International Journal of Mathematics
Vol. 34, No. 8 (2023) 2350043 (47 pages)

DOI: 10.1142/S0129167X2350043X

Quantization of symplectic fibrations and canonical metrics

Louis Ioos*
Université de Cergy-Pontoise, 95000 Cergy, France
louis.ioos@cyu.fr

Leonid Polterovich
School of Mathematical Sciences, Tel Aviv University
Ramat Aviv, Tel Aviv 69978, Israel
polterov@tauex.tau.ac.il

Received 28 September 2022
Revised 31 March 2023
Accepted 18 April 2023
Published 16 June 2023

Communicated by Xiaonan Ma

We relate Berezin–Toeplitz quantization of higher rank vector bundles to quantum-classical hybrid systems and quantization in stages of symplectic fibrations. We apply this picture to the analysis and geometry of vector bundles, including the spectral gap of the Berezin transform and the convergence rate of Donaldson's iterations toward balanced metrics on stable vector bundles. We also establish refined estimates in the scalar case to compute the rate of Donaldson's iterations toward balanced metrics on Kähler manifolds with constant scalar curvature.

Keywords: Geometric quantization; symplectic fibrations; canonical Kähler metrics; Bergman kernel.

1. Introduction

The goal of a *quantization process* is to associate a system of quantum mechanics with any given system of classical mechanics, in such a way that one recovers the laws of classical mechanics from the laws of quantum mechanics at the limit when the *Planck constant* $\hbar$ tends to 0. However, in many concrete physical situations, one also encounters *quantum-classical hybrid systems* [12], where some degrees of freedom remain quantized while the others are taken to be purely classical. The archetypical example is the celebrated *Stern–Gerlach experiment*, which describes in first approximation a classical particle with quantized spin exhibiting a highly non-classical behavior. From the point of view of quantization, such quantum-classical hybrids are naturally described as the intermediate step in the process of *quantization in stages* [15, §4.1], where instead of quantizing all degrees of freedom at

*Corresponding author.

once, one first quantizes some specific degrees of freedom, for instance the internal degrees of freedom of a classical particle, and then quantizes the remaining classical degrees of freedom of a quantum-classical hybrid.

In this paper, we consider the process of quantization in stages in the context of Berezin–Toeplitz quantization [2, 6] of a compact *quantizable* symplectic manifold (X,ω), so that the cohomology class $[\omega] \in H^2(X,\mathbb{R})$ is integral. Assuming the existence of an integrable complex structure $J \in \mathrm{End}(TX)$ compatible with ω, this implies the existence of a holomorphic Hermitian line bundle (L,h^L) *prequantizing* (X,ω), so that its *Chern curvature* $R^L \in \Omega^2(X,\mathbb{C})$ satisfies

$$\omega = \frac{\sqrt{-1}}{2\pi} R^L. \tag{1.1}$$

Given a smooth volume form $d\nu_X$ on X, one can then define the associated *Hilbert space of quantum states*

$$\mathscr{H} := (H^0(X,L), \langle\cdot,\cdot\rangle) \tag{1.2}$$

as the finite-dimensional space $H^0(X,L)$ of holomorphic sections of L endowed with the L_2-Hermitian product $\langle\cdot,\cdot\rangle$ defined for any $s_1,s_2 \in \mathscr{H}$ by

$$\langle s_1,s_2\rangle := \int_X h^L(s_1(x),s_2(x))d\nu_X(x). \tag{1.3}$$

In the context of quantization, the space of smooth functions $\mathscr{C}^\infty(X,\mathbb{R})$ over X represents classical observables, describing classical mechanics over the phase space (X,ω), while the space of Hermitian operators $\mathrm{Herm}(\mathscr{H})$ acting on $\mathscr{H}$ represents quantum observables, describing quantum mechanics over the Hilbert space of quantum states $\mathscr{H}$. The Berezin–Toeplitz quantization of $(X,p\omega)$ is the map $T : \mathscr{C}^\infty(X,\mathbb{R}) \to \mathrm{Herm}(\mathscr{H})$ defined for any $f \in \mathscr{C}^\infty(X,\mathbb{R})$ and $s \in \mathscr{H}$ by

$$T(f) = Pfs, \tag{1.4}$$

where $fs \in \mathscr{C}^\infty(X,L)$ is the multiplication of $s \in \mathscr{H} \subset \mathscr{C}^\infty(X,L)$ by f and $P : \mathscr{C}^\infty(X,L) \to \mathscr{H}$ denotes the orthogonal projection with respect to (1.3). In this context, one usually introduces a *quantum number* $p \in \mathbb{N}$, and considers the quantization of the symplectic manifold $(X,p\omega)$ for any $p \in \mathbb{N}$, which is prequantized by the pth tensor power $L^p := L^{\otimes p}$ endowed with the induced Hermitian metric. The integer $p \in \mathbb{N}$ can be thought as inversely proportional to the Planck constant $\hbar$, and the limiting regime when p tends to infinity is called the *semi-classical limit.*

Let us now consider a *symplectic fibration* $\pi : (M,\omega^M) \to (X,\omega)$, where (X,ω) is a compact quantizable symplectic manifold and M is a compact manifold endowed with a closed 2-form ω^M which restricts to a quantizable symplectic form in each fiber. Assume now that both manifolds admit integrable complex structures compatible with all the data, let ν_π be a volume form in the fibers of $\pi : M \to X$ depending smoothly on the base and let $(\mathscr{L},h^{\mathscr{L}})$ be a holomorphic Hermitian line bundle over M prequantizing the fibers of $\pi : (M, \omega^M) \to (X, \omega)$. Up to replacing ω^M by $r\omega^M$ and $\mathscr{L}$ by $\mathscr{L}^r$ for $r \in \mathbb{N}$ large enough, we can also assume that the dimension of the space of quantum states associated with $(\pi^{-1}(x), \omega^M|_{\pi^{-1}(x)})$ does not depend on $x \in X$. We then get an induced holomorphic Hermitian vector bundle (E, h^E) over (X, ω), called the *quantum-classical hybrid* associated with $\pi : (M, \omega^M) \to (X, \omega)$, whose fiber over any $x \in X$ is given by

$$E_x := H^0(\pi^{-1}(x), \mathscr{L}|_{\pi^{-1}(x)}), \tag{1.5}$$

and whose Hermitian metric h^E at $x \in X$ is given for any $s_1, s_2 \in E_x$ by

$$h^E_x(s_1, s_2) := \int_{\pi^{-1}(x)} h^{\mathscr{L}}(s_1(y), s_2(y)) d\nu_\pi(y). \tag{1.6}$$

The space $\mathscr{C}^\infty(X, \mathrm{Herm}(E))$ of Hermitian endomorphisms of E over X then represents the space of quantum observables in the fibers which remain classical along the base, and we get a natural Berezin–Toeplitz quantization map $T_\pi : \mathscr{C}^\infty(M, \mathbb{R}) \to \mathscr{C}^\infty(X, \mathrm{Herm}(E))$ in the fibers.

As explained for instance in [15, §4.1], the construction of the quantum-classical hybrid associated with a symplectic fibration is the first step of a two-step process called *quantization in stages*. As a second step, we introduce a quantum number $p \in \mathbb{N}$, a holomorphic Hermitian line bundle (L, h^L) prequantizing (X, ω) and a smooth volume form $d\nu_X$ over X, then consider the Hilbert space $\mathscr{H}_p$ of quantum states to be the space $H^0(X, E_p)$ of holomorphic sections of the tensor product

$$E_p := E \otimes L^p, \tag{1.7}$$

endowed with the L_2-Hermitian product (1.3) using instead the Hermitian metric h^{E_p} on E_p induced by h^L and h^E. Then by definition (1.5) of E, we have a natural identification

$$H^0(X, E_p) \simeq H^0(M, \mathscr{L} \otimes \pi^* L^p), \tag{1.8}$$

preserving the respective natural L_2-Hermitian products as in formula (1.3) for the smooth volume form $d\nu_M$ on M defined by

$$d\nu_M := d\nu_\pi \pi^* d\nu_X. \tag{1.9}$$

Via formula (1.1) applied to $\mathscr{L} \otimes \pi^* L^p$, the identification (1.8) thus states that the Hilbert space $\mathscr{H}_p$ of quantum states associated with (1.7) as above naturally coincides with the space of quantum states associated with the symplectic manifold $(M, \omega^M + p\omega)$, for $p \in \mathbb{N}$ big enough.

In Sec. 2, we introduce the general setting of Berezin–Toeplitz quantization as a theory of quantum measurements, following [18], then describe the process of quantization in stages in this context. We then study the basic properties of the Berezin–Toeplitz quantization map for vector bundles $T_{E_p} : \mathscr{C}^\infty(X, \mathrm{Herm}(E)) \to \mathrm{Herm}(\mathscr{H}_p)$ introduced in [23], defined as before with $\mathscr{H}_p$ the common Hilbert space (1.8) for any $p \in \mathbb{N}$.

In Sec. 3, we extend to the setting of quantization in stages our study in [18] of Berezin–Toeplitz quantization as a theory of quantum measurements. In particular, we introduce and study in Sec. 3.1 a natural notion of a Berezin transform $\mathscr{B}_{E_p}$ associated with the Berezin–Toeplitz quantization of a holomorphic Hermitian vector bundle (E, h^E) over $(X, p\omega)$. This Berezin transform is a bounded operator with discrete spectrum acting on $\mathscr{C}^\infty(X, \mathrm{Herm}(E))$, and is introduced in Definition 3.1 as the result of quantization followed by dequantization. Its spectrum thus describes the *quantum noise* introduced by Berezin–Toeplitz quantization, which can be interpreted as a global version of the *Heisenberg uncertainty principle.*

To describe our main result in this respect, consider the Riemannian metric on X defined by

$$g^{TX} := \omega(\cdot, J\cdot), \tag{1.10}$$

and recall that the induced Riemannian measure coincides with the *Liouville measure* of the symplectic manifold (X, ω), defined by

$$d\nu := \frac{\omega^n}{n!}. \tag{1.11}$$

For any holomorphic vector bundle (E, h^E), we write $\overline{\partial}$ for the holomorphic $\overline{\partial}$-operator of the holomorphic vector bundle $\mathrm{End}(E)$ of endomorphisms of E. In that case, there is a natural extension of the Laplace–Beltrami operator Δ of (X, g^{TX}), given by twice the *Kodaira Laplacian*, which acts on the smooth sections $\mathscr{C}^\infty(X, \mathrm{End}(E))$ by the formula

$$\square := 2\overline{\partial}^*\overline{\partial}, \tag{1.12}$$

where $\overline{\partial}^* : \Omega^1(X, \mathrm{End}(E)) \to \mathscr{C}^\infty(X, \mathrm{End}(E))$ denotes the formal adjoint of $\overline{\partial} : \mathscr{C}^\infty(X, \mathrm{End}(E)) \to \Omega^1(X, \mathrm{End}(E))$ with respect to the natural L_2-products on both spaces induced by h^E and g^{TX}. It has discrete positive spectrum, and we write

$$0 = \lambda_0^E \leq \lambda_1^E \leq \cdots \leq \lambda_k^E \leq \cdots \tag{1.13}$$

for the increasing sequence of its eigenvalues counted with multiplicity. On the other hand, let us write

$$1 = \gamma_{0,p}^E \geq \gamma_{1,p}^E \geq \cdots \geq \gamma_{k,p}^E \geq \cdots \geq 0 \tag{1.14}$$

for the decreasing sequence of the eigenvalues of the Berezin transform $\mathcal{B}_{E_p}$, counted with multiplicities. In the special case $E = \mathbb{C}$, the Berezin transform $\mathscr{B}_{E_p}$ reduces to the usual Berezin transform considered in [18], and the Kodaira Laplacian reduces to the usual Laplace–Beltrami operator of (X, g^{TX}). In this case, we simply remove the superscript E in the notations (1.13) and (1.14) for their eigenvalues. In Sec. 3.2, we give the proof of the following result.

Theorem 1.1. *For every integer $k \in \mathbb{N}$, we have the following asymptotic estimate as $p \to +\infty$,*

$$\gamma_{k,p}^E = 1 - \frac{1}{4\pi p}\lambda_k^E + O(p^{-2}). \tag{1.15}$$

In the special case $E = \mathbb{C}$ with g^{TX} Kähler–Einstein *and ν_X the Liouville measure, we have the following refined estimate as $p \to +\infty$,*

$$\gamma_{k,p} = 1 - \frac{\lambda_k}{4\pi p} + \frac{\lambda_k^2 + 2c\lambda_k}{32\pi^2 p^2} + O(p^{-3}). \tag{1.16}$$

This shows in particular that the Berezin transform admits a positive *spectral gap*, controlled by the first positive eigenvalue of the Kodaira Laplacian at the semiclassical limit as $p \to +\infty$. In [18, Theorem 3.1], the estimate (1.15) was established in the case $E = \mathbb{C}$. Theorem 1.1 thus extends this result in two directions, to the case of a general vector bundle E on one hand, and gives the refined estimate (1.16) in the case $E = \mathbb{C}$ but with the additional assumption that the metric Kähler–Einstein on the other. The proof makes use of the quantum-classical correspondence for the Berezin–Toeplitz quantization of vector bundles established by Ma and Marinescu in [23, 24].

In Sec. 4, we complete the study started in [18, §4; 16, 17] and of the applications of Berezin–Toeplitz quantization to *Donaldson's program* toward the search for canonical Kähler metrics in complex geometry, for the approximation of Kähler metrics of *constant scalar curvature*. In fact, Donaldson introduced in [9, 10] a finite-dimensional approximation of Kähler metrics of *constant scalar curvature*, given by a sequence of Hermitian inner products on $H^0(X, L^p)$ for all $p \in \mathbb{N}$, called *balanced products*. Writing $\mathrm{Prod}(H^0(X, L^p))$ for the symmetric space of Hermitian inner products on $H^0(X, L^p)$, balanced products are defined as the fixed points of the dynamical system $\mathscr{T}_p$ acting on $q \in \mathrm{Prod}(H^0(X, L^p))$ by the formula

$$\mathscr{T}_p(q) = \frac{\dim H^0(X, L^p)}{\mathrm{Vol}(X, \nu_q)} \int_X h_q^{\mathrm{FS}}(\cdot, \cdot) d\nu_q(x), \tag{1.17}$$

where h_q^{FS} denotes the pullback of the Fubini–Study metric induced by q with respect to the canonical *Kodaira embedding* $X \hookrightarrow \mathbb{P}(H^0(X, L^p)^*)$ and ν_q denotes the Liouville measure associated with the symplectic form ω_q defined by (1.1). In this context, Donaldson shows in [9] that these balanced metrics converge as $p \to +\infty$ toward the unique Kähler metric of constant scalar curvature, when it exists and when the holomorphic automorphism group $\mathrm{Aut}(X, L)$ of (X, J, L) is discrete. The convergence of the iterations of $\mathscr{T}_p$ has been established by Donaldson in [10] and Sano in [29, Theorem 1.2]. In Sec. 4.2, we establish and compute the exponential speed of convergence of these iterations via a refined version of Theorem 1.1 using instead the eigenvalues of the operator D acting on $f \in \mathscr{C}^\infty(X, \mathbb{R})$ by the formula

$$Df := \frac{\partial}{\partial t}\Big|_{t=0} \mathrm{scal}\Big(\omega + t\frac{\sqrt{-1}}{2\pi}\bar{\partial}\partial f\Big), \tag{1.18}$$

where ω is the Kähler form associated with the constant scalar curvature metric and the map scal sends a Kähler form to the scalar curvature of the associated Riemannian metric (1.10). As explained [31, Definition 4.3, Lemma 4.4], this operator is an elliptic self-adjoint differential operator of fourth order called the *Lichnerowicz operator*, and as such, it admits a discrete spectrum.

In Sec. 4.1, we study the extension of this program due to Wang in [33] on the approximation of *Hermite–Einstein metrics* on a holomorphic vector bundle E over (X, ω), which are the Hermitian metrics h^E on E whose the Chern curvature $R^E \in \Omega^2(X, \mathrm{End}(E))$ satisfies

$$\frac{\sqrt{-1}}{2\pi} \langle \omega, R^E \rangle_{g^{TX}} = c\mathrm{Id}_E \tag{1.19}$$

for some constant $c \in \mathbb{R}$, where $\langle \cdot, \cdot \rangle_{g^{TX}}$ denotes the pairing on $\Omega^2(X, \mathrm{End}(E))$ with values in $\mathrm{End}(E)$ induced by g^{TX}. By a fundamental result of Uhlenbeck–Yau in [32] and Donaldson in [8], a holomorphic vector bundle E is *Mumford stable* if and only if it admits a Hermite–Einstein metric h^E and if it is *simple*, meaning that it cannot be decomposed as a direct sum of holomorphic vector bundles of smaller dimension. In this context, we fix a measure ν over X and define a dynamical system $\mathscr{T}_{E_p}$ acting on a Hermitian inner product $q \in \mathrm{Prod}(H^0(X, E_p))$ by the formula

$$\mathscr{T}_{E_p}(q) := \frac{\dim H^0(X, L^p)}{\mathrm{Vol}(X, \nu)\, \mathrm{rk}(E)} \int_X h_q^{\mathrm{FS}}(\cdot, \cdot) d\nu(x), \tag{1.20}$$

where h_q^{FS} is the pullback to E of the Fubini–Study metric induced by q via the canonical *Grassmanian embedding* $X \hookrightarrow \mathbb{G}(\mathrm{rk}(E), H^0(X, E_p)^*)$. The Hermitian metric h_q^{FS} is then called *ν-balanced* if q is a fixed point of (1.20). Assuming that ν is the Liouville measure, Wang showed in [33] that ν-balanced metrics on E converge as $p \to +\infty$ to the Hermite–Einstein metric up to an explicit conformal factor, and Seyyedali established the convergence of the iterations of (1.20) in [30]. Let us point out that the fact that the volume form appearing in (1.20) is fixed makes the study of its fixed points a much simpler matter than with the original dynamical system (1.17) due to Donaldson in [10]. In Sec. 4.1, we prove the following result.

Theorem 1.2. (1) *Assume that E is Mumford stable and that ν is the Liouville measure. Then there exists $p_0 \in \mathbb{N}$ such that for any $p \geq p_0$ and $q \in \mathrm{Prod}(H^0(X, E_p))$, there exists a ν-balanced product $q_p \in \mathrm{Prod}(H^0(X, E_p))$ and constants $C_p > 0$, $\beta_p \in (0, 1)$ such that for all $r \in \mathbb{N}$, we have*

$$\mathrm{dist}(\mathscr{T}^r_{E_p}(q), q_p) \leq C_p \beta_p^r, \tag{1.21}$$

and such that as $p \to +\infty$,

$$\beta_p = 1 - \frac{\lambda_1^E}{4\pi p} + o(p^{-1}), \tag{1.22}$$

where $\lambda_1^E > 0$ *is the first positive eigenvalue of the Kodaira Laplacian* (1.12) *associated with the Hermitian metric* $e^f h^E$, *where* h^E *satisfies the Hermite–Einstein equation* (1.19) *and* $f \in \mathscr{C}^\infty(X, \mathbb{R})$ *satisfies* $\Delta f = 2\pi\big(\mathrm{scal}(\omega) - \int_X \mathrm{scal}(\omega)\frac{d\nu}{\mathrm{Vol}(X)}\big)$.
(2) *Assume now that* $E = \mathbb{C}$, *that* g^{TX} *has constant scalar curvature and that* $\mathrm{Aut}(X, L)$ *is discrete. Then* (1.21) *holds for the dynamical system* (1.17), *with constant* $\beta_p \in (0, 1)$ *satisfying the following estimate as* $p \to +\infty$,

$$\beta_p = 1 - \frac{\mu_1}{8\pi p^2} + o(p^{-2}), \tag{1.23}$$

where $\mu_1 > 0$ *is the first positive eigenvalue of the operator* (1.18).

