of positive degree can locally be written as a sum of wedge products of differentials of smooth functions. Now, by passing to a local trivialization of the fiber bundle Σ_F over a local chart of F, the claim (4.8) for $\alpha\in\mathrm{C}^\infty(M)$ reduces to the claim that for $n,m\in\mathbb{N}_0$, $m>0$, the operator

$$b_\varepsilon : \mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R}^n\times\mathbb{R}^m)\longrightarrow\mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R}^n\times S^{m-1}),\qquad f\longmapsto b_\varepsilon(f)(x,y) := f(x,\varepsilon y)\,,$$

satisfies for each $f\in\mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R}^n\times\mathbb{R}^m)$ that

$$b_\varepsilon(f)\overset{\varepsilon\to 0}{\longrightarrow} f_0 \quad \text{in } \mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R}^n\times S^{m-1}),$$

where $f_0(x,y) := f(x,0)$. One easily verifies that this holds, finishing the proof of the claim.

Combining (4.6), (4.7), and (4.8) allows us to evaluate the limit in (4.5), yielding

$$\int_{\mathscr{M}_0^{\mathrm{reg}}} e^{m\omega_0}\kappa(d_{\mathfrak{g}}\sigma) = \sum_{F\in\mathcal{F}^0_{\mathrm{indef}}}\int_{S_F} e^{m\nu^*_{S_F}\omega_F}\nu^*_{S_F}\sigma_F\big(\frac{i}{2\pi}d\Theta_{S_F}\big)\wedge\Theta_{S_F} = 0\,,$$

since $\sigma_F := \mathrm{inc}_F^*\,\sigma\equiv 0$ by assumption. This concludes the proof. ☐

Lemma (4.4) gives a partial cohomological interpretation for the regular Kirwan map (4.3), since it shows that the integral of the left-hand side of (4.4) does not depend on the choice of a connection with normal form near the singularities. Theorem 5.7 will actually show a posteriori that the condition $\mathrm{inc}_F^*\,\sigma\equiv 0$ is not necessary for (4.4) to hold.

4.2. **Kirwan map of a partial resolution.** In order to compare our results with those in [29], let us now discuss the Kirwan map of a partial resolution. Suppose that $\mathscr{J}^{-1}(\{0\})$ is of indefinite type, and recall the notations of Section 3. For any connected component $F \in \mathcal{F}^0$ of the set of fixed points M^{S^1} contained in $\mathscr{J}^{-1}(\{0\})$ we consider the *complex blow-up*

$$\beta_{\widetilde{\Sigma}_F} : \widetilde{\Sigma}_F \longrightarrow \Sigma_F \tag{4.9}$$

of Σ_F along its zero section $F \subset \Sigma_F$ in the sense of [12, Section 8], which is equivariant with respect to the S^1-action considered in Section 3. The *strict transform* of the submanifold $Z_F \subset \Sigma_F$ introduced in (3.6) is defined as the closure $\widetilde{Z}_F := \overline{\beta^{-1}_{\widetilde{\Sigma}_F}(Z_F)} \subset \widetilde{\Sigma}_F$ and inherits a natural structure of a smooth S^1-submanifold of $\widetilde{\Sigma}_F$. Setting $\mathbb{F}^0 := \sqcup_{F\in\mathcal{F}^0} F \subset M^{S^1}$ we then define the *complex blow-up* of M along $\mathbb{F}^0 \subset M$ as the unique S^1-equivariant map

$$\beta : \widetilde{M} \longrightarrow M \tag{4.10}$$

which restricts to an S^1-equivariant diffeomorphism over $M \setminus \mathbb{F}^0$ and which, over each open set $U_F \subset M$ as in Proposition 3.2 is given by the map $\beta_{\widetilde{\Sigma}_F}$ over V_F. Like $\widetilde{Z}_F$, the strict transform $\widetilde{Z}$ of the subset $\mathscr{J}^{-1}(\{0\})^{\mathrm{reg}} \subset M$ is defined as the closure $\widetilde{Z} := \overline{\beta^{-1}(\mathscr{J}^{-1}(\{0\})^{\mathrm{reg}})} \subset \widetilde{M}$ and inherits a natural structure of a smooth S^1-submanifold of $\widetilde{M}$. Since the S^1-action on $\widetilde{Z}$ is locally free, one is led to the following definition.

Definition 4.5. The *partial resolution* of the symplectic quotient $\mathscr{M}_0$ is defined as the orbifold

$$\widetilde{\mathscr{M}_0} := \widetilde{Z}/S^1 \,,$$

together with the map $\beta_0 : \widetilde{\mathscr{M}_0} \longrightarrow \mathscr{M}_0$ induced by the map (4.10) after taking quotients by the S^1-action. The unique form $\widetilde{\omega}_0 \in \Omega^2(\widetilde{\mathscr{M}_0}, \mathbb{R})$ satisfying

$$\widetilde{\pi}_0^* \widetilde{\omega}_0 = \mathrm{inc}^*_{\widetilde{Z}}\, \beta^* \omega \,,$$

where $\widetilde{\pi}_0 : \widetilde{Z} \longrightarrow \widetilde{\mathscr{M}_0}$ denotes the quotient map, is called the *degenerate symplectic form* of $\widetilde{\mathscr{M}_0}$.

Partial resolutions were introduced for arbitrary compact Lie group actions by Kirwan in [21] in the algebraic case and by Meinrenken and Sjamaar in [29, Section 4.1.2] in the symplectic case. As it is explained in [7, Corollary 3.14], our definition agrees with this definition in the case $G = S^1$.

We now define the following alternative notion of a Kirwan map for singular symplectic quotients by an S^1-action, even though $\widetilde{M}$ does not admit a canonical structure of a Hamiltonian S^1-manifold.

Definition 4.6. The *resolution Kirwan map* is the map

$$\widetilde{\kappa} := \kappa \circ \beta^* : H^{\omega}_{S^1}(M) \longrightarrow H(\widetilde{\mathscr{M}_0}) \tag{4.11}$$

where $\beta^* : H^\omega_{S^1}(M) \to H^\omega_{S^1}(\widetilde{M})$ is induced by the map (4.10) and $\kappa : H^\omega_{S^1}(\widetilde{M}) \longrightarrow H(\widetilde{\mathscr{M}_0})$ is defined at the level of complexes by the formula (4.1) with $\mathrm{inc}_0 : \mathcal{J}^{-1}(\{0\}) \to M$ replaced by the inclusion map $\mathrm{inc}_{\widetilde{Z}} : \widetilde{Z} \to \widetilde{M}$.

The following result gives another topological interpretation of the regular Kirwan map (4.3), strenghtening Lemma 4.4 under an appropriate combinatorial condition on the S^1-action around the fixed points.

Lemma 4.7. *With the notation of Section 3.1, assume that the weights of the S^1-action satisfy the condition*

$$\frac{\# W^+}{\sum_{k\in W^+}|k|} = \frac{\# W^-}{\sum_{k\in W^-}|k|} \tag{4.12}$$

for each $F \in \mathcal{F}^0_{\mathrm{indef}}$. Then, for any connection $\Theta \in \Omega^1(\mathcal{J}^{-1}(\{0\})^{\mathrm{reg}}, \mathbb{R})$ with normal form near the singularities in the sense of Definition 4.2 there exists a connection $\widetilde{\Theta} \in \Omega^1(\widetilde{Z}, \mathbb{R})$ for the S^1-action on the strict transform $\widetilde{Z}$ satisfying

$$\widetilde{\Theta}|_{\beta^{-1}(\mathcal{J}^{-1}(\{0\})^{\mathrm{reg}})} = (\beta|_{\beta^{-1}(\mathcal{J}^{-1}(\{0\})^{\mathrm{reg}})})^*\Theta\,. \tag{4.13}$$

In particular, for any $\sigma \in \Omega^*_{S^1}(M)$ we have

$$\int_{\mathscr{M}_0^{reg}} e^{\omega_0}\kappa(\sigma) = \int_{\widetilde{\mathscr{M}_0}} e^{\widetilde{\omega}_0}\widetilde{\kappa}(\sigma)\,. \tag{4.14}$$

Proof. Let $F \in \mathcal{F}^0_{\mathrm{indef}}$ and equip the complex vector bundle $\nu_F : \Sigma_F \to F$ with the Hermitian norm $\frac{1}{\sqrt{2}}\|\cdot\|_F$ defined by Equation (3.11). Write $\nu_{\mathscr{S}_F} : \mathscr{S}_F \to F$ for the associated unit sphere bundle and consider the natural diffeomorphism

$$\begin{aligned} \Psi_{\mathscr{S}_F} : \mathscr{S}_F \times (0,\infty) &\xrightarrow{\sim} \Sigma_F \backslash F \\ (w,t) &\longmapsto tw\,. \end{aligned} \tag{4.15}$$

Then, the bi-ellipsoid bundle $S_F \to F$ defined in (3.44) satisfies $S_F \subset \mathscr{S}_F$. Next, using the notation (3.35), let us set

$$\Theta_{\mathscr{S}_F} := \frac{1}{2}\,\mathrm{inc}^*_{\mathscr{S}_F}(\alpha^+_F + \alpha^-_F) \in \Omega^1(\mathscr{S}_F, \mathbb{R})\,. \tag{4.16}$$

Comparing with Equations (3.37) and (3.41) one gets the identity $\Theta_{S_F} = \mathrm{inc}^*_{S_F}\Theta_{\mathscr{S}_F}$, while via the restricted diffeomorphism (3.50) one has $\Psi_F^*\Theta_{Z_F} = \mathrm{pr}^*_{S_F}\Theta_{S_F}$. On the other hand, by Definitions (3.13) and (3.14) one readily checks that (4.16) is basic for the S^1-action over $\mathscr{S}_F$ induced by multiplication with $e^{2\pi i\theta} \in \mathbb{C}^*$ for all $\theta \in \mathbb{R}/\mathbb{Z}$ if and only if the condition (4.12) is satisfied. Write $\vartheta_F \in \Omega^1(\Sigma_F\backslash F, \mathbb{R})$ for the unique form over the complement of the zero section inside Σ_F satisfying $\Psi^*_{\mathscr{S}_F}\vartheta_F = \mathrm{pr}^*_{\mathscr{S}_F}\Theta_{\mathscr{S}_F}$. Then, according to [12, Section 11] there exists a unique form $\widetilde{\vartheta}_F \in \Omega^1(\widetilde{\Sigma}_F, \mathbb{R})$ satisfying

$$\widetilde{\vartheta}_F|_{\beta^{-1}_{\widetilde{\Sigma}_F}(\Sigma_F\backslash F)} = (\beta_{\widetilde{\Sigma}_F}|_{\beta^{-1}_{\widetilde{\Sigma}_F}(\Sigma_F\backslash F)})^*\vartheta_F\,.$$

By restricting $\widetilde{\vartheta}_F$ to $\widetilde{Z}_F \subset \widetilde{\Sigma}_F$ one gets a form $\Theta_{\widetilde{Z}_F} \in \Omega^1(\widetilde{Z}_F, \mathbb{R})$ satisfying

$$\Theta_{\widetilde{Z}_F}|_{\beta^{-1}_{\widetilde{\Sigma}_F}(Z_F)} = (\beta_{\widetilde{\Sigma}_F}|_{\beta^{-1}_{\widetilde{\Sigma}_F}(Z_F)})^* \Theta_{Z_F}.$$

In view of Definition 4.2 of a connection $\Theta \in \Omega^1(\mathcal{J}^{-1}(\{0\})^{\mathrm{reg}}, \mathbb{R})$ with normal form near the singularities this concludes the proof of (4.13). Formula (4.14) is then a straightforward consequence of the Definitions 4.3 and 4.6 of the regular and resolution Kirwan maps, using the fact that $\beta_0^{-1}(\mathscr{M}_0^{\mathrm{reg}})$ is of full measure inside $\widetilde{\mathscr{M}_0}$. □

Lemma 4.7 shows that, under the combinatorial condition (4.12), the first term in (1.10) can be interpreted topologically in terms of the partial resolution $\widetilde{\mathscr{M}_0}$ of $\mathscr{M}$ introduced in Definition 4.5. We will show in Section 6.3 that this condition is actually realised on a large class of examples.