In [18, Theorem 4.4, Remark 4.9], Part 1 of Theorem 1.2 was established in the case $E = \mathbb{C}$. Theorem 1.2 thus extends this result in two directions, to the case of a general vector bundle E on the one hand, and to the case of the original notion of balanced metrics instead of ν-balanced metrics for $E = \mathbb{C}$ on the other. While the rate of exponential convergence for ν-balanced metrics computed in [18] has been predicted by Donaldson in [10], there was in our knowledge no such predictions concerning (1.23) for the original notion of balanced metrics.

In Sec. 4.3, we establish remarkable identities relating the dynamical systems (1.20) and (1.17) to the *moment maps* for balanced embeddings introduced in [9, 33]. In particular, we show in Remarks 4.20 and 4.23 how Theorem 1.2 reduce to the lower bounds of the differential of the balanced moment maps established by Keller, Meyer and Seyyedali in [19] and Fine in [14]. The methods presented in this paper thus gives a natural interpretation of their lower bounds as the quantum noise induced by Berezin–Toeplitz quantization. Via the work of Phong and Sturm in [26], and as explained in [16], this gives a conceptual explanation for the most crucial part of the proof of the results of Donaldson in [9] and Wang in [33] on the existence of balanced metrics, replacing a delicate geometric argument by the use of the spectral gap of the Berezin transform.

Finally, in Sec. 5, we describe the Stern–Gerlach experiment as a fundamental example of a quantum-classical hybrid, and we discuss the physical interpretation of the classical-quantum correspondence for vector bundles established by Ma and Marinescu in [24, (0.3)] via the process of quantization in stages.

2. Berezin–Toeplitz Quantization in Stages

In this section, we introduce the two-step process of quantization in stages in the context of Berezin–Toeplitz quantization, then study the Berezin–Toeplitz quantization of holomorphic Hermitian vector bundles as the quantization of a quantum-classical hybrid.

In Sec. 2.1, we describe the Berezin–Toeplitz quantization of a compact symplectic manifold from the point of view of quantum measurement theory. In Sec. 2.2, we describe the process of quantization in stages in this context, and in Sec. 2.3, we describe the classical-quantum correspondence for vector bundles established by Ma and Marinescu in [24, (0.3)].

2.1. *Berezin–Toeplitz quantization*

Let (X, ω) be a *quantizable* compact symplectic manifold of dimension $2n$, meaning that the cohomology class $[\omega] \in \Omega^2(X, \mathbb{R})$ is integral. This is equivalent with the existence of a Hermitian line bundle with Hermitian connection (L, h, ∇^L) *prequantizing* (X, ω), so that the curvature $R^L \in \Omega^2(X, \mathbb{C})$ of ∇^L satisfies the *prequantization condition* (1.1). We will always assume that (X, ω) admits a compatible integrable complex structure $J \in \mathrm{End}(TX)$, making (X, J, ω) into a *Kähler manifold* and (L, h, ∇^L) into a holomorphic Hermitian line bundle equipped with its *Chern connection*. For any $p \in \mathbb{N}^*$, the symplectic manifold $(X, p\omega)$ is again Kähler for the same complex structure, and is prequantized by the pth tensor power $L^p := L^{\otimes p}$ with induced Hermitian metric h^{L^p} and connection ∇^{L^p}.

Let us fix a smooth volume form $d\nu_X$ on X, and recall that the space of *quantum states* associated with $(X, p\omega)$ is the finite-dimensional Hilbert space $\mathscr{H}_p$ of holomorphic sections of L^p endowed with the L_2-Hermitian product $\langle \cdot, \cdot \rangle_p$ defined as in (1.3).

From the point of view of quantum measurements, a quantization process is described through the following basic notion of *Positive Operator Valued Measure* (POVM), where $\mathcal{B}(X)$ denotes the σ-algebra of Borel sets of X.

Definition 2.1. A *POVM* acting on $\mathscr{H}_p$ over X is a σ-additive map

$$W_p : \mathcal{B}(X) \to \mathrm{Herm}(\mathscr{H}_p) \tag{2.1}$$

with values in positive operators, satisfying $W_p(\emptyset) = 0$ and $W_p(X) = \mathrm{Id}_{\mathscr{H}_p}$.

In the context of Berezin–Toeplitz quantization, the fundamental tool used to relate quantum states to classical ones is the following *evaluation map* at $x \in X$,

$$\begin{aligned} \mathrm{ev}_x : \mathscr{H}_p &\to L^p_x \\ s &\mapsto s(x). \end{aligned} \tag{2.2}$$

We write $\mathrm{ev}^*_x : L^p_x \to \mathscr{H}_p$ for its dual with respect to the Hermitian metric h^{L^p} on L^p_x and to the L_2-Hermitian product $\langle \cdot, \cdot \rangle_p$ on $\mathscr{H}_p$.

Proposition 2.2. *The* $\mathrm{Herm}(\mathscr{H}_p)$*-operator valued measure defined for all* $x \in X$ *by*

$$dW_p(x) := \mathrm{ev}^*_x \, \mathrm{ev}_x \, d\nu_X(x) \tag{2.3}$$

induces by integration a map $W_p : \mathcal{B}(X) \to \mathrm{Herm}(\mathscr{H}_p)$ *satisfying the axioms of Definition* 2.1.

Proof. The operator valued measure (2.3) induces by integration a σ-additive map $W_p : \mathcal{B}(X) \to \mathrm{Herm}(\mathscr{H}_p)$ satisfying $W_p(\emptyset) = 0$. Furthermore, for any $U \in \mathcal{B}(X)$

and $s_1, s_2 \in \mathscr{H}_p$, formula (2.2) implies

$$\langle W_p(U)s_1, s_2\rangle_p := \int_U \langle \operatorname{ev}_x^* \operatorname{ev}_x s_1, s_2\rangle_p d\nu_X(x) = \int_U h^p(s_1(x), s_2(x)) d\nu_X(x). \tag{2.4}$$

Taking $s_1 = s_2$ shows that $W_p(U)$ is a positive operator, and taking $U = X$ shows that $W_p(X) = \operatorname{Id}_{\mathscr{H}_p}$. □

The POVM induced by formula (2.3) is called the Berezin–Toeplitz POVM of $(X, p\omega)$. By the classical *Kodaira vanishing theorem*, for all $p \in \mathbb{N}$ big enough and all $x \in X$, the evaluation map (2.2) is surjective, so that its dual is injective. Thus there exists a unique rank-1 projector $\Pi_p(x) \in \operatorname{Herm}(\mathscr{H}_p)$, called the *coherent state projector* at $x \in X$, and a unique positive function $\rho_p \in \mathscr{C}^\infty(X, \mathbb{R})$, called the *Rawnsley function*, such that (2.3) takes the form

$$dW_p(x) = \Pi_p(x)\rho_p(x) d\nu_X(x). \tag{2.5}$$

The following proposition shows that we recover the definition [18, §3.1, Remark 3.12] of a (weighted) Berezin–Toeplitz POVM.

Proposition 2.3. *For any $x \in X$, the coherent state projector $\Pi_p(x) \in \operatorname{Herm}(\mathscr{H}_p)$ is the unique orthogonal projector satisfying*

$$\operatorname{Ker} \Pi_p(x) = \{s \in \mathscr{H}_p \mid s(x) = 0\}, \tag{2.6}$$

and the Rawnsley function at $x \in X$ satisfies

$$\rho_p(x) = \operatorname{ev}_x \operatorname{ev}_x^* \in \operatorname{Herm}(L^p) \simeq \mathbb{R}. \tag{2.7}$$

Proof. By definition, the coherent state projector at $x \in X$ is the orthogonal projection on the image of $\operatorname{ev}_x^* : L_x^p \to \mathscr{H}_p$. Now by definition of the evaluation map (2.2), an element $s \in \mathscr{H}_p$ is orthogonal to the image of $\operatorname{ev}_x^* : L_x^p \to \mathscr{H}_p$ if and only if for all $v \in L_x^p$, we have

$$0 = \langle \operatorname{ev}_x^* \cdot v, s\rangle_p = h^p(v, s(x)), \tag{2.8}$$

that is, if and only if $s(x) = 0$. As orthogonal projectors are characterized by their kernels, this completes the proof of formula (2.6).

To establish the identity (2.7), note that (2.6) implies that $\operatorname{ev}_x \Pi_p(x) = \operatorname{ev}_x$, and (2.5) that $\rho_p(x)\Pi_p(x) = \operatorname{ev}_x^* \operatorname{ev}_x$. Thus for any $s \in \mathscr{H}_p$, we get

$$(\operatorname{ev}_x \operatorname{ev}_x^*)s(x) = \operatorname{ev}_x \operatorname{ev}_x^* \operatorname{ev}_x s = \rho_p(x) \operatorname{ev}_x \Pi_p(x)s = \rho_p(x)s(x), \tag{2.9}$$

which proves formula (2.7). □

A POVM with a density of the form (2.5) is called a *rank*-1 *POVM*, and the associated coherent state projector $\Pi_p(x) \in \operatorname{Herm}(\mathscr{H}_p)$ at $x \in X$ is then interpreted as the quantization of a classical particle at $x \in X$, where quantum states are seen as positive rank-1 operators acting on $\mathscr{H}_p$. This induces the natural notion of a *coherent state quantization*, from which we recover the following Berezin–Toeplitz quantization of classical observables.

Definition 2.4. The Berezin–Toeplitz quantization of $(X, p\omega)$ is the linear map defined by

$$\begin{aligned} T_p : \mathscr{C}^\infty(X, \mathbb{R}) &\to \mathrm{Herm}(\mathscr{H}_p) \\ f &\mapsto \int_X f(x) dW_p(x). \end{aligned} \tag{2.10}$$

The argument of Eq. (2.4) shows that Definition 2.4 coincides with the usual definition (1.4).

2.2. *Quantization in stages*

In this section, we introduce the concept of quantization in stages, and show how it can be described by the Berezin–Toeplitz quantization of vector bundles introduced in [23], which we interpret as the quantization of quantum-classical hybrids.

Definition 2.5. Let $\pi : M \to X$ be a submersion between compact manifolds. A Hermitian line bundle with Hermitian connection $(\mathscr{L}, h^{\mathscr{L}}, \nabla^{\mathscr{L}})$ over M is said to *prequantize* the fibration if the 2-form $\omega^M \in \Omega^2(X, \mathbb{R})$ defined by the formula

$$\omega^M := \frac{\sqrt{-1}}{2\pi} R^{\mathscr{L}}, \tag{2.11}$$

where $R^{\mathscr{L}}$ is the curvature of $\nabla^{\mathscr{L}}$, restricts to a symplectic form on the fibers of $\pi : M \to X$.

Let $\pi : (M, \omega^M) \to (X, \omega)$ be a prequantized fibration with base a compact prequantized symplectic manifold (X, ω), and assume that both X and M admit integrable complex structures compatible with ω and the restriction of ω_M to the fibers, and making $\pi : M \to X$ holomorphic. This endows $\pi : (M, \omega^M) \to (X, \omega)$ with the structure of a *Kähler fibration* in the sense [5, Definition 1.4]. Furthermore, this makes $(\mathscr{L}, h^{\mathscr{L}}, \nabla^{\mathscr{L}})$ into a holomorphic Hermitian line bundle equipped with its Chern connection.

From now on, we fix smooth volume forms $d\nu_M$ and $d\nu_X$ over M and X, respectively, and assume that the higher cohomology groups of $\mathscr{L}$ in the fibers satisfy $H^j(\pi^{-1}(x), \mathscr{L}|_{\pi^{-1}(x)}) = \{0\}$, for all $j > 0$ and $x \in X$. The Riemann–Roch–Hirzebruch formula then implies that $\dim H^0(\pi^{-1}(x), \mathscr{L}|_{\pi^{-1}(x)})$ does not depend on $x \in X$. Thanks to the *Kodaira vanishing theorem*, this can always be achieved replacing ω^M by $r\omega^M$ for $r \in \mathbb{N}$ large enough in Definition 2.11, so that $\mathscr{L}$ is replaced by $\mathscr{L}^r$. This assumption allows us to make the following definition, which is at the basis of the concept of quantization in stages.

Definition 2.6. The *quantum-classical hybrid* associated with the prequantized Kähler fibration $\pi : (M, \omega^M) \to (X, \omega)$ is the holomorphic vector bundle E over X whose fiber at any $x \in X$ is the space (1.5) of holomorphic sections of $\mathscr{L}$ over $\pi^{-1}(x)$, endowed with the L^2-Hermitian product (1.6) induced by $h^{\mathscr{L}}$ and the smooth volume form $d\nu_\pi$ defined over any fiber of $\pi : M \to X$ via formula (1.9).

The holomorphic structure of E is defined through its holomorphic sections over any open set $U \subset X$ as the holomorphic sections of $\mathscr{L}$ over $\pi^{-1}(U)$, so that E coincides as a sheaf with the direct image of $\mathscr{L}$ by $\pi : M \to X$.

Following Definition 2.5, let us now consider $p \in \mathbb{N}$ big enough so that the 2-form $\omega^M + p\pi^*\omega \in \Omega^2(M, \mathbb{R})$ is non-degenerate. This endows M with a symplectic structure, prequantized by the line bundle $\mathscr{L} \otimes \pi^* L^p$ with induced metric and connection. Then the identification (1.8) between the Hilbert space of quantum states $\mathscr{H}_p$ associated with $(M, \omega_M + p\pi^*\omega)$ and the space of holomorphic sections of $E_p := E \otimes L^p$ follows from Definition 2.6, since holomorphic sections of $E_p := E \otimes L^p$ are precisely the holomorphic sections of the fiber depending holomorphically from the base. The L_2-Hermitian product (1.3) then satisfies

$$\langle s_1, s_2 \rangle_p = \int_X h^{E_p}(s_1(x), s_2(x)) d\nu_X(x), \tag{2.12}$$

for any $s_1, s_2 \in \mathscr{H}_p$ seen as holomorphic sections of E_p, where h^{E_p} denotes the Hermitian product on E_p induced by h^L and h^E. Hence the quantization of $(M, \omega^M + p\pi^*\omega)$ can be obtained as a two-step process, called *quantization in stages*, by first considering the holomorphic Hermitian vector bundle (E, h^E) induced by the space of quantum states of the fibers of $\pi : (M, \omega^M) \to (X, \omega)$, and then taking the quantization of the vector bundle (E, h^E) over $(X, p\omega)$ to be the space of holomorphic sections of E_p as above. Note that in the limiting regime when p tends to infinity, the horizontal form $\pi^*\omega$ dominates, and the situation is then essentially different from the previous section, as the 2-form $\pi^*\omega$ is degenerate along the fibers of $\pi : M \to X$. This regime is also called the *weak coupling limit* in [15, §4.5], referring to the fact that the symplectic form of the fibers becomes comparatively small.

Let us now extend the identification (1.8) of the spaces of quantum states to a natural identification of the respective Berezin–Toeplitz quantizations. Proceeding by analogies, let us consider for any $x \in X$ the evaluation map

$$\begin{aligned} \mathrm{ev}_{E_x} : \mathscr{H}_p &\to E_{p,x} \\ s &\mapsto s(x), \end{aligned} \tag{2.13}$$

and write $\mathrm{ev}^*_{E_x} : E_{p,x} \to \mathscr{H}_p$ for its dual with respect to h^{E_p} and (2.12).

Proposition 2.7. *The map* $W_{E_p} : \mathcal{B}(X) \to \mathrm{Herm}(\mathscr{H}_p)$ *induced by the* $\mathrm{Herm}(\mathscr{H}_p)$-*operator valued measure defined for all* $x \in X$ *by*

$$dW_{E_p}(x) := \mathrm{ev}^*_{E_x} \mathrm{ev}_{E_x} d\nu_X(x), \tag{2.14}$$

defines a POVM in the sense of Definition 2.1.

Proof. The proof is strictly analogous to the proof of Proposition 2.2. □

For any Hermitian vector bundle (E, h^E), we will write $\mathrm{Herm}(E)$ for the bundle of Hermitian endomorphisms of E over X. We will freely use the natural identification

$$\mathrm{Herm}(E) \simeq \mathrm{Herm}(E_p). \tag{2.15}$$

The following definition generalizes Definition 2.4.

Definition 2.8. The Berezin–Toeplitz quantization of (E, h^E) over $(X, p\omega)$ is the map

$$\begin{aligned} T_{E_p} : \mathscr{C}^\infty(X, \mathrm{Herm}(E)) &\to \mathrm{Herm}(\mathscr{H}_p) \\ F &\mapsto \int_X \mathrm{ev}_{E_x}^* \, F(x) \, \mathrm{ev}_{E_x} \, d\nu_X(x). \end{aligned} \tag{2.16}$$

We now have the following basic functoriality result.

Proposition 2.9. *For any $p \in \mathbb{N}$ big enough, the Berezin–Toeplitz quantization map $T_{E_p} : \mathscr{C}^\infty(X, \mathrm{Herm}(E)) \to \mathrm{Herm}(\mathscr{H}_p)$ satisfies the formula*

$$T_p = T_{E_p} \circ T_\pi, \tag{2.17}$$

where $T_\pi : \mathscr{C}^\infty(M, \mathbb{R}) \to \mathscr{C}^\infty(X, \mathrm{Herm}(E))$ is the Berezin–Toeplitz quantization of the fibers of $\pi : (M, \omega^M) \to (X, \omega)$, and where $T_p : \mathscr{C}^\infty(M, \mathbb{R}) \to \mathrm{Herm}(\mathscr{H}_p)$ is the Berezin–Toeplitz quantization of $(M, \omega^M + p\pi^*\omega)$ for the measure (1.9).

Proof. Recall from Definition 2.6 that for any $x \in X$, the fiber $E_{p,x}$ is naturally identified with the space of holomorphic sections of $\mathscr{L} \otimes \pi^* L^p$ restricted to $\pi^{-1}(x)$. Then for any $y \in \pi^{-1}(x)$, consider the evaluation maps

$$\begin{aligned} \mathrm{ev}_y^M &: \mathscr{H}_p \to (\mathscr{L} \otimes \pi^* L^p)_y \quad \text{and} \\ \mathrm{ev}_y^{\pi^{-1}(x)} &: E_{p,x} \to (\mathscr{L} \otimes \pi^* L^p)_y, \end{aligned} \tag{2.18}$$

defined by formula (2.2) over M and $\pi^{-1}(x)$, respectively. Then by definition, for all $y \in \pi^{-1}(x)$ we get

$$\mathrm{ev}_y^{\pi^{-1}(x)} \, \mathrm{ev}_{E_x} = \mathrm{ev}_y^M, \tag{2.19}$$

and for any $f \in \mathscr{C}^\infty(M, \mathbb{R})$, the Berezin–Toeplitz quantization of $(M, \omega^M + p\pi^\omega)$ as in Definition 2.4 satisfies

$$\begin{aligned} T_p(f) &= \int_M f(y) (\mathrm{ev}_y^M)^* \, \mathrm{ev}_y^M \, d\nu_M(x) \\ &= \int_{x \in X} \int_{y \in \pi^{-1}(x)} f(y) \, \mathrm{ev}_{E_x}^* (\mathrm{ev}_y^{\pi^{-1}(x)})^* \, \mathrm{ev}_y^{\pi^{-1}(x)} \, \mathrm{ev}_{E_x} \, d\nu_\pi(y) d\nu_X(x) \end{aligned}$$

$$= \int_{x\in X} \mathrm{ev}_{E_x}^* \left(\int_{y\in\pi^{-1}(x)} f(y) (\mathrm{ev}_y^{\pi^{-1}(x)})^* \, \mathrm{ev}_y^{\pi^{-1}(x)} \, d\nu_\pi(y) \right) \mathrm{ev}_{E_x} \, d\nu_X(x)$$

$$= \int_X \mathrm{ev}_{E_x}^* \, T_\pi(f) \, \mathrm{ev}_{E_x} \, d\nu_X(x) = T_{E_p}(T_\pi(f)). \tag{2.20}$$

This gives the result. □

The following definition is set by analogy with formula (2.7) in the second part of Proposition 2.3.