Remark 4.8. It is instructive to compare the resolution Kirwan map of Definition 4.6 with the Kirwan map considered by Jeffrey, Kiem, Kirwan, and Woolf in [17] in the case $G = S^1$. They work in the complex case, so that $\mathscr{M}_0$ is obtained as a GIT quotient by a $\mathbb{C}^*$-action of a smooth projective variety M, and work instead with the so-called *intersection cohomology* $IH^*(\mathscr{M}_0)$ of the singular quotient. Relying on the fact that the intersection cohomology naturally occurs as a summand inside the usual cohomology $H^*(\widetilde{\mathscr{M}_0})$ of the partial resolution $\widetilde{\mathscr{M}_0}$ of Definition 4.5, they consider the canonical map

$$\kappa_{IH} \colon H^*_G(M) \longrightarrow H^*(\widetilde{\mathscr{M}_0}) \longrightarrow IH^*(\mathscr{M}_0), \tag{4.17}$$

obtained by composition of the resolution Kirwan map $\widetilde{\kappa} : H^{\omega}_{S^1}(M) \longrightarrow H(\widetilde{\mathscr{M}_0})$ of Definition 4.6 with the canonical projection from $H^*(\widetilde{\mathscr{M}_0})$ onto the summand $IH^*(\mathscr{M}_0)$. Nevertheless, in the general symplectic setting, the relation between $IH^*(\mathscr{M}_0)$ and $H^*(\widetilde{\mathscr{M}_0})$ is not as clear as in the complex setting considered in [17], and there might not be a canonical choice of an intersection Kirwan map in general.

5. Asymptotic expansion of the Witten integral

We shall now introduce the Witten integral, which is our main tool in our approach to geometric quantization of singular symplectic quotients. We work in the setting of Section 3.1, so that $G = S^1$ and the identification $\mathfrak{g} \simeq \mathbb{R}$ of (2.6) is understood. Recall in particular the identification (2.7) of $\Omega^*_{S^1}(M)$ with the space of S^1-invariant differential forms with values in entire analytic series of the variable $x \in \mathbb{R}$, and the identification (2.11) of the moment map with a real-valued function $\mathcal{J} : M \to \mathbb{R}$.

Definition 5.1. For any equivariantly closed $\varrho \in \Omega^*_{S^1}(M)$ and any test function $\phi \in C^\infty_c(\mathbb{R})$, the associated *Witten integral* depending on a parameter $m \in \mathbb{N}$ is given by the formula

$$\langle \mathcal{W}_m(\varrho), \phi\rangle := \int_{\mathbb{R}} \left[\int_M e^{2\pi i m x \mathcal{J}} e^{m\omega} \varrho(x)\right] \phi(x)\, dx\,. \tag{5.1}$$

The precise form of the exponents in (5.1) is determined by our conventions in (2.10) and (2.1), as it is crucial that $\bar{\omega}(X) := \omega + 2\pi i \mathcal{J}(X) \in \Omega^*_{S^1}(M)$ is equivariantly closed. In (5.1), $\phi \in \mathrm{C}^\infty_\mathrm{c}(\mathbb{R})$ plays the role of a test function at which the distribution $\mathcal{W}_m(\varrho) \in \mathcal{D}'(\mathbb{R})$ is evaluated, and this distribution is the object that we are primarily interested in, the main goal being a description of the asymptotic behavior of $\mathcal{W}_m(\varrho)$ as $m \to \infty$.

The analytic difficulties underlying the study of the asymptotic behavior of the Witten integral have already been overcome in the previous work [6]. In that work the problem of deriving asymptotics of (5.1) as $m \to \infty$ is considered in the more general setting of studying the asymptotic behavior of so-called *generalized Witten integrals*

$$\int_\mathbb{R} \int_M e^{i\psi(p,x)/\varepsilon} a(p)\, dM(p) \phi(x)\, dx \tag{5.2}$$

as $\varepsilon \to 0^+$, with amplitudes $a \in \mathrm{C}^\infty_\mathrm{c}(M)$, $\phi \in \mathrm{C}^\infty_\mathrm{c}(\mathbb{R})$, and phase function $\psi \in \mathrm{C}^\infty(M \times \mathbb{R})$ given by $\psi(p,x) := 2\pi \mathcal{J}(p)(x)$, $dM := \omega^n/n!$ being the symplectic volume form on M. Since the critical set

$$\mathrm{Crit}(\psi) := \{(p,x) \in M \times \mathbb{R} : d\psi(p,x) = 0\} = \{(p,x) \in \mathcal{J}^{-1}(\{0\}) \times \mathbb{R} : \widetilde{X}_p = 0\}$$

is not clean unless 0 is a regular value of $\mathcal{J}$, the stationary phase principle cannot be applied.Instead, a complete asymptotic expansion for integrals of the form (5.2) was derived in [6, Theorem 1.1] via a process called *destratification*, the coefficients in the asymptotics being given by integrals over the strata of the singular symplectic quotient $\mathscr{M}_0$. From this, the existence of an expansion of (5.1), and also some rough properties of its coefficients, can be inferred. However, within that more general framework the equivariant-cohomological interpretation of the coefficients in the obtained expansion remains elusive. Therefore, we shall not use the results derived there but proceed from scratch, based on the fact that the amplitude in the Witten integral is an equivariantly closed form, which allows a simpler and concise treatment.

5.1. **Preliminaries.** We begin now with the study of the asymptotics of the Witten integral 5.1. Conceptually, we will follow the method of Meinrenken in [27, Proof of Theorem 3.3], the crucial idea being a retraction onto the zero level set of the moment map and the use of the equivariant homotopy Lemma 2.3 and equivariant Stokes' Lemma 2.2, combined with the classical stationary phase principle. But since we do not assume zero to be a regular value of the moment map, the situation is much more involved. In fact, we will combine a retraction onto the regular stratum of $\mathcal{J}^{-1}(\{0\})$ with retractions onto the several components of the singular stratum of $\mathcal{J}^{-1}(\{0\})$.

As a first step, we choose – once and for all – for each $F \in \mathcal{F}$ open sets $U_F \subset M$ and $V_F \subset \Sigma_F$ as in Proposition 3.2. For technical purposes, we choose them small enough such that the symplectic form (3.40) is actually non-degenerate on a slightly larger tubular neighborhood $\widetilde{V}_F \subset \Sigma_F$ of the zero section of Σ_F containing $\overline{V}_F$, so that Proposition 3.2 applies to a corresponding neighborhood $\widetilde{U}_F \subset M$ containing $\overline{U}_F$ with a symplectomorphism $\widetilde{\Phi}_F : \widetilde{V}_F \to \widetilde{U}_F$ extending Φ_F, while keeping the sets $\widetilde{U}_F$ obtained for the different F disjoint. This setup will be kept fixed in all of the following.

Lemma 5.2. *There exists $\delta > 0$ and an S^1-invariant open neighborhood $W \subset M \setminus M^{S^1}$ of $\mathcal{J}^{-1}(\{0\})^{\mathrm{reg}}$ together with a retraction $\mathrm{ret}_0 : W \to \mathcal{J}^{-1}(\{0\})^{\mathrm{reg}}$ and an S^1-equivariant diffeomorphism*

$$\Phi_W : W \xrightarrow{\sim} \mathcal{J}^{-1}(\{0\})^{\mathrm{reg}} \times (-\delta, \delta), \qquad p \longmapsto (\mathrm{ret}_0(p), \mathcal{J}(p)), \tag{5.3}$$

such that for all $F \in \mathcal{F}^0_{\mathrm{indef}}$ we have $\Phi_F^{-1}(U_F \cap W) \subset \Sigma_{F\bullet\bullet}$ and

$$\Phi_F^{-1} \circ (\mathrm{ret}_0|_{U_F \cap W}) = \mathrm{ret}_{Z_F} \circ \Phi_F^{-1}|_{U_F \cap W}, \tag{5.4}$$

while $W \cap \widetilde{U}_F = \emptyset$ for all $F \in \mathcal{F}^0_{\mathrm{def}}$.

Proof. Let g^{TM} be an S^1-invariant Riemannian metric on M satisfying $\Phi_F^*(g^{TM}|_{\widetilde{U}_F}) = g^{T\Sigma_F}$ in the local normal form coordinates of Proposition 3.2 for all $F \in \mathcal{F}$. Write $\mathrm{grad}\,\mathcal{J} \in \mathrm{C}^\infty(M, TM)$ for the associated Riemannian gradient of $\mathcal{J} : M \to \mathbb{R}$, and let $\xi \in \mathrm{C}^\infty(M \backslash M^{S^1}, TM)$ be the vector field

$$\xi := \frac{\mathrm{grad}\,\mathcal{J}}{\|\mathrm{grad}\,\mathcal{J}\|^2_{g^{TM}}}, \tag{5.5}$$

where we use that the Hamiltonian S^1-action is locally free on $M\backslash M^{S^1}$, so that $\mathrm{grad}\,\mathcal{J}$ is nowhere vanishing on $M\backslash M^{S^1}$. The vector field (5.5) is transverse to $\mathcal{J}^{-1}(\{0\})^{\mathrm{reg}}$, satisfies $d\mathcal{J} \lrcorner\, \xi = 1$ over $M\backslash M^{S^1}$, and in view of (3.27) is mapped to the vector field $\frac{\partial}{\partial q}$ by the diffeomorphism of Proposition 3.2 over $\widetilde{U}_F \subset M$ for each $F \in \mathcal{F}^0_{\mathrm{indef}}$. From this explicit description near each $F \in \mathcal{F}^0_{\mathrm{indef}}$ and the fact that the set $K := \mathcal{J}^{-1}(\{0\})^{\mathrm{reg}} \cap (M \setminus \bigcup_{F \in \mathcal{F}^0_{\mathrm{indef}}} U_F)$ is compact, it follows that there is a $\delta > 0$ such that the flow Φ^ξ_t of ξ is defined on $\mathcal{J}^{-1}(\{0\})^{\mathrm{reg}}$ for all $t \in (-\delta, \delta)$. Consequently, the smooth map

$$\begin{aligned} \Phi^\xi : \mathcal{J}^{-1}(\{0\})^{\mathrm{reg}} \times (-\delta, \delta) &\longrightarrow M \\ (p, t) &\longmapsto \Phi^\xi_t(p)\,, \end{aligned} \tag{5.6}$$

is a diffeomorphism onto its image satisfying $\mathcal{J}(\Phi^\xi(p,t)) = t$ for all $p \in \mathcal{J}^{-1}(\{0\})^{\mathrm{reg}}$ and $t \in (-\delta, \delta)$. Setting $W := \mathrm{im}(\Phi^\xi) \subset M$, we define the diffeomorphism (5.3) to be the inverse of (5.6). Recalling the definition of ret_{Z_F} in (3.49), all claimed properties except the last one are satisfied by construction. Finally, note that K is disjoint from $\bigcup_{F \in \mathcal{F}^0_{\mathrm{def}}} \widetilde{U}_F$ simply because $\mathcal{J}^{-1}(\{0\})^{\mathrm{reg}}$ is disjoint from $\widetilde{U}_F$ when F is definite. So it suffices to take δ small enough to get that $W \cap \widetilde{U}_F = \emptyset$ for all $F \in \mathcal{F}^0_{\mathrm{def}}$. □

The following simple consequence of the equivariant Stokes' Lemma 2.2 and the equivariant homotopy Lemma 2.3 will allow us to greatly simplify the computations of the next section.

Lemma 5.3. *The Witten integral of Definition 5.1 only depends on the equivariant cohomology class of $\varrho \in \Omega^*_{S^1}(M)$ inside $H^\omega_{S^1}(M)$. Moreover, each equivariant cohomology class in $H^\omega_{S^1}(M)$ has a representative $\varrho \in \Omega^*_{S^1}(M)$ satisfying*

$$\Phi_F^*(\varrho|_{U_F}) = (\nu^*_{\Sigma_F} \mathrm{inc}^*_F \varrho)|_{V_F} \quad \textit{for all } F \in \mathcal{F}^0 \tag{5.7}$$

and

$$\varrho|_W = \mathrm{ret}_0^* \,\mathrm{inc}_0^*\, \varrho, \tag{5.8}$$

where $\mathrm{ret}_0 : W \to \mathcal{J}^{-1}(\{0\})^{\mathrm{reg}}$ *is as in Lemma 5.2 with a suitable* $\delta > 0$ *and* $\mathrm{inc}_0 : \mathcal{J}^{-1}(\{0\})^{\mathrm{reg}} \to M$ *denotes the inclusion map.*

Proof. The first claim is standard. Indeed, let $\varrho \in \Omega^*_{S^1}(M)$ be equivariantly closed. In view of the fact that $e^{2\pi i m \mathcal{J}(x)} e^{m\omega} \in \Omega^*_{S^1}(M)$ is equivariantly closed, Lemma 2.2 implies for any equivariant form $\gamma \in \Omega^*_{S^1}(M)$ that

$$\begin{aligned}\langle \mathcal{W}_m(\varrho + d_{\mathfrak{g}}\gamma), \phi\rangle &= \int_{\mathbb{R}} \left[\int_M e^{2\pi i m x \mathcal{J}} e^{m\omega} \varrho(x) + \int_M e^{2\pi i m x \mathcal{J}} e^{m\omega} d_{\mathfrak{g}}\gamma(x)\right] \phi(x)\, dx \\ &= \int_{\mathbb{R}} \left[\int_M e^{2\pi i m x \mathcal{J}} e^{m\omega} \varrho(x) + \int_M d_{\mathfrak{g}}\big(e^{2\pi i m x \mathcal{J}} e^{m\omega} \gamma(x)\big)\right] \phi(x)\, dx \\ &= \int_{\mathbb{R}} \left[\int_M e^{2\pi i m x \mathcal{J}} e^{m\omega} \varrho(x)\right] \phi(x)\, dx = \langle \mathcal{W}_m(\varrho), \phi\rangle,\end{aligned}$$

proving the first claim of the lemma.