Definition 2.10. The *Rawnsley section* $\rho_{E_p} \in \mathscr{C}^\infty(X, \mathrm{Herm}(E))$ is defined for any $x \in X$ by the formula

$$\rho_{E_p}(x) = \mathrm{ev}_{E_x} \, \mathrm{ev}_{E_x}^* \in \mathrm{End}(E_x). \tag{2.21}$$

On the other hand, consider the L_2-scalar product defined on $F_1, F_2 \in \mathscr{C}^\infty(X, \mathrm{Herm}(E))$ by the formula

$$\langle F_1, F_2 \rangle_{W_{E_p}} := \int_X \mathrm{Tr}^E[F_1(x) \rho_{E_p}(x) F_2(x)] d\nu(x). \tag{2.22}$$

Definition 2.11. The *Berezin symbol* is the map

$$T_{E_p}^* : \mathrm{Herm}(\mathscr{H}_p) \to \mathscr{C}^\infty(X, \mathrm{Herm}(E)), \tag{2.23}$$

dual to the Berezin–Toeplitz quantization map of Definition 2.8 with respect to the scalar product (2.22).

The Berezin symbol satisfies the following functoriality property in the setting of quantization in stages, which can be seen as an appropriate analogue of Proposition 2.9.

Proposition 2.12. *In the setting and notations of Proposition* 2.9, *the Berezin symbol map of Definition* 2.11 *satisfies the following formula, for any* $A \in \mathrm{Herm}(\mathscr{H}_p)$ *and* $x \in M$,

$$T_p^*(A) = \mathrm{Tr}^{E_p}[(T_{E_p}^*(A) \circ \pi) \Pi_\pi], \tag{2.24}$$

where $\Pi_\pi : \mathscr{C}^\infty(M, \mathbb{R}) \to \mathscr{C}^\infty(X, \mathrm{Herm}(E))$ *is the coherent state projector of Proposition* 2.3 *associated with the Berezin–Toeplitz quantization of the fibers of* $\pi : (M, \omega^M) \to (X, \omega)$.

Proof. Using Proposition 2.9, it suffices to show that the map

$$\begin{aligned} \mathscr{C}^\infty(X, \mathrm{Herm}(E)) &\to \mathscr{C}^\infty(M, \mathbb{R}) \\ F &\mapsto \mathrm{Tr}^E[(F \circ \pi) \Pi_\pi], \end{aligned} \tag{2.25}$$

is dual to T_π with respect to the L_2-scalar product (2.22) associated with $(M, \omega^M + p\pi^*\omega)$ for $E = \mathbb{C}$ and the L^2-scalar product (2.22) associated with (E, h^E) over $(X, p\omega)$.

Using formula (2.19) for $x \in X$ and $y \in \pi^{-1}(x)$, we see that the Rawnsley function $\rho_p \in \mathscr{C}^\infty(M,\mathbb{R})$ associated with the Berezin–Toeplitz quantization of $(M, \omega^M + p\pi^*\omega)$ satisfies

$$\begin{aligned}\rho_p(y) &= \mathrm{ev}_y^{\pi^{-1}(x)}\,\rho_{E_p}(x)(\mathrm{ev}_y^{\pi^{-1}(x)})^* \\ &= \mathrm{Tr}^E[(\mathrm{ev}_y^{\pi^{-1}(x)})^*\,\mathrm{ev}_y^{\pi^{-1}(x)}\,\rho_{E_p}(x)]. \end{aligned} \tag{2.26}$$

Then for any $F \in \mathscr{C}^\infty(X, \mathrm{Herm}(E_p))$, recalling that $\Pi_\pi(y)$ is a projection on the 1-dimensional image of $\mathrm{ev}_y^{\pi^{-1}(x)}$, we can write

$$\mathrm{Tr}^E[F(x)\Pi_\pi(y)]\rho_p(y) = \mathrm{Tr}^E[F(x)(\mathrm{ev}_y^{\pi^{-1}(x)})^*\,\mathrm{ev}_y^{\pi^{-1}(x)}\,\rho_{E_p}(x)]. \tag{2.27}$$

Via formula (1.9), this implies in particular that for any $f \in \mathscr{C}^\infty(M,\mathbb{R})$, we have

$$\begin{aligned}&\langle f, \mathrm{Tr}^E[(F\circ\pi)\Pi_\pi]\rangle_{W_p} \\ &\quad = \int_{x\in X}\int_{y\in\pi^{-1}(x)} f(y)\,\mathrm{Tr}^{E_p}[(\mathrm{ev}_y^{\pi^{-1}(x)})^*\,\mathrm{ev}_y^{\pi^{-1}(x)}\,\rho_{E_p}(x)F(x)]d\nu_{\pi^{-1}(x)}(y)d\nu_X(x) \\ &\quad = \langle T_\pi(f), F\rangle_{W_{E_p}}. \end{aligned} \tag{2.28}$$

This concludes the proof. □

2.3. *Berezin–Toeplitz quantization of vector bundles*

Let (E, h^E) be a holomorphic Hermitian vector bundle over X, and fix a smooth volume form $d\nu_X$ on X. Recall that for any $p \in \mathbb{N}$, we set $E_p := E \otimes L^p$ with induced Hermitian metric h^{E_p}, and we write $\mathscr{H}_p$ for the space of holomorphic sections of E_p, endowed with the L_2-Hermitian product $\langle\cdot,\cdot\rangle_p$ defined by formula (2.12). In this section, we discuss the semi-classical properties of the Berezin–Toeplitz quantization of (E, h^E) over $(X, p\omega)$ as $p \to +\infty$.

The following basic result first shows that Definition 2.8 coincides with the usual Berezin–Toeplitz quantization of vector bundles of [23].

Proposition 2.13. *For any $F \in \mathscr{C}^\infty(X, \mathrm{Herm}(E_p))$, the restriction to $\mathscr{H}_p \subset \mathscr{C}^\infty(X, E_p)$ of the operator acting on L_2-sections of E_p by the formula*

$$T_{E_p}(F) := P_{E_p} F P_{E_p}, \tag{2.29}$$

where $P_{E_p} : \mathscr{C}^\infty(X, E_p) \to \mathscr{H}_p$ is the orthogonal projection on holomorphic sections with respect to $\langle\cdot,\cdot\rangle_p$ and where F acts pointwise on smooth sections $\mathscr{C}^\infty(X, E_p)$, coincides with the Berezin–Toeplitz quantization of Definition 2.8.

Proof. For any $s_1, s_2 \in \mathscr{H}_p$, the Berezin–Toeplitz quantization of $F \in \mathscr{C}^\infty(X, \mathrm{Herm}(E_p))$ satisfies

$$\begin{aligned}\langle T_{E_p}(F)s_1, s_2\rangle_p &= \int_X \langle (\mathrm{ev}_x^E)^* F(x)\, \mathrm{ev}_x^E\, s_1, s_2\rangle_p d\nu_X(x)\\ &= \int_X h^{E_p}(F(x)s_1(x), s_2(x))d\nu_X(x)\\ &= \langle P_{E_p} F P_{E_p} s_1, s_2\rangle_p. \end{aligned} \tag{2.30}$$

This gives formula (2.29). □

Let $\nabla^{\mathrm{End}(E)}$ denote the Chern connection of $\mathrm{End}(E)$ endowed with the Hermitian metric induced by h^E, and for any $m \in \mathbb{N}$, write $|\cdot|_{\mathscr{C}^m}$ for the associated local $\mathscr{C}^m$-norm on $\mathscr{C}^\infty(X, \mathrm{End}(E))$. Write $\overline{\partial}$ and ∂ for the $(0,1)$ and $(1,0)$-parts of $\nabla^{\mathrm{End}(E)}$, and recall that we write g^{TX} for the Riemannian metric (1.10) on X.

Theorem 2.14 ([24, Theorem 0.3]). *For any $F \in \mathscr{C}^\infty(X, \mathrm{Herm}(E))$, we have*

$$\|T_{E_p}(F)\|_{op} \xrightarrow{p\to+\infty} |F|_{\mathscr{C}^0}. \tag{2.31}$$

Furthermore, for any $F, G \in \mathscr{C}^\infty(X, \mathrm{Herm}(E))$, we have the following estimate in operator norm as $p \to +\infty$,

$$[T_{E_p}(F), T_{E_p}(G)] = T_{E_p}([F,G]) + \frac{\sqrt{-1}}{2\pi p} T_{E_p}(C(F,G)) + O(p^{-2}), \tag{2.32}$$

with

$$C(F,G) := \sqrt{-1}(\langle \partial F, \overline{\partial} G\rangle_{g^{TX}} - \langle \partial G, \overline{\partial} F\rangle_{g^{TX}}). \tag{2.33}$$

This rest of the section is dedicated to the statements of refined semi-classical properties of the Berezin–Toeplitz quantization of (E, h^E) over $(X, p\omega)$ as $p \to +\infty$, taken from [7, 24], which lie at the core of the applications of quantization in stages in the next sections. Writing $\pi_j : X \times X \to X$, with $j = 1, 2$, for the first and second projections, and setting

$$E_p \boxtimes E_p^* := \pi_1^* E_p \otimes \pi_2^* E_p^* \tag{2.34}$$

as a holomorphic Hermitian vector bundle over $X \times X$, these refined semi-classical properties involve the following fundamental notion.

Definition 2.15. The *Bergman kernel* of (E_p, h^{E_p}) over X is the Schwartz kernel of the orthogonal projection $P_{E_p} : \mathscr{C}^\infty(X, E_p) \to \mathscr{H}_p$ with respect to the L_2-Hermitian product (2.12), characterized as a section $P_{E_p}(\cdot,\cdot) \in \mathscr{C}^\infty(X \times X, E_p \boxtimes E_p^*)$ for any $s \in \mathscr{C}^\infty(X, E_p)$ and $x \in X$ by the formula

$$P_{E_p} s(x) := \int_X P_{E_p}(x,y)s(y)d\nu_X(y). \tag{2.35}$$

For all $x \in X$ and $p \in \mathbb{N}$, recall that we write $\mathrm{ev}_{E_x}^* : E_{p,x} \to \mathscr{H}_p$ for the dual of the evaluation map (2.13) with respect to h^{E_p} and $\langle\cdot,\cdot\rangle_p$.

Lemma 2.16. *For any $p \in \mathbb{N}, y \in X$ and $v \in E_{p,y}$, the holomorphic section $\mathrm{ev}_{E_y}^* \cdot v \in \mathscr{H}_p$ satisfies the following formula, for all $x \in X$,*

$$\mathrm{ev}_{E_y}^* \cdot v(x) = P_{E_p}(x,y) \cdot v. \tag{2.36}$$

Furthermore, for any $p \in \mathbb{N}$, the Rawnsley section $\rho_{E_p} \in \mathscr{C}^\infty(X, \mathrm{Herm}(E_p))$ of Definition 2.10 *satisfies the following formula, for any $x \in X$,*

$$\rho_{E_p}(x) = P_{E_p}(x,x). \tag{2.37}$$

Proof. By definition, the dual of the evaluation (2.13) at $y \in X$ is characterized by the following formula, for all $v \in E_{p,y}$ and $s \in \mathscr{H}_p$,

$$\langle s, \mathrm{ev}_{E_y}^* \cdot v \rangle_p = h^{E_p}(s(y), v). \tag{2.38}$$

On the other hand, using the characterization (2.35) of the Bergman kernel as well as the formula $P(x,y) = P(y,x)^*$, holding for the Schwartz kernel of any self-adjoint operator, we have

$$\begin{aligned} \int_X h^{E_p}(s(x), P_{E_p}(x,y) \cdot v) d\nu_X(x) &= \int_X h^{E_p}(P_{E_p}(y,x)s(x), v) d\nu_X(x) \\ &= h^{E_p}(P_{E_p} s(y), v) = h^{E_p}(s(y), v). \end{aligned} \tag{2.39}$$

This proves formula (2.36). This readily implies formula (2.37) by Definition 2.10 of the Rawnsley section, as for all $x \in X$ and $v \in E_{p,x}$ we have

$$\rho_{E_p}(x) \cdot v = \mathrm{ev}_{E_x} \mathrm{ev}_{E_x}^* \cdot v = \mathrm{ev}_{E_x}^* \cdot v(x) = P_{E_p}(x,x) \cdot v. \tag{2.40}$$

□

Let us assume from now on that ν_X is the Liouville measure. Note that there is no loss in generality with this assumption, as one can always accommodate this change by multiplying the Hermitian metric h^E by a scalar function. Such an operation leaves unchanged the induced Hermitian metric on $\mathrm{End}(E)$, so that all the results of this section and the next are valid without any modification in the general case.

For any holomorphic Hermitian vector bundle (E, h^E), recall that we write $R^E \in \Omega^2(X, \mathrm{End}(E))$ for the curvature of its Chern connection. Let $K_X = \det(T^{(1,0)}X)^*$ be the *canonical line bundle* of X endowed with the Hermitian metric h^{K_X} induced by g^{TX}. The *Ricci form* $\mathrm{Ric}(\omega) \in \Omega^2(X, \mathbb{R})$ of (X, J, ω) is defined by the formula

$$\mathrm{Ric}(\omega) := -\sqrt{-1} R^{K_X}, \tag{2.41}$$

and the *scalar curvature* $\mathrm{scal}(\omega) \in \mathscr{C}^\infty(X, \mathbb{R})$ of g^{TX} can be defined by the formula

$$\mathrm{scal}(\omega) := \langle \omega, \mathrm{Ric}(\omega) \rangle_{g^{TX}}. \tag{2.42}$$

Theorem 2.17 ([7, Theorem 1.3]). *There exist Hermitian endomorphisms $b_0, b_1, b_2 \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ such that for any $m \in \mathbb{N}$, there exists $C_m > 0$ and*

$l \in \mathbb{N}$ *such that for all* $p \in \mathbb{N}$ *big enough,*

$$\left| \frac{1}{p^n} \rho_{E_p} - \left(b_0 + \frac{1}{p} b_1 + \frac{1}{p^2} b_2 \right) \right|_{\mathscr{C}^m} \leq \frac{C_m}{p^3}, \tag{2.43}$$

uniformly in the $\mathscr{C}^m$*-norm of the derivatives of* h *and* h^E *up to order* l*. Furthermore, we have*

$$b_0 = \mathrm{Id}_E \quad \textit{and} \quad b_1 = \frac{\mathrm{scal}(\omega)}{8\pi} \mathrm{Id}_E + \frac{\sqrt{-1}}{2\pi} \langle \omega, R^E \rangle_{g^{TX}}. \tag{2.44}$$

Lemma 2.16 and Theorem 2.17 imply in particular that as $p \to +\infty$, we have

$$\dim \mathscr{H}_p = \int_X \mathrm{Tr}^{E_p}[\rho_{E_p}(x)] d\nu_X(x) = p^n \, \mathrm{Vol}_h(X) \, \mathrm{rk}(E) + O(p^{n-1}), \tag{2.45}$$

which can also be seen as a consequence of the classical Hirzebruch–Riemann–Roch formula. For any $F \in \mathscr{C}^\infty(X, \mathrm{Herm}(E))$, consider the operator $T_{E_p}(F) := P_{E_p} F P_{E_p}$ acting on $\mathscr{C}^\infty(X, E_p)$ as in Proposition 2.13, so that it coincides with the Berezin–Toeplitz quantization of F of Definition 2.8 when restricted to $\mathscr{H}_p$, and write $T_{E_p}(F)(\cdot,\cdot) \in \mathscr{C}^\infty(X \times X, E_p \boxtimes E_p^*)$ for its Schwartz kernel. By the basic composition formula for operators with smooth Schwartz kernels, for all $x, y \in X$, we have

$$T_{E_p}(F)(x,y) = \int_X P_{E_p}(x,w) F(w) P_{E_p}(w,y) d\nu_X(w). \tag{2.46}$$

Let $\langle \cdot, \cdot \rangle_{L^2}$ be the L_2-Hermitian product on $\mathrm{End}(E)$ induced by h^E and the Liouville measure ν_X, write $\| \cdot \|_{L_2}$ for the associated norm and write $L_2(X, \mathrm{End}(E))$ for the induced space of square-integrable sections of $\mathrm{End}(E)$. The *Bochner Laplacian* Δ is the second-order differential operator characterized for any $F_1, F_2 \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ with support in a local chart by the formula

$$\langle \Delta F_1, F_2 \rangle_{L_2} = \sum_{j=1}^{2n} \int_X \langle \nabla^{\mathrm{End}(E)} F_1(x), \nabla^{\mathrm{End}(E)} F_2(x) \rangle_{\mathrm{End}(E) \otimes T^*X} d\nu_X(x), \tag{2.47}$$

where the pairing on $\mathrm{End}(E) \otimes T^*X$ with values in $\mathbb{C}$ is the one induced by h^E and g^{TX}.

Theorem 2.18 ([24, Theorem 0.1, (0.13)]). *For any* $F \in \mathscr{C}^\infty(X, \mathrm{End}(E))$*, there exist Hermitian endomorphisms* $b_0(F), b_1(F) \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ *such that for any* $m \in \mathbb{N}$*, there exists* $C_m > 0$ *and* $l \in \mathbb{N}$ *such that for any* $x \in X$ *and all* $p \in \mathbb{N}$ *big enough,*

$$\left| \frac{1}{p^n} T_{E_p}(F)(x,x) - \left(F(x) + \frac{1}{p} b_1(F)(x) + \frac{1}{p^2} b_2(F)(x) \right) \right|_{\mathscr{C}^m} \leq \frac{C_m}{p^3}, \tag{2.48}$$

uniformly in the $\mathscr{C}^m$*-norm of the derivatives of* F, h *and* h^E *up to order* l*. Furthermore, we have*

$$b_1(F) = \frac{\mathrm{scal}(\omega)}{8\pi} F + \frac{\sqrt{-1}}{4\pi} (\langle \omega, R^E \rangle_{g^{TX}} F + F \langle \omega, R^E \rangle_{g^{TX}}) - \frac{1}{4\pi} \Delta F, \tag{2.49}$$

where $\langle \cdot, \cdot \rangle_{g^{TX}}$ *denotes the pairing on* $\Omega^2(X, \mathrm{End}(E))$ *with values in* $\mathrm{End}(E)$ *induced by* g^{TX}.

Finally, in case (E, h^E) *is the trivial line bundle, for all* $f \in \mathscr{C}^\infty(X, \mathbb{R})$ *we have*

$$b_2(f) = b_2 f + \frac{\Delta^2}{32\pi^2} f - \frac{\mathrm{scal}(\omega)}{32\pi^2} \Delta f - \frac{\sqrt{-1}}{8\pi^2} \langle \mathrm{Ric}(\omega), \partial\bar{\partial} f \rangle_{g^{TX}}, \tag{2.50}$$

where $b_2 \in \mathscr{C}^\infty(X, \mathbb{C})$ *is as in Theorem* 2.17 *for* $E = \mathbb{C}$.

In the context of Sec. 2.2, it is more natural to consider instead the Laplacian (1.12), whose relation with the Bochner Laplacian (2.47) is given by the following *Weitzenböck formula*, which can be found in [4, Proposition 1.2].

Proposition 2.19. *For any* $F \in \mathscr{C}^\infty(X, \mathrm{End}(E))$, *the following identity holds*,

$$\Box F = \Delta F - \sqrt{-1}[\langle \omega, R^E \rangle_{g^{TX}}, F]. \tag{2.51}$$

3. Spectral Estimates for Berezin Transforms

In this section, we use the setting of quantization in stages developed in Sec. 2 to extend the study of Berezin–Toeplitz quantization from the point of view of quantum measurement in [18] to the case of vector bundles. In particular, we introduce a natural notion of a Berezin transform in this context, and establish asymptotic estimates as $p \to +\infty$ on its *spectral gap* in the manner of [18]. We then apply these estimates to Donaldson's iterations toward ν-balanced metrics on stable vector bundles.

3.1. *Berezin transform on vector bundles*

Recall the notations of Sec. 2.3. In this section, we introduce the *Berezin transform* in the context of quantization in stages, which is a key tool for the study of the quantum-classical correspondence for Berezin–Toeplitz quantization. The main goal of this section is to give a proof of Theorem 1.2.

Recall from Proposition 2.11 that we write $T^*_{E_p} : \mathrm{Herm}(\mathscr{H}_p) \to \mathscr{C}^\infty(X, \mathrm{Herm}(E))$ for the dual of the Berezin–Toeplitz quantization map $T_{E_p} : \mathscr{C}^\infty(X, \mathrm{Herm}(E)) \to \mathrm{Herm}(\mathscr{H}_p)$ with respect to the scalar product (2.22).

Definition 3.1. The *Berezin transform* of (E, h^E) over $(X, p\omega)$ is the linear operator acting on $\mathscr{C}^\infty(X, \mathrm{Herm}(E))$ defined by

$$\mathcal{B}_{E_p} := T^*_{E_p} \circ T_{E_p} : \mathscr{C}^\infty(X, \mathrm{Herm}(E)) \to \mathscr{C}^\infty(X, \mathrm{Herm}(E)). \tag{3.1}$$

Definition 3.1 naturally extends the definition of a Berezin transform given in [18, §2] in the case of $E = \mathbb{C}$. In particular, it retains most of the essential properties of a *Markov operator*. Namely, it is by definition a positive self-adjoint operator with respect to the- L_2-scalar product (2.22), and since it factorizes through the finite-dimensional vector space $\mathscr{H}_p$, it has finite image, so that it admits a discrete spectrum inside $[0, +\infty)$ and its positive eigenvalues have finite multiplicity.

The following proposition describes the behavior of the Berezin transform under quantization in stages, where (E, h^E) over X is the quantum-classical hybrid of a prequantized Kähler fibration $\pi : (M, \omega^M) \to (X, \omega)$ as in Definition 2.11.

Proposition 3.2. *The Berezin transform of Definition* 3.1 *is characterized by the following formula, for all* $f \in \mathscr{C}^\infty(M, \mathbb{R})$,

$$\langle \mathcal{B}_{E_p} T_\pi(f), T_\pi(f) \rangle_{W_{E_p}} = \langle \mathcal{B}_p f, f \rangle_{W_p}, \tag{3.2}$$

where $\mathcal{B}_p : \mathscr{C}^\infty(M, \mathbb{R}) \to \mathscr{C}^\infty(M, \mathbb{R})$ *is the Berezin transform of* $\mathscr{L} \otimes \pi^* L^p$ *over* M, *and* $T_\pi(f) \in \mathscr{C}^\infty(X, \mathrm{Herm}(E_p))$ *is the Berezin–Toeplitz quantization of* $f \in \mathscr{C}^\infty(M, \mathbb{R})$ *in the fibers of* $\pi : M \to X$.

Proof. This is a straightforward consequence of Proposition 2.9 and of the definition of $T^*_{E_p}$ as the dual of T_{E_p} with respect to the L_2-Hermitian product (2.22). □

The following basic result explains the role played by the asymptotic expansions of Theorems 2.17 and 2.18 in the proof of Theorem 1.1.

Proposition 3.3. *The Berezin transform of Definition* 3.1 *satisfies the following formula, for all* $F \in \mathscr{C}^\infty(X, \mathrm{Herm}(E))$ *and* $x \in X$,

$$\mathcal{B}_{E_p}(F)(x) \rho_{E_p}(x) = T_{E_p}(F)(x, x). \tag{3.3}$$

Proof. Recall the definition (2.22) of the L_2-scalar product $\langle \cdot, \cdot \rangle_{W_{E_p}}$. By Proposition 2.13 and Definition 3.1, using $P_{E_p} P_{E_p} = P_{E_p}$ and the basic trace formula for operators with smooth Schwartz kernels, for any $F_1, F_2 \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ we have

$$\begin{aligned} &\langle \mathcal{B}_{E_p}(F_1) \rho_{E_p}, F_2 \rangle_{W_{E_p}} \\ &\quad = \langle \mathcal{B}_{E_p}(F_1), \rho_{E_p} F_2 \rangle_{W_{E_p}} \\ &\quad = \langle\langle T_{E_p}(F_1), T_{E_p}(\rho_{E_p} F_2) \rangle\rangle \\ &\quad = \mathrm{Tr}^{\mathscr{H}_p}[P_{E_p} F_1 P_{E_p} \rho_{E_p} F_2] \\ &\quad = \int_X \int_X \mathrm{Tr}^{E_p}[P_{E_p}(x, y) F_1(y) P_{E_p}(y, x) \rho_{E_p}(x) F_2(x)] d\nu_X(x) d\nu_X(y) \\ &\quad = \int_X \mathrm{Tr}^{E_p}[T_{E_p}(F_1)(x, x) \rho_{E_p}(x) F_2(x)] d\nu_X(x). \end{aligned} \tag{3.4}$$

This proves formula (3.3). □

Let us now describe the proof of Theorem 1.1, following the proof of the analogous result for the scalar Berezin transform in [18, §3]. The following proposition is an extension of the refined Karabegov–Schlichenmaier formula of [18, Proposition 3.8].

Proposition 3.4. *For any $m \in \mathbb{N}$, there exists $l \in \mathbb{N}$ and a constant $C_m > 0$, uniform in the $\mathscr{C}^m$-norm of the derivatives of h^L and h^E up to order l, such that for any $F \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ and all $p \in \mathbb{N}$ big enough, we have*

$$\left|\mathcal{B}_{E_p}F - F + \frac{\square}{4\pi p}F\right|_{\mathscr{C}^m} \leq \frac{C_m}{p^2}|F|_{\mathscr{C}^{m+4}}. \tag{3.5}$$

Proof. Applying Theorems 2.17 and 2.18 to Proposition 3.3, and following the proof of the analogous result for $E = \mathbb{C}$ in [18, Proposition 3.8], we deduce that for all $m \in \mathbb{N}$, there is $C_m > 0$ such that for all $f \in \mathscr{C}^\infty(X, \mathrm{End}(E))$, we have

$$\left|\mathcal{B}_{E_p}F - F - \frac{1}{p}D_2 F\right|_{\mathscr{C}^m} \leq p^{-2}C_m|F|_{\mathscr{C}^{m+4}}\ , \tag{3.6}$$

where D_2 is a second-order differential operator satisfying

$$\begin{aligned} D_2F &= b_1(F) - Fb_1 \\ &= \frac{\sqrt{-1}}{4\pi}(\langle\omega, R^E\rangle_{g^{TX}}F + F\langle\omega, R^E\rangle_{g^{TX}}) - \frac{1}{4\pi}\Delta F - \frac{\sqrt{-1}}{2\pi}F\langle\omega, R^E\rangle_{g^{TX}} \\ &= -\frac{1}{4\pi}\Delta F + \frac{\sqrt{-1}}{4\pi}(\langle\omega, R^E\rangle_{g^{TX}}F - F\langle\omega, R^E\rangle_{g^{TX}}). \end{aligned} \tag{3.7}$$

This gives formula (3.5) via the Weitzenböck formula of Proposition 2.19. □

Recall the increasing sequence (1.13) of eigenvalues of the Laplacian $\square$, and for all $j \in \mathbb{N}$, let $e_j \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ be the normalized eigensection associated with λ_j^E, so that

$$\|e_j\|_{L_2} = 1 \quad \text{and} \quad \square e_j = \lambda_j^E e_j. \tag{3.8}$$

For a function $\Psi : \mathbb{R} \to \mathbb{R}$ with at most polynomial growth at $+\infty$, we define the operator $\Psi(\square)$ acting on $F \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ by the formula

$$\Psi(\square)F = \sum_{i=0}^{+\infty}\Psi(\lambda_j)\langle F, e_j\rangle_{L_2}e_j. \tag{3.9}$$

Using the functional calculus (3.9), we write $\|\cdot\|_{H^m}$ for the Sobolev norm of order $m \in \mathbb{N}$, defined for any $f \in \mathscr{C}^\infty(X, \mathbb{C})$ by

$$\|F\|_{H^m} := \|\Delta_h^{m/2}F\|_{L_2} + \|F\|_{L_2}. \tag{3.10}$$

For any $t > 0$, we write $\exp(-t\square)$ for the *heat operator* associated with the Laplacian (1.12) acting on $\mathscr{C}^\infty(X, \mathrm{End}(E))$. For any $m \in \mathbb{N}$, let $\|\cdot\|_{H^m}$ be a Sobolev norm of order m on $\mathscr{C}^\infty(X, \mathrm{End}(E))$.

Proposition 3.5. *For any $m \in \mathbb{N}$, there exist $l \in \mathbb{N}$ and a constant $C_m > 0$, uniform in the $\mathscr{C}^m$-norm of the derivatives of h and h^E up to order l, such that for* *any* $F \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ *and all* $p \in \mathbb{N}$, *we have*

$$\left\| \exp\left(-\frac{\square}{4\pi p}\right) F - \mathcal{B}_p(F) \right\|_{H^m} \leq \frac{C_m}{p} \|F\|_{H^m}. \tag{3.11}$$

Proof. In the same way than in the proof of the analogous result in [18, Proposition 3.9], this readily follows from the off-diagonal expansion of the Bergman kernel of Definition 2.15 as $p \to +\infty$ given in [7, Theorem 4.18′] and the classical asymptotic expansion of heat kernels for generalized Laplacians as $t \to 0$, which can be found for example in [3, Theorem 2.29]. □

The following key result, inspired by a strategy of Lebeau and Michel in [21], allows to control the eigenvalues of $\mathcal{B}_p$ without controlling its L_2-norm.

Proposition 3.6. *For any* $L > 0$ *and* $m \in \mathbb{N}$, *there exist constants* $C_m > 0$ *and* $p_m \in \mathbb{N}$, *uniform in the* $\mathscr{C}^m$*-norm of the derivatives of* h *up to some finite order, such that for any* $p \geq p_m, \mu \in \mathbb{C}$ *and* $F \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ *satisfying*

$$\mathcal{B}_{E_p} F = \mu F \quad \textit{and} \quad p|1 - \mu| < L, \tag{3.12}$$

we have

$$\|F\|_{H^m} \leq C_m \|F\|_{L_2}. \tag{3.13}$$

Proof. Using Propositions 3.4 and 3.5, the result follows from a straightforward extension of the proof of [18, Theorem 3.1, §3.5]. □

3.2. *Kähler–Einstein case and proof of Theorem* 1.1

We now consider the scalar case $E = \mathbb{C}$, and when the Kähler metric g^{TX} defined by Eq. (1.10) is *Kähler–Einstein*, so that there exists a constant $c \in \mathbb{R}$ such that

$$\mathrm{Ric}(\omega) = c\omega, \tag{3.14}$$

where the *Ricci form* $\mathrm{Ric}(\omega) \in \Omega^2(X, \mathbb{R})$ is defined via formula (2.41). We also assume that ν_X is the Liouville measure. Recall that in the scalar case $E = \mathbb{C}$, the Laplacian (1.12) coincides with the usual Laplace–Beltrami operator Δ of (X, g^{TX}) acting on $\mathscr{C}^\infty(X, \mathbb{C})$. The first step of the proof of the refined estimate 1.16 is the following refinement of the Karabegov–Schlichenmaier formula of Proposition 3.4.

Proposition 3.7. *For any* $m \in \mathbb{N}$, *there exists* $C_m > 0$ *such that for any* $f \in \mathscr{C}^\infty(X, \mathbb{C})$ *and all* $p \in \mathbb{N}$, *we have*

$$\left| \mathcal{B}_p f - \left(1 - \frac{\Delta}{4\pi p} + \frac{\Delta^2}{32\pi^2 p^2} + c\frac{\Delta}{16\pi^2 p^2}\right) f \right|_{\mathscr{C}^m} \leq \frac{C_m}{p^3} |f|_{\mathscr{C}^{m+4}}. \tag{3.15}$$

Proof. Recall from Proposition 3.3 that for any $x \in X$ and $p \in \mathbb{N}$ big enough, we have

$$\mathcal{B}_p f(x) = \frac{T_p(f)(x,x)}{\rho_p(x)}. \tag{3.16}$$

Thus by Theorems 2.17 and 2.18 as in the proof of Proposition 3.4, we know that for all $m \in \mathbb{N}$, there is $C_m > 0$ such that

$$\left| \mathcal{B}_p f - \left(1 + \frac{D_2}{p} + \frac{D_4}{p^2}\right) f \right|_{\mathscr{C}^m} \leq \frac{C_m}{p^3} |f|_{\mathscr{C}^{m+6}}, \tag{3.17}$$

where the differential operator D_2 has been computed in Eq. (3.7) for $E = \mathbb{C}$ to satisfy

$$D_2 f = -\frac{\Delta}{4\pi p} f. \tag{3.18}$$

while the differential operator D_4 satisfies

$$\begin{aligned} D_4 f &= b_2(f) - b_2 f - b_1(b_1(f) - b_1 f) \\ &= \frac{\Delta^2}{32\pi^2} f - \frac{\mathrm{scal}(\omega)}{32\pi^2} \Delta f - \frac{\sqrt{-1}}{8\pi^2} \langle \mathrm{Ric}(\omega), \partial\overline{\partial} f \rangle_{g^{TX}} + \frac{\mathrm{scal}(\omega)}{8\pi} \frac{\Delta f}{4\pi} \\ &= \frac{\Delta^2}{32\pi^2} f - \frac{\sqrt{-1}}{8\pi^2} \langle \mathrm{Ric}(\omega), \partial\overline{\partial} f \rangle_{g^{TX}}. \end{aligned} \tag{3.19}$$

Now by definition (3.14) of a Kähler–Einstein metric and of the holomorphic Laplacian of (X, g^{TX}, J), which is half the Laplace–Beltrami operator Δ, we know that

$$\sqrt{-1} \langle \mathrm{Ric}(\omega), \partial\overline{\partial} f \rangle_{g^{TX}} = c\sqrt{-1} \langle \omega, \partial\overline{\partial} f \rangle_{g^{TX}} = -\frac{c}{2} \Delta f. \tag{3.20}$$

This gives the result. □

The crucial point in the asymptotic expansion (3.15) of the Berezin transform is the fact that it is a polynomial in the Laplace–Beltrami operator Δ. This is a consequence of the Kähler–Einstein hypothesis, and is at the core of the following proof of Theorem 1.1, which is a refinement of [18, §3.5].

Proof of Theorem 1.1. Let us start with the more involved estimate (1.16). Recall the L^2-norm $\| \cdot \|_{L_2}$ on $\mathscr{C}^\infty(X, \mathbb{C})$ induced by the Riemannian measure of g^{TX}. By Proposition 3.7 and by the classical Sobolev embedding theorem, for any $f \in \mathscr{C}^\infty(X, \mathbb{C})$ we get

$$\| p(1 - \mathcal{B}_p) f - a_p(\Delta) f \|_{L_2} \leq C p^{-2} \| f \|_{H^m}, \tag{3.21}$$

for some $m \in \mathbb{N}$ large enough, where $a_p \in \mathbb{C}[x]$ is a polynomial defined for all $p \in \mathbb{N}$ by

$$a_p(x) = \frac{x}{4\pi} - \frac{x^2 + 2cx}{32\pi^2 p}. \tag{3.22}$$

Recall now that by Definition 3.1 with $E = \mathbb{C}$, the Berezin transform $\mathcal{B}_p$ is a self-adjoint with respect to the L_2-scalar product (2.22). Writing $\| \cdot \|_{W_p}$ for the associated L_2-norm, Theorem 2.17 shows that there exists a constant $C > 0$ such that

$$\left(1 - \frac{C}{p}\right) \| \cdot \|_{L^2} \leq \| \cdot \|_{W_p} \leq \left(1 + \frac{C}{p}\right) \| \cdot \|_{L^2}. \tag{3.23}$$

This shows in particular that Eq. (3.21) also holds in the norm $\| \cdot \|_{W_p}$. Letting now $j \in \mathbb{N}$ be fixed and $e_j \in \mathscr{C}^\infty(X, \mathbb{C})$ satisfy $\Delta e_j = \lambda_j e_j$ and $\|e_j\|_{L_2} = 1$, estimate (3.21) reads

$$\|p(1 - \mathcal{B}_p)e_j - a_p(\lambda_j)e_j\|_{W_p} \leq Cp^{-2}. \tag{3.24}$$

Thus if $m_j \in \mathbb{N}$ is the multiplicity of λ_j as an eigenvalue of Δ, the estimate (3.24) for all eigenfunctions of Δ associated with λ_j gives a constant $C_j > 0$ such that for all $p \in \mathbb{N}$,

$$\#(\operatorname{Spec}(p(1 - \mathcal{B}_p)) \cap [a_p(\lambda_j) - C_j p^{-2}, a_p(\lambda_j) + C_j p^{-2}]) \geq m_j. \tag{3.25}$$

This immediately follows from the variational principle applied to the compact operator $p(1 - \mathcal{B}_p)$.

Conversely, fix $L > 0$ and let $\{f_p\}_{p \in \mathbb{N}}$ be the sequence of normalized eigenfunctions considered in Proposition 3.6 for $E = \mathbb{C}$. Then by (3.21), we get $C > 0$ such that

$$\|p(1 - \mu_p)f_p - a_p(\Delta)f_p\|_{L_2} \leq Cp^{-2}. \tag{3.26}$$

In particular, we get that

$$\operatorname{dist}(p(1 - \mu_p), \operatorname{Spec} a_p(\Delta)) \leq Cp^{-2}, \tag{3.27}$$

showing that all eigenvalues of $p(1-\mathcal{B}_p)$ bounded by some $L > 0$ have to be included in the left-hand side of (3.25).