Let us consider now an arbitrary equivariantly closed $\widetilde{\varrho} \in \Omega^*_{S^1}(M)$. To prove the remaining assertions of the lemma, we need to construct an equivariantly closed $\varrho \in \Omega^*_{S^1}(M)$ in the cohomology class $[\widetilde{\varrho}] \in H^\omega_{S^1}(M)$ satisfying (5.7) and (5.8). Using Lemma 2.3 with $N = F$ and $U_N = \widetilde{U}_F$ we get for each $F \in \mathcal{F}^0$ an equivariant form $\gamma_F \in \Omega^*_{S^1}(\widetilde{V}_F)$ such that

$$\widetilde{\Phi}_F^*(\widetilde{\varrho}|_{\widetilde{U}_F}) = (\nu^*_{\Sigma_F} \,\mathrm{inc}^*_F\, \widetilde{\varrho})|_{\widetilde{V}_F} + d_{\mathfrak{g}}\gamma_F\,. \tag{5.9}$$

Let $\chi_F \in \mathrm{C}^\infty(M)$ have compact support in $\widetilde{U}_F$ and be identically equal to 1 on U_F. Setting

$$\gamma := \sum_{F\in\mathcal{F}^0} \chi_F (\widetilde{\Phi}_F^{-1})^* \gamma_F \in \Omega^*_{S^1}(M)\,,$$

the equivariantly closed form $\widehat{\varrho} := \widetilde{\varrho} - d_{\mathfrak{g}}\gamma \in \Omega^*_{S^1}(M)$ satisfies $\widetilde{\Phi}_F^*(\widehat{\varrho}|_{U_F}) = (\nu^*_{\Sigma_F}\, \mathrm{inc}^*_F\, \widetilde{\varrho})|_{V_F}$. But $\nu_{\Sigma_F} \circ \widetilde{\Phi}_F^{-1} \circ \mathrm{inc}_F = \mathrm{inc}_F$, so that (5.9) implies for each $F \in \mathcal{F}^0$

$$\mathrm{inc}^*_F\, d_{\mathfrak{g}}\gamma = \mathrm{inc}^*_F \widetilde{\varrho} - \mathrm{inc}^*_F (\widetilde{\Phi}_F^{-1})^* ((\nu^*_{\Sigma_F} \mathrm{inc}^*_F \widetilde{\varrho})|_{\widetilde{V}_F}) = 0,$$

yielding $\mathrm{inc}^*_F \widehat{\varrho} = \mathrm{inc}^*_F \widetilde{\varrho}$ and consequently

$$\widetilde{\Phi}_F^*(\widehat{\varrho}|_{U_F}) = (\nu^*_{\Sigma_F}\, \mathrm{inc}^*_F\, \widehat{\varrho})|_{V_F}\,. \tag{5.10}$$

Next, choose $\delta > 0$ in Lemma 5.2 such that it applies also with a slightly larger $\widehat{\delta} > \delta$ and gives us two corresponding retractions $\widehat{\mathrm{ret}}_0 : \widehat{W} \to \mathcal{J}^{-1}(\{0\})^{\mathrm{reg}}$ and $\mathrm{ret}_0 : W \to \mathcal{J}^{-1}(\{0\})^{\mathrm{reg}}$ from open subsets $\widehat{W}, W \subset M$ onto $J^{-1}(\{0\})^{\mathrm{reg}}$ such that $\widehat{\mathrm{ret}}_0|_W = \mathrm{ret}_0$. Let $\overline{W}$ be the closure of W in M. Then the compact set $\overline{W} \setminus \bigcup_{F\in\mathcal{F}^0} U_F$ lies in $M \setminus M^{S^1}$ and its intersection with $J^{-1}(\{0\})^{\mathrm{reg}}$ is compact. By making δ, and hence W, smaller while keeping $\widehat{\delta}$ fixed, we can therefore achieve that $\widehat{W}$ contains $\overline{W} \setminus \bigcup_{F\in\mathcal{F}^0} U_F$. Using Lemma 2.3 again, now with $N = \mathcal{J}^{-1}(\{0\})^{\mathrm{reg}}$ and $U_N = \widehat{W}$, there is a form $\sigma \in \Omega^*_{S^1}(\widehat{W})$ such that

$$\widehat{\varrho}|_{\widehat{W}} = \widehat{\mathrm{ret}}_0^* \,\mathrm{inc}_0^*\, \widehat{\varrho} + d_{\mathfrak{g}}\sigma \,. \tag{5.11}$$

On the other hand, for any $F \in \mathcal{F}^0_{\mathrm{indef}}$ one has $\nu_{\Sigma_F} \circ \mathrm{inc}_{Z_F} \circ \mathrm{ret}_{Z_F} = \nu_{\Sigma_F|\Sigma_{F\bullet\bullet}}$ so that from (5.10) we deduce that $\widetilde{\Phi}^*_F(\widehat{\varrho}|_{\widehat{W}\cap U_F}) = \mathrm{ret}^*_{Z_F}\, \mathrm{inc}^*_{Z_F}\, \widetilde{\Phi}^*_F(\widehat{\varrho}|_{\widehat{W}\cap U_F})$ for such an F. Thanks to Lemma 5.2 we therefore already know that

$$\widehat{\varrho}|_{\widehat{W}\cap U_F} = (\widehat{\mathrm{ret}}_0|_{\widehat{W}\cap U_F})^*\, \mathrm{inc}_0^*(\widehat{\varrho}|_{\widehat{W}\cap U_F})\,.$$

Invoking the second part of the equivariant homotopy Lemma 2.3 and recalling that $\widehat{W}$ is disjoint from $\widetilde{U}_F$ for each $F \in \mathcal{F}^0_{\mathrm{def}}$, we can thus choose σ in Equation (5.11) such that $\sigma|_{\widehat{W}\cap U_F} \equiv 0$ for all $F \in \mathcal{F}^0$. Let $\chi \in \mathrm{C}^\infty_{\mathrm{c}}(\widehat{W})$ be identically equal to 1 on $\overline{W} \setminus \bigcup_{F\in\mathcal{F}^0} U_F$. Then we can extend σ by 0 by setting $\widehat{\sigma} := \chi\sigma \in \Omega^*_{S^1}(M)$, and $\varrho := \widehat{\varrho} - d_{\mathfrak{g}}\widehat{\sigma} \in \Omega^*_{S^1}(M)$ satisfies the properties (5.7) and (5.8). Since it differs from $\widehat{\varrho} \in \Omega^*_{S^1}(M)$, and hence from $\widetilde{\varrho} \in \Omega^*_{S^1}(M)$, by an equivariantly exact form, $[\varrho] = [\widetilde{\varrho}] \in H^\omega_{S^1}(M)$, which concludes the proof. ☐

Conceptually, this lemma implies that when computing the Witten integral we can make the following assumption, which will lead to substantial simplifications later.

Assumption 5.4. *The equivariantly closed form $\varrho \in \Omega^*_{S^1}(M)$ in the Witten integral* (5.1) *satisfies the conditions* (5.7) *and* (5.8).

In order to obtain a meaningful geometric interpretation for the asymptotic expansion of the Witten integral, we will also make the following assumption.

Assumption 5.5. *The test function $\phi \in \mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R})$ in Definition 5.1 is identically equal to* 1 *in a neighborhood of* 0.

Next, we choose a $\delta > 0$ as in Lemma 5.3 and a cutoff function $\tau \in \mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R})$ with $\operatorname{supp}\tau \subset (-\delta, \delta)$ such that $\tau \equiv 1$ on $[-\delta/2, \delta/2]$. Since $e^{m\omega}$ is a polynomial in $m \in \mathbb{N}$, the non-stationary phase principle implies that as $m \to +\infty$ we have for any equivariantly closed ϱ

$$\langle \mathcal{W}_m(\varrho), \phi\rangle = \int_{\mathbb{R}} \left[\int_M e^{2\pi i m x \mathcal{J}} e^{m\omega} \varrho(x)\, \tau\circ\mathcal{J}\right] \phi(x)\, dx + \mathcal{O}(m^{-\infty})\,, \tag{5.12}$$

so that the integral localizes around the zero level set $\mathcal{J}^{-1}(\{0\}) \subset M$. Now, since M is compact, we can take δ small enough such that with the coordinates defined by the diffeomorphisms (3.42) and (3.48) introduced in Section 3.3, we can define for every $F \in \mathcal{F}^0$ cutoff functions $\chi_F \in \mathrm{C}^\infty_{\mathrm{c}}(U_F)$ and $\tau_F \in \mathrm{C}^\infty_{\mathrm{c}}(V_F)$ by the characterizing relations

$$\begin{aligned} &(\tau_F \circ \Psi_F)(p, r) = \tau(r^2/2), && (p, r) \in S_F \times (0, \infty), && \text{for } F \text{ definite},\\ &(\tau_F \circ \Psi_F)(p, r, q) = \tau(r)\tau(q), && (p, r, q) \in S_F \times \mathbb{R} \times (0, \infty), && \text{for } F \text{ indefinite},\\ &\chi_F \circ \Phi_F = \tau_F. \end{aligned} \tag{5.13}$$

Note that, as τ equals 1 near 0, the functions χ_F and τ_F are well-defined by (5.13) even though the coordinate r (in the definite case) and the coordinates r, q (in the indefinite case) are only available on the subspaces $\Sigma_{F\bullet} \subset \Sigma_F$ and $\Sigma_{F\bullet\bullet} \subset \Sigma_F$, respectively.

We then decompose the integral in (5.12) further as

$$\langle \mathcal{W}_m(\varrho), \phi\rangle = \underbrace{\int_{\mathbb{R}} \left[\int_M e^{2\pi i m x \mathcal{J}} e^{m\omega} \varrho(x) \left(\tau \circ \mathcal{J} - \sum_{F \in \mathcal{F}^0} \chi_F\right)\right] \phi(x)\, dx}_{=:\langle \mathcal{W}_m^{\mathrm{reg}}(\varrho), \phi\rangle}$$
$$+ \sum_{F \in \mathcal{F}^0} \underbrace{\int_{\mathbb{R}} \left[\int_M e^{2\pi i m x \mathcal{J}} e^{m\omega} \varrho(x)\ \chi_F\right] \phi(x)\, dx}_{=:\langle \mathcal{W}_m^F(\varrho), \phi\rangle} + \mathcal{O}(m^{-\infty}).$$

We then obtain

$$\langle \mathcal{W}_m(\varrho), \phi\rangle = \langle \mathcal{W}_m^{\mathrm{reg}}(\varrho), \phi\rangle + \sum_{F \in \mathcal{F}^0} \langle \mathcal{W}_m^F(\varrho), \phi\rangle + \mathcal{O}(m^{-\infty})\,, \tag{5.14}$$

where by pullback along Φ_F and thanks to Assumption 5.4 we have

$$\langle \mathcal{W}_m^F(\varrho), \phi\rangle = \int_{\mathbb{R}} \left[\int_{\Sigma_F} e^{2\pi i m \frac{x}{2} Q_F} e^{m\omega_{\Sigma_F}} \nu_{\Sigma_F}^* \varrho_F(x)\, \tau_F\right] \phi(x)\, dx. \tag{5.15}$$

5.2. **Definite fixed point set components.** Let us begin with the simpler definite case, so that $F \in \mathcal{F}^0_{\mathrm{def}}$. Then either $F \in \mathcal{F}^0_+$ or $F \in \mathcal{F}^0_-$, depending on whether F is positive or negative definite. Recall also that we have $\mathcal{J}^{-1}(\{0\}) \cap U_F = F$, which is fixed by the action of S^1. Pulling back the inner integral in (5.15) along the diffeomorphism Ψ_F introduced in (3.42), taking into account Diagram (3.43) and the fact that $\Sigma_{F\bullet}$ has full measure in Σ_F, and using (5.13) and (3.20) we get

$$\langle \mathcal{W}_m^F(\varrho), \phi\rangle = \int_{\mathbb{R}} \left[\int_{S_F \times (0,\infty)} \tau(r^2/2) e^{\pm 2\pi i m \frac{x}{2} r^2} e^{m \Psi_F^*(\omega_{\Sigma_F}|_{\Sigma_{F\bullet}})} \nu_{S_F \times (0,\infty)}^* \varrho_F(x)\right] \phi(x)\, dx \quad \text{if } F \in \mathcal{F}_\pm.$$