Let us finally show that (3.25) is an equality for $p \in \mathbb{N}$ big enough. Let $l \in \mathbb{N}$ with $l \geq m_j$ be such that for all $p \in \mathbb{N}$, there exists an orthonormal family $\{f_{k,p}\}_{1 \leq k \leq l}$ of eigenfunctions of $\mathcal{B}_p$ for $\| \cdot \|_{W_p}$ with associated eigenvalues $\{\mu_{k,p} \in \mathbb{R}\}_{1 \leq k \leq l}$ satisfying

$$p(1 - \mu_{k,p}) \in [a_p(\lambda_j) - Cp^{-2}, a_p(\lambda_j) + Cp^{-2}], \quad \text{for all } 1 \leq k \leq l. \tag{3.28}$$

As the inclusion of the Sobolev space H^q in H^{q-1} is compact, by Proposition 3.6 and (3.23), there exists a subsequence of $\{f_{k,p}\}_{p \in \mathbb{N}}$ converging to a function f_k in H^{q-1}-norm, for all $1 \leq k \leq l$. In particular, taking $q > 2$ and using (3.23) again, the family $\{f_k\}_{1 \leq k \leq l}$ is orthonormal in $L_2(X, \mathbb{C})$ and satisfies $\Delta f_k = \lambda_j f_k$ for all $1 \leq k \leq l$ by (3.26). By definition of the multiplicity $m_j \in \mathbb{N}$ of λ_j and as $a_p : \mathbb{R} \to \mathbb{R}$ is strictly increasing over $[0, L]$ for all $p \in \mathbb{N}$ big enough, this forces $l = m_j$. We thus get

$$\#(\operatorname{Spec}(p(1 - \mathcal{B}_p)) \cap [a_p(\lambda_j) - Cp^{-2}, a_p(\lambda_j) + Cp^{-2}]) = m_j. \tag{3.29}$$

Using again the fact that $a_p : \mathbb{R} \to \mathbb{R}$ is increasing over $[0, L]$ for all $p \in \mathbb{N}$ big enough, so that the respective order of the eigenvalues (1.14) and (1.13) is respected, this establishes the estimate (1.16). On the other hand, the estimate (1.15) follows from Propositions 3.4 and 3.6 in the same way. □

4. Balanced Metrics

The goal of this section is to establish Theorem 1.2 on the exponential convergence of the iterations of the dynamical systems (1.17) and (1.20) toward the respective notions of balanced metrics, due to Donaldson in [9] in the scalar case and Wang in [33] in the vector bundle case. We then establish in Sec. 4.3 remarkable identities relating the dynamical systems (1.20) and (1.17) with the *moment map* for balanced embeddings used in [9, 33], recovering the lower bounds which played a crucial role in the proofs of the celebrated results of Wang [33] and Donaldson [9] on balanced embeddings.

4.1. *Vector bundle case*

Consider now E as a holomorphic vector bundle over (X, J), and fix a smooth volume form $d\nu$ over X. For any $p \in \mathbb{N}$, we write $\mathscr{H}_p$ for the space of holomorphic sections of $E_p = E \otimes L^p$. Let $\mathrm{Met}(E_p)$ be the space of Hermitian metrics on E_p and $\mathrm{Prod}(\mathscr{H}_p)$ the space of inner Hermitian products on $\mathscr{H}_p$. Then the *Hilbert map* of E_p associated with ν is defined by

$$\begin{aligned} \mathrm{Hilb}_{E_p} : \mathrm{Met}(E_p) &\to \mathrm{Prod}(\mathscr{H}_p) \\ h^{E_p} &\mapsto \frac{\dim \mathscr{H}_p}{\mathrm{Vol}(X,\nu)\,\mathrm{rk}(E)} \int_X h^{E_p}(\cdot,\cdot) d\nu. \end{aligned} \tag{4.1}$$

Recall by Kodaira's vanishing theorem that there exists $p_0 \in \mathbb{N}$ such that for all $p \geq p_0$, the evaluation map (2.13) associated with $E_p = E \otimes L^p$ is surjective for all $x \in X$. For any $p \geq p_0$, the *Fubini–Study map* associated with E_p is the map

$$\mathrm{FS} : \mathrm{Prod}(\mathscr{H}_p) \to \mathrm{Met}(E_p), \tag{4.2}$$

sending $q \in \mathrm{Prod}(\mathscr{H}_p)$ to the Hermitian metric $\mathrm{FS}(q) \in \mathrm{Met}(E_p)$ defined for any $s_1, s_2 \in \mathscr{H}_p$ by

$$\mathrm{FS}(q)(s_1(x), s_2(x)) := q(\Pi_q(x) s_1, s_2), \tag{4.3}$$

where $\Pi_q(x) \in \mathrm{Herm}(\mathscr{H}_p, q)$ is the unique orthogonal projector with respect to q satisfying

$$\mathrm{Ker}\,\Pi_q(x) = \{s \in \mathscr{H}_p \mid s(x) = 0\}. \tag{4.4}$$

Definition 4.1. *Donaldson's map* associated with E and ν is defined by

$$\mathscr{T}_{E_p} := \mathrm{Hilb}_{E_p} \circ \mathrm{FS} : \mathrm{Prod}(\mathscr{H}_p) \to \mathrm{Prod}(\mathscr{H}_p). \tag{4.5}$$

A Hermitian product $q \in \mathrm{Prod}(\mathscr{H}_p)$ is called *ν-balanced* if

$$\mathscr{T}_{E_p}(q) = q. \tag{4.6}$$

Note that for any $q \in \mathrm{Prod}(\mathscr{H}_p)$, the metric $\mathrm{FS}(q)$ tautologically coincides with the pullback h_q^{FS} of the metric induced by q via the Kodaira embedding

$X \hookrightarrow \mathbb{G}(\mathrm{rk}(E), H^0(X, E_p)^*)$. We thus recover the explicit description (1.20) for Donaldson's map $\mathscr{T}_{E_p}$.

The following lemma, which is essentially a reformulation of the language of [11], gives a first link between ν-balanced products and Berezin–Toeplitz quantization.

Lemma 4.2. *Let $q \in \mathrm{Prod}(\mathscr{H}_p)$ be a ν-balanced product, and consider the setting of Sec.* 2.3 *with $h^{E_p} := \mathrm{FS}(q)$. Then the L_2-Hermitian product* (2.12) *satisfies*

$$\frac{\dim \mathscr{H}_p}{\mathrm{Vol}(X,\nu)\,\mathrm{rk}(E)} \langle \cdot, \cdot \rangle_p = q, \tag{4.7}$$

and the Rawnsley section of Definition 2.10 *satisfies*

$$\rho_{E_p} = \frac{\dim \mathscr{H}_p}{\mathrm{Vol}(X,\nu)\,\mathrm{rk}(E)} \mathrm{Id}_E. \tag{4.8}$$

Proof. First, the identity (4.7) is a straightforward consequence of Definition 4.1. Then using (4.7), for any $s_1, s_2 \in \mathscr{H}_p$ and $x \in X$, we have

$$\begin{aligned}
\frac{\dim \mathscr{H}_p}{\mathrm{Vol}(X,\nu)\,\mathrm{rk}(E)} q(\Pi_q(x)s_1, s_2) &= \frac{\dim \mathscr{H}_p}{\mathrm{Vol}(X,\nu)\,\mathrm{rk}(E)} \mathrm{FS}(q)(\mathrm{ev}_x^E \cdot s_1, \mathrm{ev}_x^E \cdot s_2) \\
&= \frac{\dim \mathscr{H}_p}{\mathrm{Vol}(X,\nu)\,\mathrm{rk}(E)} \langle (\mathrm{ev}_x^E)^* \,\mathrm{ev}_x^E \cdot s_1, s_2 \rangle_p \\
&= q((\mathrm{ev}_x^E)^* \,\mathrm{ev}_x^E \, s_1, s_2),
\end{aligned} \tag{4.9}$$

which shows that

$$\mathrm{ev}_{E_x}^* \,\mathrm{ev}_{E_x} = \frac{\dim \mathscr{H}_p}{\mathrm{Vol}(X,\nu)\,\mathrm{rk}(E)} \Pi_q(x). \tag{4.10}$$

The identity (4.8) follows by applying ev_x^E on the right of (4.10), using the tautological formula $\mathrm{ev}_x^E \,\Pi_q(x) = \mathrm{ev}_{E_x}$ and the surjectivity of $ev_x^E : \mathscr{H}_p \to E_{p,x}$. □

Note that the first equality of formula (2.45) shows that, if the Rawnsley section is equal to a constant scalar, then it has to be given by formula (4.8). The proof of Lemma 4.2 then shows that if the identity (4.7) holds up to a multiplicative constant, then it has to hold exactly. The normalizing factor in front of the integral in the definition (4.28) of the Hilbert product is thus necessary for fixed points of Donaldson's map (4.5) to exist.

Let $q \in \mathrm{Prod}(\mathscr{H}_p)$ be a ν-balanced product, so that in particular (4.7) holds, and consider the natural identifications

$$\begin{aligned}
\mathscr{C}^\infty(X, \mathrm{Herm}(E)) &\xrightarrow{\sim} T_{\mathrm{FS}(q)} \mathrm{Met}(E_p) \\
F &\mapsto \frac{\partial}{\partial t}\Big|_{t=0} \mathrm{FS}(q)(e^{tF} \cdot, \cdot)
\end{aligned} \tag{4.11}$$

and

$$\begin{aligned} \mathrm{Herm}(\mathscr{H}_p) &\xrightarrow{\sim} T_q \,\mathrm{Prod}(\mathscr{H}_p) \\ A &\mapsto \frac{\partial}{\partial t}\Big|_{t=0} \langle e^{tA}\cdot, \cdot\rangle_p. \end{aligned} \tag{4.12}$$

The following result is essentially a reformulation of the language of [11].

Proposition 4.3. *Let $q \in \mathrm{Prod}(\mathscr{H}_p)$ be a ν-balanced product, and consider the setting of Sec.* 2.3 *with $h^{E_p} := \mathrm{FS}(q)$. Then the differential of the Hilbert map* (4.1) *at $\mathrm{FS}(q) \in \mathrm{Met}(E_p)$ is given by*

$$\begin{aligned} D_{\mathrm{FS}(q)} \,\mathrm{Hilb}_{E_p} : \mathscr{C}^\infty(X, \mathrm{Herm}(E_p)) &\to \mathrm{Herm}(\mathscr{H}_p) \\ F &\mapsto T_{E_p}(F), \end{aligned} \tag{4.13}$$

and the differential of the Fubini–Study map of (4.2) *at $q \in \mathrm{Prod}(\mathscr{H}_p)$ is given by*

$$\begin{aligned} D_q \,\mathrm{FS} : \mathrm{Herm}(\mathscr{H}_p) &\to \mathscr{C}^\infty(X, \mathrm{Herm}(E_p)) \\ A &\mapsto T^*_{E_p}(A). \end{aligned} \tag{4.14}$$

Proof. For any $F \in \mathscr{C}^\infty(X, \mathrm{Herm}(E_p))$ and $t \in \mathbb{R}$, set

$$h_t^{E_p} := h^{E_p}(e^{tF}\cdot, \cdot) \in \mathrm{Met}(E_p). \tag{4.15}$$

Then for any $s_1, s_2 \in \mathscr{H}_p$ and using Proposition 2.13, we have

$$\begin{aligned} \frac{\partial}{\partial t}\Big|_{t=0} \mathrm{Hilb}_{E_p}(h_t^{E_p})(s_1, s_2) &= \frac{\dim \mathscr{H}_p}{\mathrm{Vol}(X,\nu)\,\mathrm{rk}(E)} \int_X h^{E_p}(F(x)s_1(x), s_2(x))d\nu(x) \\ &= q(T_{E_p}(F)s_1, s_2), \end{aligned} \tag{4.16}$$

which shows (4.13).

Let us now show (4.14). For any $A \in \mathrm{Herm}(\mathscr{H}_p)$ and $t \in \mathbb{R}$, set

$$q_t := q(e^{tA}\cdot, \cdot). \tag{4.17}$$

Let us first show that

$$\frac{\partial}{\partial t}\Big|_{t=0} q_t(\Pi_{q_t}(x)s_1, s_2) = q(\Pi_q(x)A\Pi_q(x)s_1, s_2). \tag{4.18}$$

Note that for all $x \in X$, the kernel (4.4) of $\Pi_{q_t}(x)$ does not depend on $t \in \mathbb{R}$, so that both sides of formula (4.18) vanish as soon as s_1 or s_2 belongs to $\mathrm{Ker}\,\Pi_q(x)$. We thus only need to show (4.18) for s_1 and s_2 satisfying

$$s_1 = \Pi_q(x)s_1 \quad \text{and} \quad s_2 = \Pi_q(x)s_2. \tag{4.19}$$

Taking the derivative of the projection formula $\Pi_{q_t}(x) = \Pi_{q_t}(x)\Pi_{q_t}(x)$, we get

$$\left(\frac{\partial}{\partial t}\Big|_{t=0} \Pi_{q_t}(x)\right)\Pi_{q_t}(x) = (\mathrm{Id}_{\mathscr{H}_p} - \Pi_{q_t}(x))\left(\frac{\partial}{\partial t}\Big|_{t=0} \Pi_{q_t}(x)\right). \tag{4.20}$$

Then under the assumption (4.19), we get

$$\begin{aligned}
&\frac{\partial}{\partial t}\Big|_{t=0} q_t(\Pi_{q_t}(x)s_1, s_2)\\
&\quad = q(A\Pi_q(x)s_1, s_2) + q\left(\left(\frac{\partial}{\partial t}\Big|_{t=0} \Pi_{q_t}(x)\right) s_1, s_2\right)\\
&\quad = q(\Pi_q(x)A\Pi_q(x)s_1, s_2) + q\left(\Pi_q(x)\left(\frac{\partial}{\partial t}\Big|_{t=0} \Pi_{q_t}(x)\right)\Phi_q(x)s_1, s_2\right)\\
&\quad = q(\Pi_q(x)A\Pi_q(x)s_1, s_2) \qquad (4.21)
\end{aligned}$$

which shows (4.18).

Now using Definition 4.1 and Lemma 4.2, the identity (4.18) implies

$$\begin{aligned}
\frac{\partial}{\partial t}\Big|_{t=0} \mathrm{FS}(q_t)(s_1(x), s_2(x)) &= q(\Pi_q(x)A\Pi_q(x)s_1, s_2)\\
&= \frac{\dim \mathscr{H}_p}{\mathrm{Vol}(X,\nu)\,\mathrm{rk}(E)} \langle\, \mathrm{ev}_x^* \,\mathrm{ev}_x\, A \,\mathrm{ev}_x^*\, \mathrm{ev}_x\, s_1, s_2\rangle_p\\
&= \frac{\dim \mathscr{H}_p}{\mathrm{Vol}(X,\nu)\,\mathrm{rk}(E)}\, \mathrm{FS}(q)(\mathrm{ev}_x\, A\, \mathrm{ev}_x^*\, s_1(x), s_2(x)),
\end{aligned} \qquad (4.22)$$

so that to establish (4.14), it suffices to show

$$\frac{\dim \mathscr{H}_p}{\mathrm{Vol}(X,\nu)\,\mathrm{rk}(E)}\, \mathrm{ev}_x\, A\, \mathrm{ev}_x^* = T^*(A)(x). \qquad (4.23)$$

Using Definition 2.8, for any $F \in \mathscr{C}^\infty(X, \mathrm{Herm}(E_p))$ we compute

$$\mathrm{Tr}^{\mathscr{H}_p}[AT_p(F)] = \int_X \mathrm{Tr}^{\mathscr{H}_p}[A\,\mathrm{ev}_x\, F(x)\,\mathrm{ev}_x^*] d\nu(x) = \int_X \mathrm{Tr}^{E_p}[\mathrm{ev}_x^*\, A\,\mathrm{ev}_x\, F(x)] d\nu(x). \qquad (4.24)$$

By definition of the dual with respect to the L_2-scalar product (2.22) and using Lemma 4.2 again, this shows (4.23). □

Corollary 4.4. *Let $q \in \mathrm{Prod}(\mathscr{H}_p)$ be a ν-balanced product, and consider the setting of Sec.* 2.3 *with $h^{E_p} := \mathrm{FS}(q)$. Then the differential of Donaldson's map $\mathscr{T}_{E_p}$ at $q \in \mathrm{Prod}(\mathscr{H}_p)$ satisfies*

$$D_q \mathscr{T}_{E_p} = T_{E_p} \circ T_{E_p}^* : \mathrm{Herm}(\mathscr{H}_p) \to \mathrm{Herm}(\mathscr{H}_p). \qquad (4.25)$$

In particular, its positive eigenvalues with multiplicity as an endomorphism of $\mathrm{Herm}(\mathscr{H}_p)$ coincide with the positive eigenvalues of the Berezin transform associated with $h^{E_p} = \mathrm{FS}(q)$.

Proof. The identity (4.25) is an immediate consequence of Proposition 4.3. For the second statement, note that $T_{E_p}^*$ maps isomorphically the eigenspace associated with a nonzero eigenvalue of $T_{E_p}^* \circ T_{E_p}$ acting on $\mathrm{Herm}(\mathscr{H}_p)$ to the eigenspace associated with the same eigenvalue of $\mathcal{B}_p := T_{E_p} \circ T_{E_p}^*$ acting on $\mathscr{C}^\infty(X, \mathrm{Herm}(E_p))$. $\square$

Recall from [8, 32] that E is *Mumford stable* if and only if E is simple and admits a Hermitian metric h^E satisfying the Hermite–Einstein equation (1.19). For any Hermitian metric $h^{E_p} \in \mathrm{Met}(E_p)$, write $h^{L^{-p}} \otimes h^{E_p} \in \mathrm{Met}(E)$ for the Hermitian metric on E induced by h^L and h^E. We then have the following straightforward reformulation of a fundamental result of Wang in [33].

Theorem 4.5 ([33, Theorem 1.2]). *Assume that E is Mumford stable and that ν is the Liouville measure. Then there is $p_0 \in \mathbb{N}$ such that for any $p \geq p_0$, there exists a ν-balanced product $q_p \in \mathrm{Prod}(\mathscr{H}_p)$, unique up to multiplicative constant. Furthermore, there is an explicit constant $c_p > 0$ for all $p \geq p_0$ such that the following smooth convergence holds*:

$$c_p h^{L^{-p}} \otimes \mathrm{FS}(q_p) \xrightarrow{p \to +\infty} e^f h^E, \tag{4.26}$$

where $h^E \in \mathrm{Met}(E)$ satisfies the Hermite–Einstein equation (1.19) *and $f \in \mathscr{C}^\infty(X, \mathbb{R})$ satisfies $\Delta f = 2\pi\big(\mathrm{scal}(\omega) - \int_X \mathrm{scal}(\omega) \frac{d\nu}{\mathrm{Vol}(X)}\big)$.*

Note that the limit metric in (4.26) is precisely what is called a *weak Hermite–Einstein metric* by Wang in [33, Theorem 1.2], which can readily be seen by comparing the Hermite–Einstein equation (1.19) with his weak Hermite–Einstein equation [33, Theorem 1.2, (2)] and using the last equality of (3.20).

The following result on the convergence of Donaldson's iterations has then been established by Seyyedali in [30].

Theorem 4.6 ([30]). *Assume that E is Mumford stable and that ν is the Liouville measure. Then there is $p_0 \in \mathbb{N}$ such that for any $p \geq p_0$ and any $q \in \mathrm{Prod}(\mathscr{H}_p)$, there exists a ν-balanced Hermitian product $q_p \in \mathrm{Prod}(\mathscr{H}_p)$ such that*

$$\mathscr{T}_{E_p}^r(q) \xrightarrow{r \to +\infty} q_p. \tag{4.27}$$

Let us now take for (E, h^E) the trivial Hermitian line bundle. For any holomorphic line bundle L, write $\mathrm{Met}^+(L) \subset \mathrm{Met}(L)$ for the space of *positive* Hermitian metrics on L, which are the Hermitian metrics $h \in \mathrm{Met}(L)$ such that the associated 2-form $\omega_h \in \Omega^2(X, \mathbb{R})$ defined by formula (1.1) is Kähler, so that the bilinear product g_h^{TX} defined by formula (1.10) is a Riemannian metric. We write $\mathrm{Vol}_h(X) > 0$ for the Riemannian volume of (X, g_h^{TX}) and Δ_h for the associated Laplace–Beltrami operator acting on $\mathscr{C}^\infty(X, \mathbb{C})$. The *scalar Hilbert map* is the map

$$\begin{aligned} \mathrm{Hilb}_p : \mathrm{Met}^+(L^p) &\to \mathrm{Prod}(\mathscr{H}_p) \\ h_p &\mapsto \frac{\dim \mathscr{H}_p}{\mathrm{Vol}_{h_p}(X)} \int_X h_p(\cdot, \cdot) \frac{\omega_{h_p}^n}{n!}. \end{aligned} \tag{4.28}$$

Using *Kodaira's embedding theorem*, let us take $p \in \mathbb{N}$ big enough so that the Fubini–Study map (4.2) associated with L^p takes values in the space

$\mathrm{Met}^+(L^p)$ of *positive* Hermitian metrics. We then have the following analogue of Definition 4.1.

Definition 4.7. *Donaldson's map* associated with (X, L) is defined by

$$\mathscr{T}_p := \mathrm{Hilb}_p \circ \mathrm{FS} : \mathrm{Prod}(\mathscr{H}_p) \to \mathrm{Prod}(\mathscr{H}_p). \tag{4.29}$$

A Hermitian product $q \in \mathrm{Prod}(\mathscr{H}_p)$ is called *balanced* if

$$\mathscr{T}_p(q) = q, \tag{4.30}$$

that is, if $q \in \mathrm{Prod}(\mathscr{H}_p)$ is $\frac{\omega^n_{\mathrm{FS}(q)}}{n!}$-balanced in the sense of Definition 4.1.

As in Sec. 4.1, we recover the explicit description (1.17) for Donaldson's map $\mathscr{T}_p$.

Let $q \in \mathrm{Prod}(\mathscr{H}_p)$ be balanced, and recall the natural identifications (4.11) and (4.12) with $E = \mathbb{C}$.

Proposition 4.8. *Let $q \in \mathrm{Prod}(\mathscr{H}_p)$ be a balanced product, and consider the setting of Sec.* 2.1 *with $h^p := \mathrm{FS}(q)$ and $d\nu_X := \omega^n_{\mathrm{FS}(q)}/n!$. Then the differential of the Hilbert map* (4.28) *at $\mathrm{FS}(q) \in \mathrm{Met}(L^p)$ is given by*

$$\begin{aligned} D_{\mathrm{FS}(q)} \mathrm{Hilb}_p : \mathscr{C}^\infty(X, \mathbb{R}) &\to \mathrm{Herm}(\mathscr{H}_p) \\ f &\mapsto T_p\left(f + \frac{1}{4\pi}\Delta_{\mathrm{FS}(q)} f\right), \end{aligned} \tag{4.31}$$

where $\Delta_{\mathrm{FS}(q)}$ is the Laplace–Beltrami operator of $(X, g^{TX}_{\mathrm{FS}(q)})$ acting on $\mathscr{C}^\infty(X, \mathbb{C})$.

Proof. For any $f \in \mathscr{C}^\infty(X, \mathbb{R})$ and $t \in \mathbb{R}$, set

$$h_t := e^{tf} h^p \in \mathrm{Met}(L^p). \tag{4.32}$$

We will use the following classical formula of Kähler geometry,

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \frac{\omega^n_{h_t}}{n!} = \frac{1}{4\pi}\Delta_{h^p} f \, \frac{\omega^n_{h^p}}{n!}. \tag{4.33}$$

Then for any $s_1, s_2 \in \mathscr{H}_p$ and by Proposition 2.13, we have

$$\begin{aligned} &\left.\frac{\partial}{\partial t}\right|_{t=0} \mathrm{Hilb}_p(h_t)(s_1, s_2) \\ &\quad = \frac{\dim \mathscr{H}_p}{\mathrm{Vol}_{h^p}(X)}\left(\int_X \left.\frac{\partial}{\partial t}\right|_{t=0} h_t(s_1, s_2) dv_{h^p} + \int_X h^p(s_1, s_2) \left.\frac{\partial}{\partial t}\right|_{t=0} dv_{h_t}\right) \\ &\quad = \frac{\dim \mathscr{H}_p}{\mathrm{Vol}_{h^p}(X)} \int_X \left(f + \frac{1}{4\pi}\Delta_{h^p} f\right) h^p(s_1, s_2) \frac{\omega^n_{h^p}}{n!} \\ &\quad = q\left(T_p\left(f + \frac{1}{4\pi}\Delta_{h^p} f\right) s_1, s_2\right), \end{aligned} \tag{4.34}$$

which proves (4.31) taking $h^p = \mathrm{FS}(q)$. □

Corollary 4.9. *Let $q \in \mathrm{Prod}(\mathscr{H}_p)$ be a balanced product, and consider the setting of Sec.* 2.1 *with $h^p := \mathrm{FS}(q)$ and $d\nu_X := \omega^n_{\mathrm{FS}(q)}/n!$. Then the differential of Donaldson's map $\mathscr{T}_p$ at $q \in \mathrm{Prod}(\mathscr{H}_p)$ satisfies*

$$D_q\mathscr{T}_p = T_p\left(1 + \frac{1}{4\pi}\Delta_{\mathrm{FS}(q)}\right)T_p^*. \tag{4.35}$$

Proof. This is a consequence of Definition 4.7 for $\mathscr{T}_p$, together with Proposition 4.3 on the differential of the Fubini–Study map associated with L^p and Proposition 4.8 on the differential of the scalar Hilbert map. □

Let $\mathrm{Aut}(X, L)$ denote the group of holomorphic automorphisms of X lifting holomorphically to L. The relevance of the balanced products of Definition 4.7 in Kähler geometry is illustrated by the following celebrated result of Donaldson.

Theorem 4.10 ([9]). *Assume that* $\mathrm{Aut}(X, L)$ *is discrete and that there exists a positive Hermitian metric $h_\infty \in \mathrm{Met}^+(L)$ such that the induced Kähler metric g^{TX}_∞ has* constant scalar curvature. *Then there is $p_0 \in \mathbb{N}$ such that for any $p \geq p_0$, there exists a unique balanced metric g^{TX}_p associated with L^p. Furthermore, the following smooth convergence holds*:

$$\frac{1}{p}g^{TX}_p \xrightarrow{p\to+\infty} g^{TX}_\infty. \tag{4.36}$$

On the other hand, the following result on the convergence of the iterations of Donaldson's map (4.29) has been established by Donaldson in [10] and Sano in [29].

Theorem 4.11 ([10, 29]). *Under the assumptions of Theorem* 4.10, *there exist $p_0 \in \mathbb{N}$ such that for any $p \geq p_0$ and any $q \in \mathrm{Prod}(\mathscr{H}_p)$, there exists a balanced product $q_p \in \mathrm{Prod}(\mathscr{H}_p)$ such that*

$$\mathscr{T}^r_p(q) \xrightarrow{r\to+\infty} q_p. \tag{4.37}$$

4.2. *Scalar case and proof of Theorem* 1.2

Recall that in the case $E = \mathbb{C}$, the Laplacian (1.12) is the Laplace–Beltrami operator Δ_h of (X, g^{TX}_h) acting on $\mathscr{C}^\infty(X, \mathbb{C})$. Recall the L^2-Hermitian product $\langle\cdot,\cdot\rangle_{L_2}$ on $\mathscr{C}^\infty(X, \mathbb{C})$ induced by the Riemannian measure of g^{TX}_h, and let us write $\|\cdot\|_{H^m}$ for the Sobolev norm (3.10) of order $m \in \mathbb{N}$. Using the functional calculus (3.9), we define an operator on $L_2(X, \mathbb{C})$ by the formula

$$S_p := \left(1 + \frac{\Delta_h}{4\pi p}\right)^{1/2}\mathcal{B}_p\left(1 + \frac{\Delta_h}{4\pi p}\right)^{1/2}, \tag{4.38}$$

where $\mathcal{B}_p$ is the Berezin transform of Definition 3.1 for $E = \mathbb{C}$. This definition is motivated by the following lemma.

Lemma 4.12. *Let $q \in \mathrm{Prod}(\mathscr{H}_p)$ be a balanced Hermitian product, and let $h \in \mathrm{Met}^+(L)$ be defined by $h^p := h_{\mathrm{FS}(q)}$. Then S_p is a smoothing self-adjoint operator with respect to $\langle\cdot,\cdot\rangle_{L_2}$, and its positive eigenvalues coincide with the positive eigenvalues of Donaldson's map $\mathscr{T}_p$ of Definition 4.7.*

Proof. Under the assumptions of the statement, Lemma 4.2 shows that the L_2-products (1.3) and (2.22) for $E = \mathbb{C}$ satisfy

$$\langle\cdot,\cdot\rangle_{W_p} = \frac{\dim \mathscr{H}_p}{\mathrm{Vol}_h(X)}\langle\cdot,\cdot\rangle_{L_2}. \tag{4.39}$$

In particular, as $\mathcal{B}_p$ is smoothing and self-adjoint with respect to $\langle\cdot,\cdot\rangle_{W_p}$, it is also self-adjoint with respect to $\langle\cdot,\cdot\rangle_{L_2}$, so that S_p is itself a smoothing and self-adjoint operator with respect to $\langle\cdot,\cdot\rangle_{L_2}$.

Recall on the other hand that as $h^p = \mathrm{FS}(q)$, we have $p\Delta_{\mathrm{FS}(q)} = \Delta_h$. Using Corollary 4.9, this implies that $D_q\mathscr{T}_p$ is a self-adjoint operator satisfying

$$\begin{aligned} D_q\mathscr{T}_p &= T_p\left(1+\frac{\Delta_h}{4\pi p}\right)T_p^* \\ &= \left(T_p\left(1+\frac{\Delta_h}{4\pi p}\right)^{1/2}\right)\left(T_p\left(1+\frac{\Delta_h}{4\pi p}\right)^{1/2}\right)^*. \end{aligned} \tag{4.40}$$

Following the argument of the proof of Corollary 4.9, we then see that the map $\big(T_p\big(1+\frac{\Delta_h}{4\pi p}\big)^{1/2}\big)^*$ maps isomorphically the eigenspace associated with a nonzero eigenvalues of $D_q\mathscr{T}_p$ acting on $\mathrm{Herm}(\mathscr{H}_p)$ to the eigenspace associated with the same eigenvalue of S_p acting on $\mathscr{C}^\infty(X,\mathbb{C})$, which gives the result. □

Let us now consider a general $h \in \mathrm{Met}^+(L)$, and let us study the behavior of the operator S_p of (4.38) as $p \to +\infty$. Let us denote D_h for the operator (1.18) acting on $\mathscr{C}^\infty(X,\mathbb{R})$ describing the variation of the scalar curvature of g_h^{TX}.

Proposition 4.13. *For any $m \in \mathbb{N}$, there exists $C_m > 0$ and $l \in \mathbb{N}$ such that for any $f \in \mathscr{C}^\infty(X,\mathbb{C})$ and all $p \in \mathbb{N}$, we have*

$$\left\|S_p f - \left(1 - \frac{D_h}{8\pi p^2}\right)f\right\|_{H^m} \le \frac{C_m}{p^3}\|f\|_{H^{m+8}}, \tag{4.41}$$

uniformly in the $\mathscr{C}^m$-norm of the derivatives of h up to order l.

Proof. Following the proof of Proposition 3.7, we know that for all $m \in \mathbb{N}$, there is $C_m > 0$ such that

$$\left|\mathcal{B}_p f - \left(1 + \frac{D_2}{p} + \frac{D_4}{p^2}\right)f\right|_{\mathscr{C}^m} \le \frac{C_m}{p^3}|f|_{\mathscr{C}^{m+6}}, \tag{4.42}$$

where D_2 and D_4 are the second-order and fourth-order differential operators given by formulas (3.18) and (3.19). Now following e.g. [14, (5)] with our conventions, we have the following formula for the variation of scalar curvature operator,

$$D_h f = \frac{\Delta_h^2}{4\pi} f + \frac{\sqrt{-1}}{\pi} \langle \operatorname{Ric}(\omega), \partial\overline{\partial} f \rangle_{g_h}, \tag{4.43}$$

so that we get

$$D_4 = \frac{\Delta_h^2}{16\pi^2} - \frac{D_h}{8\pi}. \tag{4.44}$$

On the other hand, Δ_h commutes with $\big(1+\frac{\Delta_h}{4\pi p}\big)^{1/2}$, and by definition (3.10) of the Sobolev norms, there exists a constant $C_0 > 0$ such that, for all $f \in \mathscr{C}^\infty(X,\mathbb{R})$ and $p \in \mathbb{N}$, we have $\|f - \big(1+\frac{\Delta_h}{4\pi p}\big)^{1/2} f\|_{H^m} \leq C_0 p^{-1} \|f\|_{H^{m+2}}$. Using formulas (3.18) and (4.44), we thus get a constant $C > 0$ such that for any $f \in \mathscr{C}^\infty(X,\mathbb{R})$ and $p \in \mathbb{N}$,

$$\begin{aligned}
&\left\| \left(1+\frac{\Delta_h}{4\pi p}\right)^{1/2} \left(1+\frac{D_2}{p}+\frac{D_4}{p^2}\right) \left(1+\frac{\Delta_h}{4\pi p}\right)^{1/2} f \right\|_{H^m} \\
&\quad = \left\| f + \frac{\Delta_h^3}{64\pi^3 p^3} f - \left(1+\frac{\Delta_h}{4\pi p}\right)^{1/2} \frac{D_h}{8\pi p^2} \left(1+\frac{\Delta_h}{4\pi p}\right)^{1/2} f \right\|_{H^m} \\
&\quad \leq \left\| f - \frac{D_h}{8\pi p^2} f \right\|_{H^m} + \frac{1}{p^3} \left\| \frac{\Delta_h^3}{64\pi^3} f \right\|_{H^m} \\
&\qquad + \frac{1}{p^2} \left\| \frac{D_h}{8\pi} f - \left(1+\frac{\Delta_h}{4\pi p}\right)^{1/2} \frac{D_h}{8\pi} \left(1+\frac{\Delta_h}{4\pi p}\right)^{1/2} f \right\|_{H^m} \\
&\quad \leq \left\| f - \frac{D_h}{8\pi p^2} f \right\|_{H^m} + \frac{C}{p^3} \|f\|_{H^{m+8}}.
\end{aligned} \tag{4.45}$$

Taking the expansion (4.42) into definition (4.38) of S_p, formula (4.41) for $m = 0$ follows from the estimate (4.45). The case of general $m \in \mathbb{N}$ follows in the same way using formula (3.10) for the Sobolev norm. □

Proposition 4.14. *For any $m, k_1, k_2 \in \mathbb{N}$, there exists $C > 0$ and $l \in \mathbb{N}$ such that for any $f \in \mathscr{C}^\infty(X,\mathbb{C})$ and all $p \in \mathbb{N}$, we have*

$$\left\| \left(\frac{\Delta_h}{p}\right)^{k_1} \left(e^{-\frac{\Delta_h}{4\pi p}} - \mathcal{B}_p\right) \left(\frac{\Delta_h}{p}\right)^{k_2} f \right\|_{H^m} \leq \frac{C}{p} \|f\|_{H^m}, \tag{4.46}$$

uniformly in the $\mathscr{C}^m$-norm of the derivatives of h up to order l.

Proof. The case $m = 0$ follows from the uniformity in the estimates of the Bergman kernel of [22, Theorem 4.2.1] and the analogous result of Lu and Ma in the appendix of [13, Theorem 25] for the Q_K-operator. This proof readily extends to the case of general $m \in \mathbb{N}$, following the analogous extension in the proof of [18, Proposition 3.9]. □

Corollary 4.15. *For any $m \in \mathbb{N}$, there exists $C > 0$ and $l \in \mathbb{N}$ such that for any $f \in \mathscr{C}^\infty(X, \mathbb{C})$ and all $p \in \mathbb{N}$, we have*

$$\left\| \left(\left(1 + \frac{\Delta_h}{4\pi p} \right) e^{-\frac{\Delta_h}{4\pi p}} - S_p \right) f \right\|_{H^m} \leq \frac{C_m}{p} \|f\|_{H^m}, \tag{4.47}$$

uniformly in the $\mathscr{C}^m$-norm of the derivatives of h up to order l.