Corollary 3.6 implies that

$$\Psi_F^*(\omega_{\Sigma_F}|_{\Sigma_{F\bullet}}) = \nu_{S_F \times (0,\infty)}^* \omega_F \pm r\, dr \wedge \mathrm{pr}_{S_F}^* \Theta_{S_F} \pm \frac{r^2}{2} \mathrm{pr}_{S_F}^* d\Theta_{S_F} \quad \text{if } F \in \mathcal{F}_\pm\,, \tag{5.16}$$

so that expanding the exponential series and taking into account the fact that the first and last terms of the right-hand side of (5.16) have no dr part as they are pulled back from F

and S_F respectively, we get

(5.17)

$$\begin{aligned}\langle \mathcal{W}_m^F(\varrho),\phi\rangle &= \sum_{k=0}^{\infty}\frac{(\pm m)^{k+1}}{2^k}\int_{\mathbb{R}}\Big[\int_{S_F\times(0,\infty)}\tau(r^2/2)e^{\pm\pi i m x r^2}e^{m\nu^*_{S_F\times(0,\infty)}\omega_F}\\ &\qquad \nu^*_{S_F\times(0,\infty)}\varrho_F(x)\frac{d\Theta_{S_F}^k}{k!}\,dr\wedge\Theta_{S_F}r^{2k+1}\Big]\phi(x)\,dx\\ &= \sum_{k=0}^{\infty}\left(\frac{\pm 1}{2\pi}\right)^{k+1}\int_{\mathbb{R}}\int_0^{\infty}e^{\pm ixs}s^k\tau(\pm s/(2\pi m))\Big[\int_{S_F}e^{m\nu^*_{S_F}\omega_F}\nu^*_{S_F}\varrho_F(x)\frac{d\Theta^k_{S_F}}{k!}\Theta_{S_F}\Big]\phi(x)\,ds\,dx.\end{aligned}$$

Here we performed the change of variable $s = m\pi r^2$ in the last line. Note also that the infinite sum in (5.17) has only finitely many non-zero summands because $d\Theta^k_{S_F} = 0$ for $k > \ell_F$.

Recall now that the *Heaviside function* $H : \mathbb{R} \to \mathbb{R}$ is defined by $H(s) = 1$ for all $s \geq 0$ and $H(s) = 0$ otherwise. Using a standard formula for its Fourier transform as a tempered distribution [14, Example 7.1.17] and the fact that $\tau \in \mathrm{C}^\infty_c(\mathbb{R})$ is identically equal to 1 on $(-\delta/2, \delta/2)$, one gets for any $k \in \mathbb{N}$ and any Schwartz function $\psi \in \mathcal{S}(\mathbb{R})$ as $m \to \infty$ the equality

$$\begin{aligned}\int_0^\infty\int_{\mathbb{R}}e^{\pm isx}s^k\tau(\pm s/(2\pi m))\psi(x)\,dx\,ds &= \int_0^\infty s^k\,\hat\psi(\mp s)\;ds + \int_0^\infty s^k\left(\tau(\pm s/(2\pi m))-1\right)\hat\psi(\mp s)\,ds\\ &= (\mp 1)^k\left\langle q^k H(\mp q),\hat\psi\right\rangle + \mathcal{O}\Big(\int_{m\pi\delta}^\infty s^k|\hat\psi(\mp s)|\,ds\Big)\\ &= (\pm i)^{k+1}k!\left\langle \underline{x}_\pm^{-(k+1)},\psi\right\rangle + \mathcal{O}(m^{-\infty}),\end{aligned} \tag{5.18}$$

where we used that $\hat\psi$ is rapidly decreasing and introduced the distribution $\underline{x}_\pm^{-k}$ defined by

$$\underline{x}_\pm^{-k} := \lim_{\varepsilon\to 0^+}(x\pm i\varepsilon)^{-k}\,, \tag{5.19}$$

which satisfies the standard relation $\frac{d}{dx}\underline{x}_\pm^{-k} = -k\underline{x}_\pm^{-k-1}$ for all $k \in \mathbb{N}$ as distributions. More generally, for any absolutely converging Laurent series $P(x) = \sum_{j=-N}^{+\infty} a_j\, x^j$, we will write $P(\underline{x}_\pm)$ for the distribution defined by

$$\left\langle P(\underline{x}_\pm),\psi\right\rangle := \sum_{j=-N}^{-1}a_j\left\langle \underline{x}_\pm^j,\psi\right\rangle + \left\langle \sum_{j=0}^{\infty}a_j x^j,\psi\right\rangle. \tag{5.20}$$

We thus get from (5.17) that as $m \to \infty$

$$\begin{aligned}\langle \mathcal{W}_m^F(\varrho),\phi\rangle &= \left\langle \int_{S_F}\nu^*_{S_F}\varrho_F(x)\,e^{m\nu^*_{S_F}\omega_F}\sum_{k=0}^{\infty}\frac{(\frac{i}{2\pi}d\Theta_{S_F})^k}{\underline{x}_\pm^{k+1}}\frac{i}{2\pi}\Theta_{S_F},\phi\right\rangle + \mathcal{O}(m^{-\infty})\\ &= \left\langle \int_{S_F}e^{m\nu^*_{S_F}\omega_F}\nu^*_{S_F}\varrho_F(x)\frac{\frac{i}{2\pi}\Theta_{S_F}}{\underline{x}_\pm - \frac{i}{2\pi}d\Theta_{S_F}},\phi\right\rangle + \mathcal{O}(m^{-\infty})\quad \text{if } F\in\mathcal{F}_\pm\,.\end{aligned} \tag{5.21}$$

Remark 5.6. Adapting the proof of Duistermaat and Heckman in [9, (2.11)-(2.31)] one gets from the first line of (5.21) that

$$\langle \mathcal{W}_m^F(\varrho), \phi \rangle = \left\langle \int_F \frac{e^{m\omega_F}\varrho_F(\underline{x}_\pm)}{e_F(\underline{x}_\pm)}, \phi \right\rangle + \mathcal{O}(m^{-\infty}) \quad \text{if } F \in \mathcal{F}_\pm\,, \tag{5.22}$$

where $e_F(x) \in \Omega_T^*(F)$ is the *equivariant Euler class* of the normal bundle $\nu_{\Sigma_F} : \Sigma_F \to F$. Note that the integrand coincides with the integrand appearing in Berline-Vergne localization formula [3], and one can actually see that our method does provide an alternative proof of it. In the application to the computation of the Riemann-Roch numbers in Section 6.1, we will not use Formula (5.22) but instead provide a direct proof based on (5.21) and the Kirillov formula of Theorem 2.10.

5.3. **Indefinite fixed point set components.** Let us now deal with the indefinite case $F \in \mathcal{F}_{\text{indef}}$, so that $\ell_F^+ > 0$ and $\ell_F^- > 0$. Using the diffeomorphism Ψ_F introduced in (3.48) and the coordinates (3.27), the integral (5.15) becomes

(5.23)

$$\langle \mathcal{W}_m^F(\varrho), \phi \rangle = \int_{\mathbb{R}} \left[\int_{S_F \times (0,\infty)\times\mathbb{R}} \tau(r)\tau(q) e^{2\pi i m x q} e^{m\Psi_F^*(\omega_{\Sigma_F}|_{\Sigma_{F\bullet\bullet}})} \nu^*_{S_F\times(0,\infty)\times\mathbb{R}} \varrho_F(x) \right] \phi(x)\, dx.$$

Now, by Corollary 3.7 we know that

$$\Psi_F^*(\omega_{\Sigma_F}|_{\Sigma_{F\bullet\bullet}}) = \pi_{Z_F}^* \mathrm{inc}_{Z_F}^* \omega_{\Sigma_F} + d\big(q\, \mathrm{pr}_{S_F}^* \Theta_{S_F}\big) + d_{\mathfrak{g}} \bar{\beta}_F \tag{5.24}$$

where

$$\bar{\beta}_F := (\sqrt{r^4+q^2} - r^2)\mathrm{pr}_{S_F}^* \overline{\Theta}_{S_F} \in \Omega^1(S_F \times (0,\infty)\times \mathbb{R})^{S^1}_{\text{bas}}$$

is a basic form in the sense of (2.5), so that in particular $d\bar{\beta}_F = d_{\mathfrak{g}}\bar{\beta}_F$ by Definition 2.1. Since

$$d_{\mathfrak{g}}\bar{\beta}_F = d\bar{\beta}_F = (\sqrt{r^4+q^2} - r^2)\mathrm{pr}_{S_F}^* d\overline{\Theta}_{S_F} + \Big(\frac{2r^3\, dr + q\, dq}{\sqrt{r^4+q^2}} - 2r\, dr\Big) \wedge \mathrm{pr}_{S_F}^* \overline{\Theta}_{S_F}, \tag{5.25}$$

both $\bar{\beta}_F$ and $d_{\mathfrak{g}}\bar{\beta}_F$ have bounded extensions to the manifold with boundary $S_F \times [0,\infty)\times\mathbb{R}$. On the other hand, the summand $d\big(q\, \mathrm{pr}_{S_F}^* \Theta_{S_F}\big)$ is constant with respect to r, and the fact that the summand $\pi_{Z_F}^* \mathrm{inc}_{Z_F}^* \omega_{\Sigma_F}$ is bounded as $r \to 0^+$ follows from (3.30) in view of (3.40) and (3.39). Inserting (5.24) into (5.23) therefore yields

$$\langle \mathcal{W}_m^F(\varrho), \phi \rangle = I_1^F(m) + I_2^F(m), \tag{5.26}$$

where the integrals

$$I_1^F(m) := \int_{\mathbb{R}} \left[\int_{S_F\times(0,\infty)\times\mathbb{R}} \tau(r)\tau(q) e^{2\pi i m x q} e^{m\pi_{Z_F}^*\mathrm{inc}_{Z_F}^*\omega_{\Sigma_F} + md(q\,\mathrm{pr}_{S_F}^*\Theta_{S_F})} \nu_{S_F\times(0,\infty)\times\mathbb{R}}^* \varrho_F(x) \right] \phi(x)\,dx,$$

$$I_2^F(m) := \int_{\mathbb{R}} \left[\int_{S_F\times(0,\infty)\times\mathbb{R}} \tau(r)\tau(q) e^{2\pi i m x q} e^{m\pi_{Z_F}^*\mathrm{inc}_{Z_F}^*\omega_{\Sigma_F} + md(q\,\mathrm{pr}_{S_F}^*\Theta_{S_F})} \nu_{S_F\times(0,\infty)\times\mathbb{R}}^* \varrho_F(x) \right.$$
$$\left. \left(e^{md_{\mathfrak{g}}\bar{\beta}_F} - 1\right)\right] \phi(x)\,dx$$

are absolutely convergent.

5.3.1. *The integral* $I_1^F(m)$. Let us first turn to the integral $I_1^F(m)$. Here we closely follow the method of [27, Proof of Theorem 3.3]. Isolating the dq part in (5.24) and expanding the corresponding exponential series we get

$$I_1^F(m) = \sum_{k=0}^{\infty} \frac{m^{k+1}}{k!} \int_{\mathbb{R}} \left[\int_{S_F\times(0,\infty)\times\mathbb{R}} q^k e^{2\pi i m x q} \tau(r)\tau(q) \nu_{S_F\times(0,\infty)\times\mathbb{R}}^* \varrho_F(x) \right.$$
$$\left. e^{m\pi_{Z_F}^*\mathrm{inc}_{Z_F}^*\omega_{\Sigma_F}}\, dq\, d\Theta_{S_F}^k \Theta_{S_F} \right] \phi(x)\,dx$$
$$= \sum_{k=0}^{\infty} \frac{m^{k+1}}{k!} \int_{S_F\times(0,\infty)} \Bigg[\underbrace{\int_{\mathbb{R}^2} q^k e^{2\pi i m x q} \nu_{S_F\times(0,\infty)}^* \varrho_F(x)\tau(q)\phi(x)\,dq\,dx}_{=:\tilde{I}(m)} \Bigg] \tau(r) e^{m\Psi_F^*\mathrm{inc}_{Z_F}^*\omega_{\Sigma_F}} d\Theta_{S_F}^k \Theta_{S_F}.$$

Here we took Diagram (3.51) into account. Note that, as before, the infinite sums occurring here are actually finite because $d\Theta_{S_F}^k = 0$ for $k \in \mathbb{N}$ large enough. Moreover, the integrand of $\tilde{I}(m)$ depends on the external parameters $(p, r) \in S_F \times (0,\infty)$ only via the point $\nu_{S_F\times(0,\infty)}(p, r) = \nu_{S_F}(p) \in F$, so that $\tilde{I}(m)$ is constant with respect to $r \in (0,\infty)$.

Applying the classical stationary phase principle [14, Lemma 7.7.3] to $\tilde{I}(m)$ while taking into account Assumption 5.5, we get

$$\tilde{I}(m) = \frac{i^k}{(2\pi)^k m^{k+1}} \frac{\partial^k \nu_{S_F\times(0,\infty)}^* \varrho_F}{\partial x^k}(0) + \mathcal{O}(m^{-\infty}), \qquad m \to +\infty,$$

the estimate being uniform on $S_F \times (0,\infty)$ because S_F is compact and $\tilde{I}(m)$ is constant with respect to $r \in (0,\infty)$. This yields

$$\begin{aligned} I_1^F(m) &= \int_{S_F\times(0,\infty)} \tau(r)\, e^{m\Psi_F^*\mathrm{inc}_{Z_F}^*\omega_{\Sigma_F}} \nu_{S_F\times(0,\infty)}^* \varrho_F\Big(\frac{i}{2\pi} d\Theta_{S_F}\Big)\, \Theta_{S_F} + \mathcal{O}(m^{-\infty}) \\ &= \int_{Z_F} \tau(r)\, e^{m\,\mathrm{inc}_{Z_F}^*\omega_{\Sigma_F}} \nu_{Z_F}^* \varrho_F\Big(\frac{i}{2\pi} d\Theta_{Z_F}\Big)\, \Theta_{Z_F} + \mathcal{O}(m^{-\infty}) \\ &= \int_{\mathcal{J}^{-1}(\{0\})^{\mathrm{reg}}} \chi_F\, e^{m\,\mathrm{inc}_{\mathcal{J}^{-1}(\{0\})^{\mathrm{reg}}}^*\omega}\, \mathrm{inc}_{\mathcal{J}^{-1}(\{0\})^{\mathrm{reg}}}^* \varrho\Big(\frac{i}{2\pi} d\Theta\Big)\, \Theta + \mathcal{O}(m^{-\infty}). \end{aligned} \tag{5.27}$$

Here, in the second line, we consider r as a coordinate on Z_F using the diffeomorphism Ψ_F from (3.50). The third line is obtained by pullback along the diffeomorphism Φ_F^{-1}, taking into account Assumption 5.4, (3.7), the last line in Equation (5.13), and the fact that χ_F is supported in U_F.

5.3.2. *The integral* $I_2^F(m)$. Let us now turn to the integral $I_2^F(m)$. First note from (5.24) that in the coordinates (3.27) defined by the diffeomorphism (3.48), the equivariant form (5.28)

$$e^{2\pi imxq}e^{\pi_{Z_F}^*\mathrm{inc}_{Z_F}^*\omega_{\Sigma_F}+d(q\,\mathrm{pr}_{S_F}^*\Theta_{S_F})} = \Psi_F^*(e^{2\pi im\frac{x}{2}Q_F}e^{m\omega_{\Sigma_F}})e^{-d_{\mathfrak{g}}\bar{\beta}_F} \in \Omega_{S^1}^*(S_F\times(0,\infty)\times\mathbb{R})$$

is equivariantly closed. Note also that

$$e^{md_{\mathfrak{g}}\bar{\beta}_F}-1=\sum_{k=1}^{n}\frac{1}{k!}(md_{\mathfrak{g}}\bar{\beta}_F)^k = d_{\mathfrak{g}}\Big(\underbrace{m\sum_{k=1}^{n}\frac{m^k}{k!}\bar{\beta}_F(d_{\mathfrak{g}}\bar{\beta}_F)^{k-1}}_{=:\bar{\beta}_{m,F}}\Big) \tag{5.29}$$

is an equivariantly exact form. By (5.25) one concludes that

$$\bar{\beta}_{m,F}=m\sum_{k=1}^{n}\frac{m^k}{k!}(\sqrt{r^4+q^2}-r^2)^k(\mathrm{pr}_{S_F}^*d\overline{\Theta}_{S_F})^{k-1}\mathrm{pr}_{S_F}^*\overline{\Theta}_{S_F} \tag{5.30}$$

has a continuous extension to the manifold with boundary $S_F\times[0,\infty)\times\mathbb{R}$. Consider further for $\varepsilon\geq 0$ the manifold with boundary $S_F\times[\varepsilon,\infty)\times\mathbb{R}$, together with the canonical parametrization of its boundary given by the orientation preserving map

$$\begin{aligned} b_{F,\varepsilon}: S_F\times\mathbb{R} &\xrightarrow{\cong} S_F\times\{\varepsilon\}\times\mathbb{R}=\partial(S_F\times[\varepsilon,\infty)\times\mathbb{R}),\\ (p,q)&\longmapsto (p,\varepsilon,q)\,. \end{aligned} \tag{5.31}$$

Using Lebesgue's convergence theorem, Lemma 2.2, and the fact that the form (5.28) is equivariantly closed we get as $m\to+\infty$

$$\begin{aligned} I_2^F(m) &= \lim_{\varepsilon\to 0}\int_{\mathbb{R}}\Big[\int_{S_F\times[\varepsilon,\infty)\times\mathbb{R}}\tau(r)\tau(q)e^{2\pi imxq}e^{m\pi_{Z_F}^*\mathrm{inc}_{Z_F}^*\omega_{\Sigma_F}+md(q\,\mathrm{pr}_{S_F}^*\Theta_{S_F})}\\ &\qquad d_{\mathfrak{g}}\bar{\beta}_{m,F}\,\nu_{S_F\times(0,\infty)\times\mathbb{R}}^*\varrho_F(x)\Big]\phi(x)\,dx\\ &= \lim_{\varepsilon\to 0}\int_{\mathbb{R}}\Big[\int_{S_F\times[\varepsilon,\infty)\times\mathbb{R}}d_{\mathfrak{g}}\Big(\tau(r)\tau(q)e^{2\pi imxq}e^{m\pi_{Z_F}^*\mathrm{inc}_{Z_F}^*\omega_{\Sigma_F}+md(q\,\mathrm{pr}_{S_F}^*\Theta_{S_F})}\\ &\qquad \bar{\beta}_{m,F}\,\nu_{S_F\times(0,\infty)\times\mathbb{R}}^*\varrho_F(x)\Big)\Big]\phi(x)\,dx-R_2^F(m)\\ &= \lim_{\varepsilon\to 0}\int_{\mathbb{R}}\Big[\int_{S_F\times\mathbb{R}}b_{F,\varepsilon}^*\Big(\tau(r)\tau(q)e^{2\pi imxq}e^{m\pi_{Z_F}^*\mathrm{inc}_{Z_F}^*\omega_{\Sigma_F}+md(q\,\mathrm{pr}_{S_F}^*\Theta_{S_F})}\\ &\qquad \bar{\beta}_{m,F}\,\nu_{S_F\times\{\varepsilon\}\times\mathbb{R}}^*\varrho_F(x)\Big)\Big]\phi(x)\,dx-R_2^F(m), \end{aligned}$$

where we wrote

$$R_2^F(m) := \int_{\mathbb{R}} \int_{S_F\times(0,\infty)\times\mathbb{R}} d(\tau(r)\tau(q)) e^{2\pi i m x q} e^{m\pi_{Z_F}^* \mathrm{inc}_{Z_F}^* \omega_{\Sigma_F} + m d(q\, \mathrm{pr}_{S_F}^* \Theta_{S_F})} \bar{\beta}_{m,F}\, \nu_{S_F\times(0,\infty)\times\mathbb{R}}^* \varrho_F(x)\phi(x)\, dx. \tag{5.32}$$

Now, notice that (3.38), (3.33), and (3.32) imply that ω_{vert} is homogeneous of degree 2 with respect to scalar multiplication in the fiber, so that according to the definition (3.40) of ω_{Σ_F} one has

$$b_{F,\varepsilon}^* \pi_{Z_F}^* \mathrm{inc}_{Z_F}^* \omega_{\Sigma_F} = \varepsilon^2 \mathrm{pr}_{S_F}^* \mathrm{inc}_{S_F}^* \omega_{\mathrm{vert}} + \mathrm{pr}_{S_F}^* \nu_{S_F}^*\, \omega_F. \tag{5.33}$$

Taking into account (5.30) and expanding the exponential series we therefore see that with

$$\tilde{I}_2^F(m) := \int_{\mathbb{R}^2} e^{2\pi i m x q} \sum_{k=1}^{n} \frac{|q|^k m^{k+1}}{k!} \left[\int_{S_F} e^{m\nu_{S_F}^* \omega_F + mq d\Theta_{S_F}} (d\overline{\Theta}_{S_F})^{k-1}\, \nu_{S_F}^* \varrho_F(x)\, \Theta_{S_F}\, \overline{\Theta}_{S_F} \right] \tau(q)\phi(x)\, dx\, dq$$

we obtain

$$I_2^F(m) = \tilde{I}_2^F(m) - R_2^F(m). \tag{5.34}$$

Separating the sum over k into even and odd indices and using the *sign function* $\mathrm{sgn} : \mathbb{R} \to \mathbb{R}$ defined for all $q \in \mathbb{R}$ by $\mathrm{sgn}(q) := H(q) - H(-q)$, we rewrite $\tilde{I}_2^F(m)$ as

$$\begin{aligned}
\tilde{I}_2^F(m) &= m \int_{\mathbb{R}^2} e^{2\pi i m x q} \tau(q) \int_{S_F} e^{m\nu_{S_F}^* \omega_F}\, \nu_{S_F}^* \varrho_F(x)\, \phi(x)\, \Theta_{S_F}\, \overline{\Theta}_{S_F} \\
&\qquad e^{mq\, d\Theta_{S_F}} \Bigg[\mathrm{sgn}(q) \sum_{\substack{k=1 \\ k \text{ odd}}}^{n} \frac{q^k m^k}{k!} d\overline{\Theta}_{S_F}^{k-1} + \sum_{\substack{k=1 \\ k \text{ even}}}^{n} \frac{q^k m^k}{k!} d\overline{\Theta}_{S_F}^{k-1} \Bigg]\, dq\; dx \\
&= m \int_{\mathbb{R}^2} e^{2\pi i m x q} \tau(q) \int_{S_F} e^{m\nu_{S_F}^* \omega_F}\, \nu_{S_F}^* \varrho_F(x)\, \phi(x)\, \Theta_{S_F}\, \overline{\Theta}_{S_F} \\
&\qquad \Bigg[\mathrm{sgn}(q) \sum_{\substack{j,k=0 \\ k \text{ odd}}}^{n} \frac{q^{j+k} m^{j+k}}{j!\,k!} d\Theta_{S_F}^j d\overline{\Theta}_{S_F}^{k-1} + \frac{e^{mq\, d\Theta_{S_F}^+} + e^{mq\, d\Theta_{S_F}^-} - 2\, e^{mq\, d\Theta_{S_F}}}{2\, d\overline{\Theta}_{S_F}} \Bigg]\, dq\, dx\,,
\end{aligned} \tag{5.35}$$

where we used the fact that, by the definition of Θ_{S_F} and $\overline{\Theta}_{S_F}$ in (3.46), one has

$$\begin{aligned}
e^{mq\, d\Theta_{S_F}} \sum_{\substack{k=1 \\ k \text{ even}}}^{n} \frac{q^k m^k}{k!} d\overline{\Theta}_{S_F}^{k-1} &= \frac{e^{mq\, d\Theta_{S_F}}}{2} \left(\frac{e^{mq\, d\overline{\Theta}_{S_F}} - 1}{d\overline{\Theta}_{S_F}} + \frac{e^{-mq\, d\overline{\Theta}_{S_F}} - 1}{d\overline{\Theta}_{S_F}} \right) \\
&= \frac{e^{mq\, d\Theta_{S_F}^+} + e^{mq\, d\Theta_{S_F}^-} - 2\, e^{mq\, d\Theta_{S_F}}}{2\, d\overline{\Theta}_{S_F}}\,.
\end{aligned}$$