Proof. Fix $p \in \mathbb{N}$, and consider the functional calculus (3.9) with $\Psi : \mathbb{R} \to \mathbb{R}$ given by $\Psi(s) = (1 + s/4\pi p)^{-1/2}$ for $s \geq 0$ and $\Psi(s) = 0$ otherwise, so that in particular $\Psi(s) \leq 1$ for all $s \in \mathbb{R}$. Then using the elliptic estimates (3.10) and the fact that Δ_h commutes with $\Psi(\Delta_h)$, for any $f \in \mathscr{C}^\infty(X, \mathbb{C})$ and $m \in \mathbb{N}$, we get

$$\left\| \left(1 + \frac{\Delta_h}{4\pi p} \right)^{-1/2} f \right\|_{H^m} \leq \|f\|_{H^m}. \tag{4.48}$$

Now by definition (4.38) of the operator S_p, using Proposition 3.5 and the fact that any function of Δ_h commutes with the heat operator $e^{-\frac{\Delta_h}{4\pi p}}$, this implies that for all $m \in \mathbb{N}$, there exists $C_m > 0$ such that for all $f \in \mathscr{C}^\infty(X, \mathbb{C})$ and $p \in \mathbb{N}$,

$$\begin{aligned} &\left\| \left(\left(1 + \frac{\Delta_h}{4\pi p} \right) e^{-\frac{\Delta_h}{4\pi p}} - S_p \right) f \right\|_{H^m} \\ &\quad = \left\| \left(1 + \frac{\Delta_h}{4\pi p} \right)^{1/2} \left(e^{-\frac{\Delta_h}{4\pi p}} - \mathscr{B}_p \right) \left(1 + \frac{\Delta_h}{4\pi p} \right)^{1/2} f \right\|_{H^m} \\ &\quad \leq \left\| \left(1 + \frac{\Delta_h}{4\pi p} \right) \left(e^{-\frac{\Delta_h}{4\pi p}} - \mathscr{B}_p \right) \left(1 + \frac{\Delta_h}{4\pi p} \right)^{1/2} f \right\|_{H^m} \\ &\quad \leq \frac{C_m}{p} \left\| \left(1 + \frac{\Delta_h}{4\pi p} \right)^{-1/2} f \right\|_{H^m} \leq \frac{C_m}{p} \|f\|_{H^m}. \end{aligned} \tag{4.49}$$

This proves the result. □

Proposition 4.16. *For any $L > 0$ and $m \in \mathbb{N}$, there exist constants $C_m > 0$ and $p_m \in \mathbb{N}$, uniform in the $\mathscr{C}^m$-norm of the derivatives of h up to some finite order, such that for any $p \geq p_m, \mu \in \mathbb{C}$ and $f \in \mathscr{C}^\infty(X, \mathbb{C})$ satisfying*

$$S_p f = \mu f \quad \textit{and} \quad p|1 - \mu| < L, \tag{4.50}$$

we have

$$\|f\|_{H^m} \leq C_m \|f\|_{L_2}. \tag{4.51}$$

Proof. For any $p \in \mathbb{N}$, $f \in \mathscr{C}^\infty(X, \mathbb{C})$ and $\mu \in \mathbb{C}$ such that $S_p f = \mu f$, we have

$$\begin{aligned} p \left(\left(1 + \frac{\Delta_h}{4\pi p} \right) e^{-\frac{\Delta_h}{4\pi p}} - S_p \right) f &= p(1 - \mu) f - p \left(1 - \left(1 + \frac{\Delta_h}{4\pi p} \right) e^{-\frac{\Delta_h}{4\pi p}} \right) f \\ &= p(1 - \mu) f - \frac{\Delta_h}{4\pi} \Psi(\Delta_h / 4\pi p) f, \end{aligned} \tag{4.52}$$

where the bounded operator $\Psi(\Delta_h/4\pi p)$ acting on $L_2(X,\mathbb{C})$ is defined as in (3.9) for the continuous function $\Psi:\mathbb{R}\to\mathbb{R}$ given for any $s\in\mathbb{R}^*$ by

$$\Psi(s) := \frac{1-e^{-s}(1+s)}{s}. \tag{4.53}$$

Using Corollary 4.15 and formula (3.10), formula (4.52) implies that for any $L>0$ and $m\in\mathbb{N}$, there is a constant $C>0$, uniform in the $\mathscr{C}^m$-norm of the derivatives of h up to some finite order, such that for any $f\in\mathscr{C}^\infty(X,\mathbb{C})$ and $\mu\in\mathbb{C}$ satisfying (4.50), we have

$$\|\Psi(\Delta_h/4\pi p)f\|_{H^{m+2}} \leq C\|f\|_{H^m}. \tag{4.54}$$

On the other hand, using Proposition 4.14 again, we get

$$\begin{aligned}\|\Psi(\Delta_h/4\pi p)f\|_{H^m} \geq {} & \left\|\Psi(\Delta_h/4\pi p)f + \left(S_p - \left(1+\frac{\Delta_h}{4\pi p}\right)e^{-\frac{\Delta_h}{4\pi p}}\right)f\right\|_{H^m} \\ & - \left\|\left(S_p - \left(1+\frac{\Delta_h}{4\pi p}\right)e^{-\frac{\Delta_h}{4\pi p}}\right)f\right\|_{H^m} \\ & \geq \inf_{s>0}\{\Psi(s)+\mu-(1+s)e^{-s}\}\|f\|_{H^m} - C_m p^{-1}\|f\|_{H^m}.\end{aligned} \tag{4.55}$$

Combining (4.54) and (4.55), we see that for any $L>0$ and $m\in\mathbb{N}$, there exists $p_0\in\mathbb{N}$ and $C>0$, uniform in the $\mathscr{C}^m$-norm of the derivatives of h up to some finite order, such that for any $p\geq p_0$, $f\in\mathscr{C}^\infty(X,\mathbb{C})$ and $\mu\in\mathbb{C}$ satisfying (4.50), we have

$$\|f\|_{H^{m+2}} \geq C\|f\|_{H^m}. \tag{4.56}$$

This implies formula (4.51) by induction on $m\in\mathbb{N}$. □

Let now $h_\infty\in\mathrm{Met}^+(L)$ be such that $g^{TX}_{h_\infty}$ has constant scalar curvature, and recall that the associated variation of scalar curvature operator D_{h_∞} of (1.18) is then a positive elliptic self-adjoint operator. Write

$$0=\mu_0\leq\mu_1\leq\cdots\leq\mu_j\leq\cdots \tag{4.57}$$

for the increasing sequence of its eigenvalues.

On the other hand, Theorem 4.10 gives us Hermitian metrics $h_p\in\mathrm{Met}^+(L)$ for all $p\in\mathbb{N}$ big enough, such that the Kähler metrics $g^{TX}_{h_p^p}$ associated with $h_p^p\in\mathrm{Met}^+(L^p)$ are balanced and such that $h_p\to h_\infty$ as $p\to+\infty$. Then by Lemma 4.12, the smoothing operator S_p defined in (4.38) for $h_p\in\mathrm{Met}^+(L)$ is self-adjoint with respect to $\langle\cdot,\cdot\rangle_{L_2}$. Write

$$\mu_{0,p}\geq\mu_{1,p}\geq\cdots\geq\mu_{j,p}\geq\cdots\geq 0 \tag{4.58}$$

for the decreasing sequence of its eigenvalues.

Theorem 4.17. *Assume that $h_p \in \mathrm{Met}^+(L)$ is such that $g^{TX}_{h_p^p}$ is balanced, for all $p \in \mathbb{N}$ big enough. Then the eigenvalues* (4.58) *of the associated operator S_p as in* (4.38) *satisfy the following estimate as $p \to +\infty$:*

$$1 - \mu_{j,p} = \frac{\mu_j}{8\pi p^2} + o(p^{-2}). \tag{4.59}$$

Proof. By Theorem 4.10 and using the uniformity in Proposition 4.13, the Sobolev embedding theorem implies that there exists a sequence $\varepsilon_p \to 0$ as $p \to +\infty$ such that for any $f \in \mathscr{C}^\infty(X, \mathbb{C})$, we have

$$\left\| p(1 - S_p)f - \frac{D_{h_\infty}}{8\pi p} f \right\|_{L_2} < \varepsilon_p \|f\|_{H^m}, \tag{4.60}$$

for some $m \in \mathbb{N}$ large enough. For any $j \in \mathbb{N}$, write $e(\mu_j) \in \mathscr{C}^\infty(X, \mathbb{C})$ for the normalized eigenfunction associated with the jth eigenvalue of the operator D_{h_∞} as in (4.57). Then for all $p \in \mathbb{N}$, we have

$$\left\| p(1 - S_p)e(\mu_j) - \frac{D_{h_\infty}}{8\pi p} e(\mu_j) \right\|_{L_2} < \varepsilon_p. \tag{4.61}$$

Conversely, fix $L > 0$ and consider a sequence of normalized eigenfunctions $\{f_p\}_{p\in\mathbb{N}}$ of the sequence of operators $\{S_p\}_{p\in\mathbb{N}}$ associated with eigenvalues $\{\mu_p\}_{p\in\mathbb{N}}$ satisfying $p|1-\mu_p| < L$. Using the uniformity in Proposition 4.16 and by the inequality (4.60), we get a sequence $\varepsilon_p \to 0$ as $p \to +\infty$ such that

$$\left\| p(1 - \mu_p)f_p - \frac{D_{h_\infty}}{8\pi p} f_p \right\|_{L_2} < \varepsilon_p. \tag{4.62}$$

The rest of the proof is then strictly analogous to the proof of Theorem 1.1 at the end of Sec. 3.2. □

Proof of Theorem 1.2. Let us first deal with Statement (2), since Statement (1) is essentially a vector bundle version of the same argument. By [31, Definition 4.3, Lemma 4.4], if there does not exist any holomorphic vector fields over X, the kernel of D_{h_∞} is generated by the constant function. As $\mathrm{Aut}(X, L)$ is discrete, there is no holomorphic vector fields over X, and its two first eigenvalues in (4.57) satisfy

$$\mu_0 = 0 \quad \text{and} \quad \mu_1 > 0. \tag{4.63}$$

Combining Lemma 4.12 with Theorem 4.17, we then get that the differential $D_{q_p}\mathscr{T}_p$ of Donaldson's map (4.29) at a balanced product $q_p \in \mathrm{Prod}(\mathscr{H}_p)$, which satisfies $D_{q_p}\mathscr{T}_p(\mathrm{Id}_{\mathscr{H}_p}) = \mathrm{Id}_{\mathscr{H}_p}$ by definition, has a sequence of decreasing eigenvalues $\{\beta_{k,p}\}_{k\in\mathbb{N}}$ such that as $p \to +\infty$,

$$\beta_{0,p} = 1 \quad \text{and} \quad \beta_{1,p} = 1 - \frac{\mu_1}{8\pi p^2} + o(p^{-2}). \tag{4.64}$$

In particular, Eq. (4.63) implies that $\beta_{k,p} < 1$ for all $k \geq 1$ and all $p \in \mathbb{N}$ big enough. Finally, following for instance [18, Proposition 4.8], we know that the dual map T_p^* is injective for all $p \in \mathbb{N}$ big enough, while the operator $\left(1 + \frac{\Delta_p}{4\pi p}\right)^{1/2}$ is strictly positive, hence injective. Thus by Corollary 4.9, we see that $D_{q_p}\mathscr{T}_p$ is injective as well, for all $p \in \mathbb{N}$ big enough. Hence for any such $p \in \mathbb{N}$, we can apply the classical Grobman–Hartman theorem as in [18, §4] to find coordinates around q_p in $\mathrm{Prod}(\mathscr{H}_p)$ in which $\mathscr{T}_p$ coincide with its linearization $D_{q_p}\mathscr{T}_p$. Using Theorem 4.6 and formula (4.64), we then get the exponential convergence (1.21), with rate $\beta_p := \beta_{1,p}$.

To establish Statement (1), let h^E satisfy the Hermite–Einstein equation (1.19), let $f \in \mathscr{C}^\infty(X, \mathbb{R})$ satisfy $\Delta f = 2\pi\big(\mathrm{scal}(\omega) - \int_X \mathrm{scal}(\omega)\frac{d\nu}{\mathrm{Vol}(X)}\big)$, and write ∇^E for the Chern connection on $(E, e^f h^E)$. Recall that for any $F \in \mathscr{C}^\infty(X, \mathrm{End}(E))$, the induced Chern connection $\nabla^{\mathrm{End}(E)}$ on $\mathrm{End}(E)$ satisfies

$$\nabla^{\mathrm{End}(E)}F = [\nabla^E, F]. \tag{4.65}$$

Now if $F \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ satisfies $[\nabla^E, F] = 0$, then its characteristic subbundles are holomorphic subbundles of E, and as E is simple, this implies that $F = c\mathrm{Id}_E$ for some $c \in \mathbb{C}$. In particular, the kernel of the Bochner Laplacian (2.47) is 1-dimensional, generated by $\mathrm{Id}_E \in \mathscr{C}^\infty(X, \mathrm{End}(E))$. On the other hand, note that as h^E satisfies (1.19) and $f \in \mathscr{C}^\infty(X, \mathbb{R})$ satisfies $\Delta f = 2\pi\big(\mathrm{scal}(\omega) - \int_X \mathrm{scal}(\omega)\frac{d\nu}{\mathrm{Vol}(X)}\big)$, writing $R_{e^f h^E} \in \Omega^2(X, \mathrm{End}(E))$ for the Chern curvature of $(E, e^f h^E)$, we get

$$\frac{\sqrt{-1}}{2\pi}\langle \omega, R_{e^f h^E}\rangle_{g^{TX}} = \left(c + \int_X \frac{\mathrm{scal}(\omega)}{2}\frac{d\nu}{\mathrm{Vol}(X)} - \frac{\mathrm{scal}(\omega)}{2}\right)\mathrm{Id}_E. \tag{4.66}$$

Proposition 2.19 then shows that the associated Bochner Laplacian (2.47) coincides with twice the associated Kodaira Laplacian (1.12), and its first eigenvalues in (1.13) satisfy $\lambda_0^E = 0$ and $\lambda_1^E > 0$. The rest of the proof of Statement (1) then follows the same argument as the proof of Statement (2) above. □

4.3. *Moment map for balanced embeddings*

In this section, we study the behavior of the moment maps associated with *balanced embeddings*, introduced by Donaldson [9] and Wang [33] in their study of canonical metrics in complex geometry. We relate the derivative of these moment maps at a balanced embedding with the derivative of Donaldson's maps, and show how Theorem 1.2 gives a lower bound on their spectral gap, recovering the results of [19, Theorem 5; 14, Theorem 7].

Consider first the setting of Sec. 4.1, fix $p \in \mathbb{N}$ big enough and let $q \in \mathrm{Prod}(\mathscr{H}_p)$ be a ν-balanced Hermitian product in the sense of Definition 4.1. Let $\mathbb{G}(\mathrm{rk}(E), \mathscr{H}_p)$ be the Grassmanian of rank-$\mathrm{rk}(E)$ planes inside $\mathscr{H}_p$, and let us consider the natural injection

$$\begin{aligned} \mathbb{G}(\mathrm{rk}(E), \mathscr{H}_p) &\hookrightarrow \mathrm{Herm}(\mathscr{H}_p) \\ z &\mapsto \Pi_z, \end{aligned} \tag{4.67}$$

sending a rank-rk(E) plane $z \in \mathbb{G}(\mathrm{rk}(E), \mathscr{H}_p)$ to the orthogonal projection $\Pi_z \in \mathrm{Herm}(\mathscr{H}_p)$ on this plane. By Kodaira's embedding theorem, there exists $p_0 \in \mathbb{N}$ such that for all $p \geq p_0$, the evaluation map (2.13) is surjective for all $x \in X$, and induces a natural embedding

$$\mathrm{Kod}_p^E : X \to \mathbb{G}(\mathrm{rk}(E), \mathscr{H}_p)). \tag{4.68}$$

Let us write $\mathrm{GL}(\mathscr{H}_p)$ for the group of invertible endomorphisms of $\mathscr{H}_p$, and $U(\mathscr{H}_p)$ for its unitary group. The following definition is a reformulation of the *moment map* used in [33].

Definition 4.18. The *moment map for ν-balanced embeddings* is the map

$$\begin{aligned} \mu^E : \mathrm{GL}(\mathscr{H}_p)/U(\mathscr{H}_p) &\to \mathrm{Herm}(\mathscr{H}_p) \\ G &\mapsto \int_X \Pi_{Gz} d\nu(z), \end{aligned} \tag{4.69}$$

where we identified X with its image in the Grassmanian $\mathbb{G}(\mathrm{rk}(E), \mathscr{H}_p)$ under the Kodaira map (4.68).

Comparing with Definition 4.1 of $\mathscr{T}_{E_p}$ and by Definition 4.1 of a ν-balanced Hermitian product, we have

$$\mu^E(\mathrm{Id}_{\mathscr{H}_p}) = \frac{\mathrm{Vol}(X,\nu)\,\mathrm{rk}(E)}{\dim \mathscr{H}_p}\mathrm{Id}_{\mathscr{H}_p}, \tag{4.70}$$

so that the moment map at a balanced embedding is in fact equal a constant multiple of the identity. This is the characterization of a ν-balanced metric used by Wang in [33, Theorem 1.1].

Consider the identification

$$\begin{aligned} \mathrm{GL}(\mathscr{H}_p)/U(\mathscr{H}_p) &\xrightarrow{\sim} \mathrm{Prod}(\mathscr{H}_p) \\ G &\mapsto q_G := q(G\cdot, G\cdot), \end{aligned} \tag{4.71}$$

and recall the natural identification (4.12). The following result establishes a link between the differential of Donaldson's map $\mathscr{T}_{E_p}$ and the moment map μ^E at a ν-balanced product.

Proposition 4.19. *Assume that $q \in \mathrm{Prod}(\mathscr{H}_p)$ is ν-balanced. Then the differentials of the moment map and Donaldson's map at q satisfy*

$$\frac{\dim \mathscr{H}_p}{\mathrm{Vol}(X,\nu)\,\mathrm{rk}(E)} D_q\mu^E = \mathrm{Id}_{\mathscr{H}_p} - D_q\mathscr{T}_{E_p}. \tag{4.72}$$

Proof. Fix $B \in \mathrm{End}(\mathscr{H}_p)$, and $z \in \mathbb{G}(\mathrm{rk}(E), \mathscr{H}_p)$. Take $s \in \mathscr{H}_p$ such that $\Pi_z s = s$, so that $\Pi_{e^{tB}z} e^{tB}s = e^{tB}s$ for all $t \in \mathbb{R}$, and differentiating, we get

$$\left(\frac{\partial}{\partial t}\Big|_{t=0} \Pi_{e^{tB}z}\right) s = (\mathrm{Id}_{\mathscr{H}_p} - \Pi_z)Bs. \tag{4.73}$$

Take now $s^\perp \in \mathscr{H}_p$ such that $\Pi_z s^\perp = 0$, so that $\Pi_{e^{tB}z} e^{-tB} s^\perp = 0$ for all $t \in \mathbb{R}$, and differentiating, we get

$$\left(\frac{\partial}{\partial t}\Big|_{t=0} \Pi_{e^{tB}z} \right) s^\perp = \Pi_z B s^\perp. \tag{4.74}$$

By Eqs. (4.73) and (4.74), we thus get that

$$\begin{aligned} \frac{\partial}{\partial t}\Big|_{t=0} \Pi_{e^{tB}z} &= \left(\frac{\partial}{\partial t}\Big|_{t=0} \Pi_{e^{tB}z} \right) \Pi_z + \left(\frac{\partial}{\partial t}\Big|_{t=0} \Pi_{e^{tB}z} \right) (\mathrm{Id}_{\mathscr{H}_p} - \Pi_z) \\ &= B\Pi_z + \Pi_z B - 2\Pi_z B \Pi_z. \end{aligned} \tag{4.75}$$

Now consider a Hermitian endomorphism $A \in \mathrm{Herm}(\mathscr{H}_p)$ as a tangent vector in $T_q \operatorname{Prod}(\mathscr{H}_p)$. Via the differential of (4.71), it is the image of the endomorphism $B \in \mathrm{End}(\mathscr{H}_p)$ satisfying $B = A/2$. Using formulas (4.18) and (4.23), one can use Definition 4.1 and Proposition 4.3 to get

$$D_q \mathscr{T}_{E_p} \cdot A = \frac{\dim \mathscr{H}_p}{\mathrm{Vol}(X,\nu)\,\mathrm{rk}(E)} \int_X \Pi_z A \Pi_z \, d\nu(z). \tag{4.76}$$

Thus using identities (4.70) and (4.75), we get

$$\begin{aligned} D_q \mu^E \cdot A &= \frac{1}{2} A \mu^E(\mathrm{Id}_{\mathscr{H}_p}) + \frac{1}{2}\mu^E(\mathrm{Id}_{\mathscr{H}_p}) A - \int_X \Pi_z A \Pi_z \, d\nu(z) \\ &= \frac{\mathrm{Vol}(X,\nu)\,\mathrm{rk}(E)}{\dim \mathscr{H}_p} (A - D_q \mathscr{T}_{E_p} \cdot A). \end{aligned} \tag{4.77}$$

This completes the proof. □

Remark 4.20. Assume that E is Mumford stable and ν is the Liouville measure. For any $p \in \mathbb{N}$ big enough, let $q_p \in \operatorname{Prod}(\mathscr{H}_p)$ be a ν-balanced Hermitian product furnished by Theorem 4.5. Then by Theorem 1.2 and Proposition 4.19, for all $A \in \mathrm{Herm}(\mathscr{H}_p)$, we have as $p \to +\infty$,

$$\frac{\dim \mathscr{H}_p}{\mathrm{Vol}(X,\nu)} \mathrm{Tr}^{\mathscr{H}_p}[A D_{q_p} \mu^E(A)] \geq \left(\frac{\lambda_1^E}{2\pi p} + o(p^{-1}) \right) \mathrm{Tr}^{\mathscr{H}_p}[A^2]. \tag{4.78}$$

Via formula (2.45) for $\dim \mathscr{H}_p$, we then recover the result of Keller, Meyer and Seyyedali in [19, Theorem 3]. The differential of the moment map is interpreted in [19] as a quantization of the Bochner Laplacian. However, its relevance in the study of Hermite–Einstein metrics is better seen from its interpretation as the Hessian of the *energy functional* associated with this moment map problem, as defined for instance in [30, (2.3)] under the name of *Donaldson's functional*. The lower bound (4.78) thus gives an estimate on the convexity of this functional, which plays an instrumental role in the convergence results of Theorem 4.6 and gives a natural explanation for the key lower bound appearing in the work of Wang [33].