Expanding the exponential series and applying the classical stationary phase principle [14, Lemma 7.7.3] to the second term in the square bracket of (5.35) yields the asymptotic expansion

$$m \int_{\mathbb{R}^2} e^{2\pi i m x q} \tau(q) \nu_{S_F}^* \varrho_F(x)\, \phi(x) \frac{e^{mq\, d\Theta^+_{S_F}} + e^{mq\, d\Theta^-_{S_F}} - 2\, e^{mq\, d\Theta_{S_F}}}{2\, d\overline{\Theta}_{S_F}}\, dq\, dx$$
$$= \frac{\nu_{S_F}^* \varrho_F(\frac{i}{2\pi} d\Theta^+_{S_F}) + \nu_{S_F}^* \varrho_F(\frac{i}{2\pi} d\Theta^-_{S_F}) - 2\, \nu_{S_F}^* \varrho_F(\frac{i}{2\pi} d\Theta_{S_F})}{2\, d\overline{\Theta}_{S_F}} + \mathcal{O}(m^{-\infty}),$$

where the remainder estimate is uniform in the base point in S_F because the latter is compact. To deal with the first term in the square bracket of (5.35) we note that with the substitution $s = 2\pi m q$ and (5.18) one computes

$$\int_{\mathbb{R}^2} e^{2\pi i m x q} \tau(q)\, \nu_{S_F}^* \varrho_F(x)\, \phi(x) \mathrm{sgn}(q)\, q^{j+k}\ dq\, dx$$
$$= 2\Big(\frac{i}{2\pi m}\Big)^{j+k+1} (j+k)!\, \big\langle \underline{x}^{-(j+k+1)}, \nu_{S_F}^* \varrho_F(x)\, \phi(x) \big\rangle + \mathcal{O}(m^{-\infty}),$$

where we introduced the distributions

$$\underline{x}^{-k} := \frac{1}{2}\big(\underline{x}_+^{-k} + \underline{x}_-^{-k}\big), \qquad k \in \mathbb{N}_0. \tag{5.36}$$

We will also use the notation (5.20) with the distributions (5.36). Expanding $e^{m\nu_{S_F}^* \omega_F}$, separating powers of m, and recalling Assumption 5.5 we get from (5.35) the asymptotic expansion

(5.37)
$$\tilde{I}_2^F(m) = \int_{S_F} e^{m\nu_{S_F}^* \omega_F} \frac{\nu_{S_F}^* \varrho_F(\frac{i}{2\pi} d\Theta^+_{S_F}) + \nu_{S_F}^* \varrho_F(\frac{i}{2\pi} d\Theta^-_{S_F}) - 2\, \nu_{S_F}^* \varrho_F(\frac{i}{2\pi} d\Theta_{S_F})}{2\, d\overline{\Theta}_{S_F}} \Theta_{S_F}\, \overline{\Theta}_{S_F}$$
$$+ 2 \sum_{\substack{j,k=0 \\ k \text{ odd}}}^{n} \frac{(\frac{i}{2\pi})^{j+k+1} (j+k)!}{j!\, k!} \Big\langle \underline{x}^{-(j+k+1)}, \int_{S_F} e^{m\nu_{S_F}^* \omega_F} \nu_{S_F}^* \varrho_F(x) d\Theta^j_{S_F} (d\overline{\Theta}_{S_F})^{k-1}\, \Theta_{S_F}\, \overline{\Theta}_{S_F} \phi(x) \Big\rangle$$

up to terms of order $\mathcal{O}(m^{-\infty})$ as $m\to\infty$. Let us now simplify (5.37) with the help of the binomial theorem by writing for $x\neq 0$

$$
\begin{aligned}
2\sum_{\substack{j,k=0\\ k \text{ odd}}}^{n} &\frac{(j+k)!}{j!\,k!}\frac{(\frac{i}{2\pi})^{j+k+1}}{x^{j+k+1}} d\Theta_{S_F}^j (d\overline{\Theta}_{S_F})^{k-1}\Theta_{S_F}\,\overline{\Theta}_{S_F} \\
&= \sum_{j,k=0}^{n}\frac{(j+k)!}{j!\,k!}\frac{d\Theta_{S_F}^j(d\overline{\Theta}_{S_F})^k - d\Theta_{S_F}^j(-d\overline{\Theta}_{S_F})^k}{(-2\pi i x)^{j+k+1}\,d\overline{\Theta}_{S_F}}\Theta_{S_F}\,\overline{\Theta}_{S_F}\\
&= \sum_{r=0}^{n}\frac{(d\Theta_{S_F}+d\overline{\Theta}_{S_F})^r-(d\Theta_{S_F}-d\overline{\Theta}_{S_F})^r}{(-2\pi i x)^{r+1}\,d\overline{\Theta}_{S_F}}\Theta_{S_F}\,\overline{\Theta}_{S_F}\\
&= \frac{\Theta_{S_F}\,\overline{\Theta}_{S_F}}{d\overline{\Theta}_{S_F}}\left(\frac{1}{-2\pi i x-(d\Theta_{S_F}+d\overline{\Theta}_{S_F})}-\frac{1}{-2\pi i x-(d\Theta_{S_F}-d\overline{\Theta}_{S_F})}\right)\\
&= \frac{\frac{i}{2\pi}\Theta^+_{S_F}}{x-\frac{i}{2\pi}d\Theta^+_{S_F}}\frac{\frac{i}{2\pi}\Theta^-_{S_F}}{x-\frac{i}{2\pi}d\Theta^-_{S_F}},
\end{aligned}
$$

where in the last step we used the definitions of Θ_{S_F} and $\overline{\Theta}_{S_F}$ given in (3.46). Using the exceptional Kirwan map introduced in (1.11) we finally arrive at

(5.38)

$$
\tilde{I}_2^F(m) = \int_{S_F} e^{m\nu^*_{S_F}\omega_F}\kappa_F(\varrho_F) + \left\langle \int_{S_F} e^{m\nu^*_{S_F}\omega_F}\nu^*_{S_F}\varrho_F(\underline{x})\frac{\frac{i}{2\pi}\Theta^+_{S_F}}{\underline{x}-\frac{i}{2\pi}d\Theta^+_{S_F}}\frac{\frac{i}{2\pi}\Theta^-_{S_F}}{\underline{x}-\frac{i}{2\pi}d\Theta^-_{S_F}},\phi\right\rangle + \mathcal{O}(m^{-\infty}).
$$

This deals with the first term of the right-hand side of (5.34). Using equations (5.13) and (5.28) the second term $R_2^F(m)$ given in (5.32) is seen to be equal to

$$
R_2^F(m) = \int_{\mathbb{R}}\int_M (d\chi_F)\, e^{2\pi i m x\mathcal{J}} e^{m\omega} e^{-d_{\mathfrak{g}}\tilde{\beta}_F}\tilde{\beta}_{m,F}\,\varrho(x)\phi(x)dx\,, \tag{5.39}
$$

where the basic forms $\tilde{\beta}_F$, $\tilde{\beta}_{m,F}\in\Omega^*(U_F\backslash F)^{S^1}_{\text{bas}}$ are characterized by $\Phi_F^*\tilde{\beta}_F := (\Psi_F^{-1})^*\overline{\beta}_F$ and $\Phi_F^*\tilde{\beta}_{m,F} := (\Psi_F^{-1})^*\overline{\beta}_{m,F}$ with Φ_F restricted to $V_F\cap\Sigma_{F\bullet\bullet}$; they can be extended smoothly from $\Phi_F(V_F\cap\Sigma_{F\bullet\bullet})$ to $U_F\backslash F$ thanks to Formulas (5.3) and (5.30). Collecting everything we finally get

(5.40)

$$
I_2^F(m) = \int_{S_F} e^{m\nu^*_{S_F}\omega_F}\kappa_F(\varrho_F) + \left\langle \int_{S_F} e^{m\nu^*_{S_F}\omega_F}\nu^*_{S_F}\varrho_F(\underline{x})\frac{\frac{i}{2\pi}\Theta^+_{S_F}}{\underline{x}-\frac{i}{2\pi}d\Theta^+_{S_F}}\frac{\frac{i}{2\pi}\Theta^-_{S_F}}{\underline{x}-\frac{i}{2\pi}d\Theta^-_{S_F}},\phi\right\rangle - R_2^F(m) + \mathcal{O}(m^{-\infty})\,.
$$

5.4. **Full asymptotic expansion of the Witten integral.** Let us now combine the computations of the whole section to establish the full asymptotic expansion of the Witten integral (5.1).

Theorem 5.7. *Let ϕ be identically equal to 1 in a neighborhood of 0. Then, for any equivariantly closed form $\varrho \in \Omega^*_{S^1}(M)$ one has the asymptotic expansion*

(5.41)
$$\langle \mathcal{W}_m(\varrho), \phi\rangle = \int_{\mathscr{M}_0^{\mathrm{reg}}} e^{m\omega_0}\kappa(\varrho) + \sum_{F\in\mathcal{F}^0_{\mathrm{def}}, F\in\mathcal{F}^0_{\pm}} \left\langle \int_{S_F} e^{m\nu^*_{S_F}\omega_F}\nu^*_{S_F}\varrho_F(\underline{x}_{\pm})\frac{\frac{i}{2\pi}\Theta_{S_F}}{\underline{x}_{\pm} - \frac{i}{2\pi}d\Theta_{S_F}}, \phi\right\rangle$$
$$+ \sum_{F\in\mathcal{F}^0_{\mathrm{indef}}} \int_{S_F} e^{m\nu^*_{S_F}\omega_F}\kappa_F(\varrho_F) + \left\langle \int_{S_F} e^{m\nu^*_{S_F}\omega_F}\nu^*_{S_F}\varrho_F(\underline{x})\frac{\frac{i}{2\pi}\Theta^+_{S_F}}{\underline{x} - \frac{i}{2\pi}d\Theta^+_{S_F}}\frac{\frac{i}{2\pi}\Theta^-_{S_F}}{\underline{x} - \frac{i}{2\pi}d\Theta^-_{S_F}}, \phi\right\rangle + \mathcal{O}(m^{-\infty})$$

as $m \to +\infty$. Furthermore, each summand in (5.41) *only depends on the equivariant cohomology class $[\varrho] \in H^\omega_{S^1}(M)$.*

Proof. First, we prove (5.41) under Assumption 5.4. Having treated in the previous subsections all terms in (5.14) except $\langle \mathcal{W}^{\mathrm{reg}}_m(\varrho), \phi\rangle$, we are left with studying the latter integral. To this end, we use the S^1-equivariant diffeomorphism $\Phi_W : W \xrightarrow{\sim} \mathcal{J}^{-1}(\{0\})^{\mathrm{reg}} \times (-\delta, \delta)$ of Lemma 5.2, which provides the coordinate $q := \mathcal{J}(p)$ on W. This coordinate coincides for all $F \in \mathcal{F}^0_{\mathrm{indef}}$ with the coordinate q of (3.27) via the diffeomorphism (3.48) when restricted to $U_F \cap W$. Now, following [27, Proof of Theorem 3.3], and analogously to Equation (5.24) one has

$$\omega|_W = \mathrm{ret}_0^*\mathrm{inc}_0^*\omega + d(q\,\mathrm{ret}_0^*\Theta) + d_{\mathfrak{g}}\tilde{\beta}, \tag{5.42}$$

where $\mathrm{inc}_0 : \mathcal{J}^{-1}(\{0\})^{\mathrm{reg}} \to M$ is the inclusion, $\Theta \in \Omega^1(\mathcal{J}^{-1}(\{0\})^{\mathrm{reg}}, \mathbb{R})$ is any connection for the S^1-action on $\mathcal{J}^{-1}(\{0\})^{\mathrm{reg}}$ in the sense of (2.8), and $\tilde{\beta} \in \Omega^*(W)^{S^1}_{\mathrm{bas}}$ is a basic form in the sense of (2.5) satisfying $\mathrm{inc}_0^*\tilde{\beta} \equiv 0$. Furthermore, thanks to (5.4) and comparing with (5.24), we can choose the forms in (5.42) in such a way that for all $F \in \mathcal{F}^0_{\mathrm{indef}}$ we have

$$\Phi_F^*(\Theta|_{U_F\cap\mathcal{J}^{-1}(\{0\})^{\mathrm{reg}}}) = (\Psi_F^{-1})^*\mathrm{pr}^*_{S_F}\Theta_{S_F} \quad \text{and} \quad \Phi_F^*(\tilde{\beta}|_{U_F\cap W}) = (\Psi_F^{-1})^*\bar{\beta}_F\,,$$

with Φ_F suitably restricted. In the same way as in (5.29), let us write

$$e^{md_{\mathfrak{g}}\tilde{\beta}} - 1 = d_{\mathfrak{g}}\tilde{\beta}_m\,,$$

with $\tilde{\beta}_m \in \Omega^*(W)^{S^1}_{\mathrm{bas}}$ satisfying $\Phi_F^*(\tilde{\beta}_m|_{U_F\cap W}) = (\Psi_F^{-1})^*\bar{\beta}_{m,F}$ for all $F \in \mathcal{F}^0_{\mathrm{indef}}$. In particular, recalling the definitions of the forms appearing in (5.39), we have that $\tilde{\beta} = \tilde{\beta}_F$ and $\tilde{\beta}_m = \tilde{\beta}_{m,F}$ on $U_F \cap W$. Following [27, Proof of Theorem 3.3], which boils down to carrying over the computations from Section 5.3 to the present situation, and using Assumption 5.4, we then readily obtain

$$\begin{aligned}\langle \mathcal{W}_m^{\mathrm{reg}}(\varrho),\phi\rangle &= \int_{\mathcal{J}^{-1}(\{0\})^{\mathrm{reg}}} \Big(1-\sum_{F\in\mathcal{F}^0}\chi_F\Big)\, e^{m\,\mathrm{inc}_0^*\omega}\mathrm{inc}_0^*\varrho\Big(\frac{i}{2\pi}d\Theta\Big)\wedge\Theta \\ &\quad + \sum_{F\in\mathcal{F}^0}\int_{\mathbb{R}}\int_M (d\chi_F)\, e^{2\pi i m x\mathcal{J}} e^{m\omega} e^{-d_{\mathfrak{g}}\tilde{\beta}}\tilde{\beta}_m \varrho(x)\phi(x)dx + \mathcal{O}(m^{-\infty}) \\ &= \int_{\mathcal{J}^{-1}(\{0\})^{\mathrm{reg}}} \Big(1-\sum_{F\in\mathcal{F}^0}\chi_F\Big)\, e^{m\,\mathrm{inc}_0^*\omega}\mathrm{inc}_0^*\varrho\Big(\frac{i}{2\pi}d\Theta\Big)\wedge\Theta + R_2^F(m) + \mathcal{O}(m^{-\infty}) \qquad (5.43)\end{aligned}$$

as $m\to+\infty$, with $R_2^F(m)$ as in (5.39). Note also that the sums on the right-hand side of (5.43) can be restricted to $\mathcal{F}^0_{\mathrm{indef}}$, since for definite F one has $\chi_F = 1$ on $\mathcal{J}^{-1}(\{0\})\cap \operatorname{supp}\chi_F = F$. Therefore, when inserting (5.43) into the original expression (5.14) for $\langle\mathcal{W}_m(\varrho),\phi\rangle$ while taking into account (5.26), the first term on the right-hand side of (5.43) combines with the integrals $I_1^F(m)$ computed in (5.27) in such a way that all terms involving the cutoff functions χ_F disappear. Similarly, the remainder term $R_2^F(m)$ appears in (5.43) and in (5.40) with opposite signs and thus gets cancelled out, which leaves only the term (5.38). Summing up, we get

$$\begin{aligned}\langle \mathcal{W}_m(\varrho),\phi\rangle &= \langle \mathcal{W}_m^{\mathrm{reg}}(\varrho),\phi\rangle + \sum_{F\in\mathcal{F}^0_{\mathrm{def}}}\langle \mathcal{W}_m^F(\varrho),\phi\rangle + \sum_{F\in\mathcal{F}^0_{\mathrm{indef}}}\big(I_1^F(m)+I_2^F(m)\big) + \mathcal{O}(m^{-\infty}) \\ &= \int_{\mathcal{J}^{-1}(\{0\})^{\mathrm{reg}}} e^{m\,\mathrm{inc}_0^*\omega}\mathrm{inc}_0^*\varrho\Big(\frac{i}{2\pi}d\Theta\Big)\wedge\Theta + \sum_{F\in\mathcal{F}^0_{\mathrm{def}}}\langle \mathcal{W}_m^F(\varrho),\phi\rangle + \sum_{F\in\mathcal{F}^0_{\mathrm{indef}}}\tilde{I}_2^F(m) \\ &\quad + \mathcal{O}(m^{-\infty}) \qquad (5.44)\end{aligned}$$

as $m\to+\infty$. If we now insert the expression (5.21) for the integrals $\langle\mathcal{W}_m^F(\varrho),\phi\rangle$ with definite F and the expression (5.38) for the integrals $\tilde{I}_2^F(m)$ and apply Lemma 2.6 to the regular Kirwan map (4.3), we get the asymptotics (5.41) for any equivariantly closed $\varrho\in\Omega^*_{S^1}(M)$ satisfying Assumption 5.4.

Let us now deal with the case of a general equivariantly closed form $\varrho\in\Omega^*_{S^1}(M)$ which not necessarily satisfies Assumption 5.4. First, note that thanks to Lemma 5.3 the left-hand side of (5.41) only depends on the equivariant cohomology class of ϱ. On the other hand, by the same lemma one can write $\varrho = \tilde{\varrho} + d_{\mathfrak{g}}\beta$ with $\tilde{\varrho}\in\Omega^*_{S^1}(M)$ satisfying Assumption 5.4, so that we can apply (5.41) to $\langle\mathcal{W}_m(\varrho),\phi\rangle = \langle\mathcal{W}_m(\tilde{\varrho}),\phi\rangle$. Since the equivariant homotopy Lemma 2.3 implies that we can choose β such that $\beta_F := \mathrm{inc}_F^*\beta\equiv 0$, it follows from Lemma 4.4 that the first term on the right hand side of (5.41) remains unchanged if we replace ϱ by $\tilde{\varrho}$; as $\varrho_F = \tilde{\varrho}_F$, all other terms on the right hand side are also unchanged, and we obtain (5.41) for ϱ.

Finally note that Lemma 2.6, applied to the S^1-actions on $S_F^{\pm}$ in (1.9) with connections $\Theta^{\pm}_{S_F}$, respectively, implies that all terms on the right-hand side of (5.41), except maybe the first one, depend only on the equivariant cohomology class of ϱ. Since this is also true for the left-hand side by Lemma 5.3, this implies that the first term of the right-hand side only depends on the equivariant cohomology class of ϱ up to an error of $\mathcal{O}(m^{-\infty})$ as $m \to +\infty$. But as this term is polynomial in $m \in \mathbb{N}$, this error actually vanishes identically in $m \in \mathbb{N}$. This concludes the proof of the theorem. □

Remark 5.8. Recalling the definition of S_F from (3.44), we see that $S_F = S_F^+ \times_F S_F^-$ as a fiberwise product, where we set $S_F^\pm := P_F \times_{K_F} S_\pm^{2l^\pm+1}$. On the other hand, the 1-forms $\Theta_{S_F}^\pm$ are connections for the S^1-actions on $S_F^\pm$, respectively. Thus, adapting the proof of Duistermaat and Heckman in [9, (2.11)-(2.31)], and using the multiplicativity of the Euler class one gets from (5.44) that as $m \to +\infty$ one has

$$\begin{aligned}(5.45)\quad \langle \mathcal{W}_m(\varrho), \phi\rangle = &\int_{\mathscr{M}_0^{\mathrm{reg}}} e^{m\omega_0}\kappa(\varrho) + \sum_{F\in\mathcal{F}^0_{\mathrm{indef}}} \int_{S_F} e^{m\nu^*_{S_F}\omega_F}\kappa_F(\varrho_F) \\ &+ \sum_{F\in\mathcal{F}^0_{\mathrm{def}}, F\in\mathcal{F}^0_\pm} \left\langle \int_F \frac{e^{m\omega_F}\varrho_F(\underline{x}_\pm)}{e_F(\underline{x}_\pm)}, \phi\right\rangle + \sum_{F\in\mathcal{F}^0_{\mathrm{indef}}} \left\langle \int_F \frac{e^{m\omega_F}\varrho_F(\underline{x})}{e_F(\underline{x})}, \phi\right\rangle + \mathcal{O}(m^{-\infty}),\end{aligned}$$

where $e_F(x) \in \Omega^*_{S^1}(F)$ is the equivariant Euler class of the normal bundle $\nu_{\Sigma_F} : \Sigma_F \to F$. Note again that the integrands in the last terms coincide with the integrand appearing in Berline-Vergne localization formula [3]. As already pointed out in Remark 5.6, we will not use this fact for the computation of the Riemann-Roch numbers in the next section, but instead provide a direct proof based on (5.41) and the Kirillov formula of Theorem 2.10.

6. Invariant Riemann-Roch formula

In this section, we will proceed to the proof of our main result Theorem 1.1.

6.1. Asymptotics of Riemann-Roch numbers.

In what follows, we will relate the asymptotics as $m \to +\infty$ of the G-invariant Riemann-Roch numbers of Definition 2.9 for $G = S^1$ with the asymptotics of the Witten integral (5.1). As in Theorem 1.1, we assume that the action of S^1 is free on $\mathcal{J}^{-1}(\{0\}) \setminus M^{S^1}$. Recall the identification (2.6) sending $X \in \mathfrak{g}$ to $x \in \mathbb{R}$ and let V_0 be a small neighbourhood of $0 \in \mathfrak{g}$ such that $\exp : V_0 \to U_e := \exp(V_0) \subset S^1$ is a diffeomorphism. Take $\psi \in \mathrm{C}^\infty(S^1, \mathbb{R})$ to be identically 1 around $e \in S^1$ and compactly supported in U_e. We then have the following

Proposition 6.1. *Under the assumptions of Theorem 1.1 we have*

$$(6.1)\quad \mathrm{RR}^{S^1}(M, L^m) = \langle \mathcal{W}_m(\varrho), \phi\rangle + \sum_{F\in\mathcal{F}^0} \int_{S^1} (1-\psi(g)) \int_F \frac{\mathrm{Td}(F)\, e^{m\omega_F}}{\det{}_{\Sigma_F}(1 - g\exp(R^{\Sigma_F}/2\pi i))}\, dg + \mathcal{O}(m^{-\infty})$$

as $m \to +\infty$, where $\langle \mathcal{W}_m(\varrho), \phi\rangle$ is the Witten integral (5.1) *evaluated on $\varrho(x) := \mathrm{Td}_{\mathfrak{g}}(M, X) \in \Omega^*_{S^1}(M)$ and $\phi(x) := \psi(\exp(X)) \in \mathrm{C}^\infty_{\mathrm{c}}(V_0, \mathbb{R})$.*

Proof. Applying Theorems 2.10 and 2.11 to (2.16) we get with (2.18)

$$
\begin{aligned}
\mathrm{RR}^{S^1}(M, L^m) &= \int_{S^1} \chi^{(m)}(g)\, dg = \int_{U_e} \chi^{(m)}(g)\psi(g)\, dg + \int_{S^1} \chi^{(m)}(g)(1-\psi(g))\, dg \\
&= \underbrace{\int_{V_0}\int_M \mathrm{Td}_{\mathfrak{g}}(M,X)\ \mathrm{ch}_{\mathfrak{g}}(L^m,X)\,\psi(\exp(X))\, dX}_{=\langle \mathcal{W}_m(\varrho),\phi\rangle} \\
&\quad + \int_{S^1}\int_{M^{S^1}} \frac{\mathrm{Td}(M^{S^1})\ \mathrm{Tr}_{L^m}[g^{-1}]\exp(-mR^L/2\pi i)}{\det_{\Sigma_{M^{S^1}}}(1-g\exp(R^{\Sigma_{M^{S^1}}}/2\pi i))}(1-\psi(g))\, dg.
\end{aligned} \tag{6.2}
$$

To identify the second term of the right-hand side of (6.2) with the second term of the right-hand side of (6.1), first note that by the Kostant formula (2.12) and the identification (2.6) we have on M^{S^1}

$$\mathrm{Tr}_{L^m}[\exp(-X)] = e^{2\pi i m x \mathcal{J}}\,.$$

In addition, for each $F \in \mathcal{F}$ recall the isotypic decomposition (3.2) of its normal bundle $\nu_{\Sigma_F} : \Sigma_F \to F$ with respect to the induced S^1-action, and write $R^{\Sigma_F} = \sum_{k\in W} R^{\Sigma_F^{(k)}}$ for the splitting of the curvature of the connection ∇^{Σ_F} in this decomposition. Recall from Notation 3.1 that $\mathcal{J} : M \to \mathbb{R}$ takes a constant value on F denoted by $\mathcal{J}(F) \in \mathbb{R}$. Then, partial integration with respect to x yields with (1.2) as $m \to +\infty$

$$
\begin{aligned}
&\int_{S^1}\int_{M^{S^1}} \frac{\mathrm{Td}(M^{S^1})\ \mathrm{Tr}_{L^m}[g^{-1}]\exp(-mR^L/2\pi i)}{\det_{\Sigma_{M^{S^1}}}(1-g\exp(R^{\Sigma_{M^{S^1}}}/2\pi i))}(1-\psi(g))\, dg \\
&= \sum_{F\in\mathcal{F}} \int_{-1/2}^{1/2} e^{2\pi i m \mathcal{J}(F) x}(1-\phi(x)) \int_F \frac{\mathrm{Td}(F)\, e^{m\omega_F}}{\prod_{k\in W}\det_{\Sigma_F^{(k)}}(1-e^{2\pi i k x}\exp(R^{\Sigma_F^{(k)}}/2\pi i))}\, dx \\
&= \sum_{F\in\mathcal{F}^0} \int_{-1/2}^{1/2} (1-\phi(x)) \int_F \frac{\mathrm{Td}(F)\, e^{m\omega_F}}{\prod_{k\in W}\det_{\Sigma_F^{(k)}}(1-e^{2\pi i k x}\exp(R^{\Sigma_F^{(k)}}/2\pi i))}\, dx + \mathcal{O}(m^{-\infty}) \\
&= \sum_{F\in\mathcal{F}^0} \int_{S^1} (1-\psi(g)) \int_F \frac{\mathrm{Td}(F)\, e^{m\omega_F}}{\det_{\Sigma_F}(1-g\exp(R^{\Sigma_F}/2\pi i))}\, dg + \mathcal{O}(m^{-\infty})\,.
\end{aligned} \tag{6.3}
$$

Bringing the computations (6.2) and (6.3) together gives the result. □

6.2. **Proof of Theorem 1.1.** We are now ready to prove Theorem 1.1 combining Proposition 6.1 with the full asymptotic expansion of the Witten integral of Theorem 5.7. Let ϱ and ϕ be as in the previous proposition, and take $\tilde{\psi} \in \mathrm{C}^\infty(S^1,\mathbb{R})$ with compact support in the small neighborhood U_e of $e \in S^1$, but such that $e \notin \operatorname{supp}\tilde{\psi}$, and write $\tilde{\phi}(x) := \tilde{\psi}(\exp(X)) \in \mathrm{C}^\infty_{\mathrm{c}}(V_0,\mathbb{R})$ for the induced function. Since $\psi + \tilde{\psi} \in \mathrm{C}^\infty_{\mathrm{c}}(U_e,\mathbb{R})$ is identically 1 close to e, Formula (6.1) also holds with the cut-off function ψ replaced by $\psi+\tilde{\psi}$. Therefore, taking the difference of the two resulting formulas, whose left-hand sides do not depend on the cut-off functions, Formula (5.41) applied to $\phi+\tilde{\phi}$ and $\tilde{\phi}$ implies

$$\sum_{F\in\mathcal{F}^0}\int_{S^1}\tilde{\psi}(g)\int_F\frac{\mathrm{Td}(F)\,e^{m\omega_F}}{\det_{\Sigma_F}(1-g\exp(R^{\Sigma_F}/2\pi i))}\,dg \tag{6.4}$$
$$=\sum_{F\in\mathcal{F}^0_{\mathrm{def}}}\int_{\mathbb{R}}\int_{S_F} e^{m\nu^*_{S_F}\omega_F}\nu^*_{S_F}\varrho_F(x)\frac{\frac{i}{2\pi}\Theta_{S_F}}{x-\frac{i}{2\pi}d\Theta_{S_F}}\tilde{\phi}(x)\,dx$$
$$+\sum_{F\in\mathcal{F}^0_{\mathrm{indef}}}\int_{\mathbb{R}}\int_{S_F} e^{m\nu^*_{S_F}\omega_F}\nu^*_{S_F}\varrho_F(x)\frac{\frac{i}{2\pi}\Theta^+_{S_F}}{x-\frac{i}{2\pi}d\Theta^+_{S_F}}\frac{\frac{i}{2\pi}\Theta^-_{S_F}}{x-\frac{i}{2\pi}d\Theta^-_{S_F}}\tilde{\phi}(x)\,dx+\mathcal{O}(m^{-\infty})$$

as $m\to+\infty$, where we took into account that the singular support of the distributions appearing in (5.41) is given by $\{0\}\subset\mathbb{R}$, so that they turn into regular integrals over $\mathbb{R}$ as $0\notin\operatorname{supp}\tilde{\phi}$ by assumption. Now, as (6.4) holds for all test functions $\tilde{\psi}\in \mathrm{C}^\infty_{\mathrm{c}}(U_e\setminus\{e\},\mathbb{R})$, one deduces from this for $x\neq 0$ the identity of Laurent polynomials

$$\sum_{F\in\mathcal{F}^0}\int_F\frac{\mathrm{Td}(F)\,e^{m\omega_F}}{\prod_{k\in W}\det_{\Sigma_F^{(k)}}(1-e^{2\pi ikx}\exp(R^{\Sigma_F^{(k)}}/2\pi i))}=\sum_{F\in\mathcal{F}^0_{\mathrm{def}}}\int_{S_F} e^{m\nu^*_{S_F}\omega_F}\nu^*_{S_F}\varrho_F(x)\frac{\frac{i}{2\pi}\Theta_{S_F}}{x-\frac{i}{2\pi}d\Theta_{S_F}}$$
$$+\sum_{F\in\mathcal{F}^0_{\mathrm{indef}}}\int_{S_F} e^{m\nu^*_{S_F}\omega_F}\nu^*_{S_F}\varrho_F(x)\frac{\frac{i}{2\pi}\Theta^+_{S_F}}{x-\frac{i}{2\pi}d\Theta^+_{S_F}}\frac{\frac{i}{2\pi}\Theta^-_{S_F}}{x-\frac{i}{2\pi}d\Theta^-_{S_F}},$$

since both sides are polynomials in $m\in\mathbb{N}$, so that the error term $\mathcal{O}(m^{-\infty})$ vanishes. Comparing this with the full asymptotic expansion of the Witten integral (5.41) and using the notation introduced in (5.20), this implies

$$\langle\mathcal{W}_m(\varrho),\phi\rangle=\int_{\mathscr{M}_0^{\mathrm{reg}}}e^{m\omega_0}\kappa(\varrho)+\sum_{F\in\mathcal{F}^0_{\mathrm{def}},F\in\mathcal{F}^0_{\pm}}\left\langle\int_F\frac{\mathrm{Td}(F)\,e^{m\omega_F}}{\prod_{k\in W}\det_{\Sigma_F^{(k)}}(1-e^{2\pi ik\underline{x}_\pm}\exp(R^{\Sigma_F^{(k)}}/2\pi i))},\phi\right\rangle$$
$$+\sum_{F\in\mathcal{F}^0_{\mathrm{indef}}}\int_{S_F}e^{m\nu^*_{S_F}\omega_F}\kappa_F(\varrho_F)+\left\langle\int_F\frac{\mathrm{Td}(F)\,e^{m\omega_F}}{\prod_{k\in W}\det_{\Sigma_F^{(k)}}(1-e^{2\pi ik\underline{x}}\exp(R^{\Sigma_F^{(k)}}/2\pi i))},\phi\right\rangle+\mathcal{O}(m^{-\infty}).$$

Plugging this into Proposition 6.1 and expressing the distributions (5.19) and (5.36) as distributions on the Lie group S^1 we get

$$\mathrm{RR}^{S^1}(M,L^m)=\int_{\mathscr{M}_0^{\mathrm{reg}}}e^{m\omega_0}\kappa(\varrho)+\sum_{F\in\mathcal{F}^0_+}\lim_{\varepsilon\to0^+}\int_0^1\widetilde{\chi}_F^{(m)}(e^{2\pi ix-\varepsilon})\,dx \tag{6.5}$$
$$+\sum_{F\in\mathcal{F}_-}\lim_{\varepsilon\to0^+}\int_0^1\widetilde{\chi}_F^{(m)}(e^{2\pi ix+\varepsilon})\,dx+\sum_{F\in\mathcal{F}^0_{\mathrm{indef}}}\int_{S_F}e^{m\nu^*_{S_F}\omega_F}\kappa_F(\varrho_F)$$
$$+\frac{1}{2}\left(\lim_{\varepsilon\to0^+}\int_0^1\widetilde{\chi}_F^{(m)}(e^{2\pi ix-\varepsilon})\,dx+\lim_{\varepsilon\to0^+}\int_0^1\widetilde{\chi}^{(m)}(e^{2\pi ix+\varepsilon})\,dx\right)+\mathcal{O}(m^{-\infty}),$$

where $\widetilde{\chi}_F^{(m)}$ is the meromorphic function defined for any $F \in \mathcal{F}_0$ and $z \in \mathbb{C}$ by

$$\widetilde{\chi}_F^{(m)}(z) := \int_F \frac{\operatorname{Td}(F)\, e^{m\omega}}{\prod_{k\in W} \det_{\Sigma_F^{(k)}}(1 - z^k \exp(R^{\Sigma_F^{(k)}}/2\pi i))} .$$

Notice that the dependence on the test function ψ has cancelled out in (6.5). Now, by a result of Meinrenken in [27, Theorem 5.1], the left-hand side of (6.5) is an arithmetic poynomial in $m \in \mathbb{N}$, while the terms of the right-hand side except $\mathcal{O}(m^{-\infty})$ are polynomials in $m \in \mathbb{N}$, so that the error term in (6.5) actually vanishes identically in $m \in \mathbb{N}$.

Note that, as seen for instance from [8, (2.5)], the poles of $\widetilde{\chi}_F^{(m)}(z)$ are contained in $\{0, 1\} \subset \mathbb{C}$. Therefore, with a change of coordinates the residue theorem implies that

$$\lim_{\varepsilon\to 0^+} \int_0^1 \widetilde{\chi}_F^{(m)}(e^{2\pi i x - \varepsilon})\, dx = \frac{1}{2\pi i} \lim_{\varepsilon\to 0^+} \int_{\{|z|=e^{-\varepsilon}\}} \widetilde{\chi}_F^{(m)}(z)\, \frac{dz}{z} = \operatorname*{Res}_{z=0} \frac{\widetilde{\chi}_F^{(m)}(z)}{z} .$$

Using a change of variable $z \mapsto z^{-1}$ we get in the same way

$$\lim_{\varepsilon\to 0^+} \int_0^1 \widetilde{\chi}_F^{(m)}(e^{2\pi i x + \varepsilon})\, dx = \operatorname*{Res}_{z=0} \frac{\widetilde{\chi}_F^{(m)}(z^{-1})}{z} . \tag{6.6}$$

Inserting this in (6.5) and taking $m = 1$ then finally yields (1.10). □

Remark 6.2. Note that Formula (6.4) can also be obtained from the formula for the Witten integral given in Remark 5.8 by using the explicit Formula (2.17) for the equivariant Todd class and the explicit formula for the equivariant Euler class following for instance [2, § 8.1]. This fact is actually at the basis of the deduction of the Kirillov formula of Theorem (2.10) from the equivariant index theorem of Theorem 2.11. In the definite setting of Section 5.2, our method actually gives an alternative proof of this fact, using the theory of distributions instead of the standard methods of equivariant cohomology.

6.3. **Examples.** Let us close by illustrating Theorem 1.1 through a family of examples. Fix an m-tuple $(k_1, \cdots, k_m) \in \mathbb{Z}^m$ with $m \in \mathbb{N}$. We consider the product $M = \Pi_{j=1}^m S^2$ of m 2-spheres S^2 endowed with the symplectic form ω whose pullback to each S^2-factor is the standard volume form of volume 1. We regard M as equipped with the diagonal S^1-action such that $\phi \in \mathbb{R}/\mathbb{Z} \simeq S^1$ acts on the j-th sphere by a rotation of angle $2\pi k_j \phi$ around the z-axis of $S^2 \subset \mathbb{R}^3$. This action is Hamiltonian for the moment map

$$\mathcal{J} := \sum_{j=1}^m \pi_j^* \mathcal{J}_{k_j} - \sum_{j=1}^m k_j , \tag{6.7}$$

where for any $k \in \mathbb{Z}$, the map $\mathcal{J}_k : S^2 \to \mathbb{R}$ denotes the moment map associated with the S^1-action of weight k, which is explicitly given by $\mathcal{J}_k(x, y, z) = kz$ in the Euclidean coordinates x, y, z of $\mathbb{R}^3 \supset S^2$. The connected components of the fixed point set M^{S^1} are all isolated points, given by products of north and south poles, and the constant term subtracted in (6.7) ensures that at least the product of all north poles (where $z = 1$) is contained in $\mathcal{J}^{-1}(\{0\})$, with weights counted with multiplicity given by $(k_1, \ldots, k_m) \in \mathbb{Z}^m$. This shows in particular that any given set of weights $W \subset \mathbb{Z}$ with any multiplicities can occur. Theorem 1.1 then provides an explictly computable formula for the S^1-invariant Riemann-Roch numbers $RR(M,L)^{S^1}$, and under the combinatorial condition (4.12) the regular term can be expressed in terms of characteristic classes involving only the topology of the resolution $\widetilde{\mathscr{M}_0}$.

In the particular case of $M = S^2 \times S^2$ with weights $k_1 = -1$ and $k_2 = 1$, the partial resolution $\widetilde{\mathscr{M}_0}$ is diffeomorphic to S^2, and the quotient map $\widetilde{Z} \to \widetilde{\mathscr{M}_0}$ is a trivial S^1-principal bundle. One then checks that we recover the usual Riemann-Roch formula for the sphere S^2 as the right-hand side of (1.10), and that the second and third terms of the right-hand side of Formula (1.10) vanish, so that Formula (1.10) reads $RR(S^2 \times S^2, L^m \boxtimes L^m)^{S^1} = RR(S^2, L^m)$ for all $m \in \mathbb{N}$ and L the prequantizing line bundle of (S^2, ω), as one can easily compute explicitly.

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