Assume now that $E = \mathbb{C}$, fix $p \in \mathbb{N}$ big enough and let $q \in \operatorname{Prod}(\mathscr{H}_p)$ be a balanced Hermitian product in the sense of Definition 4.7. Recall the identification (4.71). The following definition is a reformulation of the moment map used in [9, (1)].

Definition 4.21. The *moment map for balanced metrics* is the map

$$\mu : \mathrm{GL}(\mathscr{H}_p)/U(\mathscr{H}_p) \to \mathrm{Herm}(\mathscr{H}_p) \\ H \mapsto \int_X \Pi_{Gz} \frac{\omega^n_{\mathrm{FS}(q_G)}}{n!}(z). \tag{4.79}$$

As for the moment map for ν-balanced metrics, formula (4.7) for $\mathscr{T}_p$ implies

$$\mu(\mathrm{Id}_{\mathscr{H}_p}) = \frac{\mathrm{Vol}_{\mathrm{FS}(q)}(X)}{\dim \mathscr{H}_p} \mathrm{Id}_{\mathscr{H}_p}. \tag{4.80}$$

This is the characterization of a balanced metric used by Donaldson in [9].

The following result is the analogue of Proposition 4.19 for balanced metrics.

Proposition 4.22. *Assume that* $q \in \operatorname{Prod}(\mathscr{H}_p)$ *is balanced. Then we have*

$$\frac{\dim \mathscr{H}_p}{\mathrm{Vol}_{\mathrm{FS}(q)}(X)} D_q\mu = \mathrm{Id}_{\mathscr{H}_p} - D_q\mathscr{T}_p. \tag{4.81}$$

Proof. Fix $A \in \mathrm{Herm}(\mathscr{H}_p)$, and let $B \in \mathrm{End}(\mathscr{H}_p)$ be such that $B = A/2$ as in the proof of Proposition 4.22. Then using formulas (4.18) and (4.23) as in the proof of Proposition 4.19, we get from Corollary 4.9,

$$\begin{aligned} D_q\mu.A &= \frac{\mathrm{Vol}_{\mathrm{FS}(q)}(X)}{\dim \mathscr{H}_p} A - \int_X \Pi_z A \Pi_z \frac{\omega^n_{\mathrm{FS}(q_G)}}{n!}(z) - \int_X \Pi_z \left.\frac{\partial}{\partial t}\right|_{t=0} \frac{\omega^n_{\mathrm{FS}(q_{e^{tB}})}}{n!}(z) \\ &= \frac{\mathrm{Vol}_{\mathrm{FS}(q)}(X)}{\dim \mathscr{H}_p} \left(A - \int_X T_p^*(A)\Pi_z \frac{\omega^n_{\mathrm{FS}(q_G)}}{n!} - \frac{1}{4\pi}\int_X \Delta_{\mathrm{FS}(q)} T_p^*(A)\Pi_z \frac{\omega^n_{\mathrm{FS}(q_G)}}{n!} \right) \\ &= \frac{\mathrm{Vol}_{\mathrm{FS}(q)}(X)}{\dim \mathscr{H}_p} (A - D_q\mathscr{T}_p.A), \end{aligned} \tag{4.82}$$

where we used formulas (4.14) and (4.33) for the differential of the Fubini–Study volume form. This completes the proof. ◻

Remark 4.23. Assume that the assumptions of Theorem 4.10 are satisfied, and for any $p \in \mathbb{N}$ big enough, let $q_p \in \operatorname{Prod}(\mathscr{H}_p)$ be a balanced Hermitian product. Then by Theorem 1.2 and Proposition 4.22, for all $A \in \mathrm{Herm}(\mathscr{H}_p)$, we have as $p \to +\infty$,

$$\frac{\dim \mathscr{H}_p}{\mathrm{Vol}_{\mathrm{FS}(q_p)}(X)} \mathrm{Tr}^{\mathscr{H}_p}[A D_{q_p}\mu(A)] \geq \frac{\mu_1}{8\pi p^2} + o(p^{-2}). \tag{4.83}$$

Via formula (2.45) for $\dim \mathscr{H}_p$ and the fact that $\mathrm{Vol}_{\mathrm{FS}(q_p)} = p^n \mathrm{Vol}_h(X)$ for $h^p = \mathrm{FS}(q_p)$, we then recover the result of Fine in [14, Theorem 7], where the differential of the moment map is interpreted as the Hessian of the associated energy functional in the same way as in Remark 4.20. As explained in [14, Corollary 5], the lower bound (4.83) gives a natural explanation for the lower bound playing a key role in the works of Donaldson [9] and Phong and Sturm in [26, Theorem 2].

5. Physical Interpretation and Examples

In this section, we interpret our results on Berezin–Toeplitz quantization of vector bundles in terms of *quantum-classical hybrids* in physics, as considered for instance in [12]. In Sec. 5.1, we explain how the celebrated Stern–Gerlach experiment can be interpreted as the fundamental example of a quantum-classical hybrid, and in Sec. 5.2, we illustrate the significance of Theorem 1.1 in this context. Finally, in Sec. 5.3, we discuss the physical interpretation of Theorem 2.14 via the process of quantization in stages described in Sec. 2.2.

5.1. *Stern–Gerlach experiment as a quantum-classical hybrid*

In the physical context of quantum-classical hybrids, one only quantizes some specified degrees of freedom, while keeping the others classical. At the fully classical level, this separation of degrees of freedom is described by a symplectic fibration, as in Definition 2.5. In this context, the limiting regime when p tends to infinity is called the *weak coupling limit* in [15, §4.5]. In particular, one expects to recover the geometry of the quantum-classical hybrid from the quantization of $(M, \omega^M + p\omega)$ as $p \to +\infty$. In the case when $\pi : (M, \omega^M) \to (X, \omega)$ is a fibration of coadjoint orbits, this idea has already been applied to representation theory by Guillemin, Sternberg and Lerman in [15, §4.5]. Proposition 2.9 on the functoriality of quantization in stages naturally fits into this setting.

To describe the basic physical example of this set-up, recall that any Hermitian vector space (V, q) can be realized canonically as the Hilbert space of quantum states associated with the *projectivization* $\mathbb{P}(V^*)$ of its dual. In fact, the dual of the *tautological line bundle* of complex lines of V^* over $\mathbb{P}(V^*)$ is endowed with the tautological Hermitian metric induced by q, and formula (2.11) defines a symplectic form over $\mathbb{P}(V^*)$. On the other hand, holomorphic sections of this line bundle naturally correspond to elements $v \in V$, and the associated space of quantum states (1.2) is naturally identified with (V, q). In case V is of complex dimension 2, this space corresponds to the space of *spin*-1/2, representing *quantum angular momentum*, while the projectivization is naturally identified with the sphere S^2, representing *classical angular momentum*. The quantum number $p \in \mathbb{N}$ thus corresponds to the *spin*, and the pth tensor power of the prequantizing line bundle gives rise to the Hilbert space of spin-$p/2$, which is naturally identified with the pth symmetric $\mathrm{Sym}^p V$. Finally, the Berezin–Toeplitz quantization of the Cartesian coordinate functions of $S^2 \subset \mathbb{R}^3$ coincides up to a universal constant with the usual *spin operators* acting on $\mathrm{Sym}^p V$, also called *Pauli matrices* in the spin-1/2 case. This leads to the following fundamental example of a symplectic fibration.

Example 5.1. Given a holomorphic Hermitian vector bundle (E, h^E) over a prequantized Kähler manifold (X, J, ω), the fibration $\pi : \mathbb{P}(E^*) \to X$ obtained by the projectivization of its dual is prequantized by the dual of the *tautological line bundle* over $\mathbb{P}(E^*)$, endowed with the natural Hermitian metric induced by h^E. Then the associated quantum-classical hybrid of Definition 2.6 naturally coincides with (E, h^E) over (X, ω). The particular case of a rank-2 vector bundle gives a fibration of spheres over X, reproducing a situation of Stern-Gerlach type.

5.2. *Quantum noise in the Stern–Gerlach experiment*

From the point of view of quantum-classical hybrids, Theorem 1.1 says that the quantum noise of its Berezin–Toeplitz quantization is controlled by the spectrum of the associated Kodaira Laplacian (1.12). In this section, we describe this spectrum in various cases in the physical context of the Stern–Gerlach experiment introduced in Example 5.1.

In the case when $E = \mathbb{C}$, the Kodaira Laplacian (1.12) coincides with the Laplace–Beltrami operator Δ of (X, g^{TX}) acting on $\mathscr{C}^\infty(X, \mathbb{C})$, and for general (E, h^E), the Laplacian (1.12) can be obtained as the operator induced by the *horizontal* Laplace–Beltrami operator of $\pi : \mathbb{P}(E^*) \to X$, seen as a Kähler fibration in the sense of [5, Definition 1.4]. In many physical situations involving quantum-classical hybrids, the relevant observables $F \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ under consideration can be diagonalized in a direct sum $E = \oplus_{j=1}^m E_j$ of holomorphic Hermitian subbundles of (E, h^E). Writing $F = \sum_{j=1}^m f_j P_j$, with $f_j \in \mathscr{C}^\infty(X, \mathbb{C})$ and where $P_j \in \mathrm{End}(E)$ are the orthogonal projectors on E_j, the Laplacian (1.12) is then given by the formula

$$\Box F = \sum_{j=1}^m (\Delta f_j) P_j. \tag{5.1}$$

On the other hand, in the fundamental case of the Stern–Gerlach experiment, one considers $E = \mathbb{C}^2$ the trivial rank-2 vector bundle, which describes classical particles over X of quantized spin-$\frac{1}{2}$. Then all $F \in \mathscr{C}^\infty(X, \mathrm{End}(\mathbb{C}^2))$ can be decomposed as $F = \sum_{j=1}^3 f_j \sigma_j$, with $f_j \in \mathscr{C}^\infty(X, \mathbb{C})$ and where $\sigma_j \in \mathrm{End}(\mathbb{C}^2)$ are the standard *Pauli matrices*, for all $1 \leq j \leq 3$. In that case, the Laplacian (1.12) is again given by the formula

$$\Box F = \sum_{j=1}^3 (\Delta f_j) \sigma_j. \tag{5.2}$$

This gives a natural interpretation to the Kodaira Laplacian, and thus to the lower bound (1.15) on the quantum noise induced by the Berezin–Toeplitz quantization of the Stern–Gerlach experiment.

Example 5.2. To illustrate the relevance of Theorem 1.1 in a concrete situation, we will compute the spectrum of the Laplacian (1.12) in the case of the holomorphic

Hermitian vector bundle $E_{(k)} := \mathbb{C} \oplus L^k$ over $X = \mathbb{CP}^1$ for $k \in \mathbb{N}$, where L is the dual of the tautological line bundle equipped with the natural Fubini–Study Hermitian metric. Following [20], let us first describe the Laplacian $\Box_k$ acting on the space of sections of the holomorphic line bundle L^k over $\mathbb{C}P^1$, defined by (1.12) with L^k instead of E. Set $G = SU(2)$, and write S^1 for the maximal torus of G so that $\mathbb{C}P^1 = G/S^1$. The total space of the line bundle L^k is given by $(G \times \mathbb{C})/S^1$, where S^1 acts on $\mathbb{C}$ via the representation $\phi \mapsto e^{-ik\phi}$. The sections of L^k then identify with the functions $f : G \to \mathbb{C}$ equivariant under the action of S^1. By the Peter–Weyl theorem, $L_2(G)$ splits as $\oplus(m+1)V_m$, where $m \geq 0$ and V_m denotes the $(m+1)$-dimensional irreducible unitary representation of G. Write $e_{j,m}$ for a vector of height $j \in [-m, m]$ in V_m, with $j - m \in 2\mathbb{N}$. With this notation and the identification above, we get

$$L_2(\mathbb{CP}^1, L^k) = \bigoplus_{m \in 2\mathbb{N}+|k|} (m+1)\,\mathrm{Span}(e_{k,m}). \tag{5.3}$$

Denote by C the *Casimir operator* of G on $L_2(G)$, which acts as the scalar operator $-m(m+2)\mathrm{Id}$ on V_m, for all $m \geq 0$. By the general version of the Weitzenböck formula of Proposition 2.19, the Laplacian (5.2) on $L_2(\mathbb{CP}^1, L^k)$ is given by

$$\Box_k = -\frac{1}{2}C - \frac{k(k+2)}{2}\mathrm{Id}. \tag{5.4}$$

Setting $m = |k| + p$ for $p \in 2\mathbb{N}$, it follows that the spectrum S_k of $\Box_k$ has the following form:

For $k = 0$, the spectrum S_0 consists of eigenvalues $p(p+2)/2$ with multiplicity $p+1$, where $p \in 2\mathbb{N}$.

For $k > 0$, the spectrum S_k consists of eigenvalues $(2pk + p(p+2))/2$ with multiplicity $k + p + 1$, where $p \in 2\mathbb{N}$.

For $k < 0$, the spectrum S_k consists of eigenvalues $((2p+4)|k| + p(p+2))/2$ with multiplicity $|k| + p + 1$, where $p \in 2\mathbb{N}$.

We apply this consideration to the quantum-classical hybrid given by the Hirzebruch surface $M_k := \mathbb{P}(E_{(k)})$ over $X = \mathbb{CP}^1$. It follows that

$$\mathrm{End}(E_{(k)}) = \mathbb{C} \oplus \mathbb{C} \oplus L^k \oplus L^{-k},$$

and the Laplacian (1.12) splits as $\Box = \Box_0 \oplus \Box_0 \oplus \Box_k \oplus \Box_{-k}$. The corresponding spectrum with multiplicities is

$$\Sigma_k = 2S_0 \cup S_k \cup S_{-k}.$$

In particular, the spectra are different for different $k \in \mathbb{N}$: this can be seen by looking at the minimal positive eigenvalue of $S_k \cup S_{-k}$ which, for $k > 1$ equals $2k$. Thus, the spectrum of the Kodaira Laplacian on $\mathrm{End}(E_{(k)})$ determines the number $k \in \mathbb{N}$. The latter is a holomorphic invariant of the Hirzebruch surface M_k, for all $k \in \mathbb{N}$. Namely, the only strictly negative self-intersection of an irreducible curve on M_k equals $-k$, see [1, p. 141]. It would be interesting to understand which biholomorphic invariants of more sophisticated complex manifolds "can be heard" with the Kodaira Laplacian.

5.3. *Quantum-classical correspondence for quantum-classical hybrids*

Theorem 2.14 has a natural interpretation in the context of quantization in stages, where one considers the quantum-classical hybrid associated with a prequantized fibration $\pi : (M, \omega^M) \to (X, \omega)$ as in Definition 2.11 at the weak coupling limit, when $p \to +\infty$. To explain this point, let us consider the setting of Proposition 2.9, where $T_\pi : \mathscr{C}^\infty(M, \mathbb{R}) \to \mathscr{C}^\infty(X, \mathrm{Herm}(E))$ denotes the Berezin–Toeplitz quantization of the fibers of $\pi : (M, \omega^M) \to (X, \omega)$, and for any $f, g \in \mathscr{C}^\infty(M, \mathbb{R})$, set $F := T_\pi(f)$ and $G := T_\pi(g)$ in Theorem 2.14. For any $p \in \mathbb{N}$ big enough, write $\{\cdot,\cdot\}$ for the vertical Poisson bracket induced by ω^M in the fibers of $\pi : M \to X$, write $\{\cdot,\cdot\}_p$ for the Poisson bracket over $\mathscr{C}^\infty(M, \mathbb{R})$ induced by the symplectic form $\omega^M + p\pi^*\omega$, and write g_p^{TX} for the associated Kähler metric on M, inducing a Hermitian product $\langle\cdot,\cdot\rangle_{g_p^{TX}}$ on T^*M. Then at the weak coupling limit $p \to +\infty$, we get by definition

$$
\begin{aligned}
\{f, g\}_p &= \sqrt{-1}(\langle \partial f, \overline{\partial} g\rangle_{g_p^{TM}} - \langle \partial f, \overline{\partial} g\rangle_{g_p^{TM}})\\
&= \{f, g\} + \sqrt{-1}(\langle \partial^H f, \overline{\partial}^H g\rangle_{g_p^{TM}} - \langle \partial^H f, \overline{\partial}^H g\rangle_{g_p^{TM}})\\
&= \{f, g\} + \frac{\sqrt{-1}}{p}(\langle \partial^H f, \overline{\partial}^H g\rangle_{\omega} - \langle \partial^H f, \overline{\partial}^H g\rangle_{\omega}) + O(p^{-2})\\
&= \{f, g\} + \frac{1}{p}\pi^*\omega(\xi_f^H, \xi_g^H) + O(p^{-2}),
\end{aligned} \tag{5.5}
$$

where $d^H f = \partial^H f + \overline{\partial}^H f$ denotes the restriction to the horizontal tangent space $T^H M \subset TM$ of the fibration, where $\langle\cdot,\cdot\rangle_\omega$ denotes the Hermitian metric on $T^{H,*}M$ induced by $\pi^*\omega$ and where $\xi_f^H \in \mathscr{C}^\infty(M, T^H M)$ is defined as the unique horizontal vector field satisfying $d^H f = \pi^*\omega(\cdot, \xi_f^H)$. We now claim that the first- and second-order coefficients in the expansion (2.32) can be, respectively, interpreted as the quantization of the first and second order coefficients in the expansion (5.5). To see this, replace ω^M by $r\omega^M$ for some $r \in \mathbb{N}$ in Definition 2.11, so that the expansion (5.5) holds with $\{\cdot,\cdot\}$ replaced by $r^{-1}\{\cdot,\cdot\}$. Then Proposition 2.9 and Theorem 2.14, together with formula (2.33), imply that as $r \to +\infty$, we get

$$
\begin{aligned}
T_{E_p}([F, G]) &= \frac{\sqrt{-1}}{2\pi r} T_{E_p}(T_\pi(\{f, g\}) + O(r^{-2}) = \frac{\sqrt{-1}}{2\pi r} T_p(\{f, g\}) + O(r^{-2}),\\
T_{E_p}(C(F, G)) &= \sqrt{-1} T_{E_p}(\langle T_\pi(\partial^H f), T_\pi(\overline{\partial}^H g)\rangle_\omega - \langle T_\pi(\partial^H f), T_\pi(\overline{\partial}^H g)\rangle_\omega) + O(r^{-1})\\
&= \sqrt{-1} T_{E_p}(T_\pi(\langle \partial^H f, \overline{\partial}^H g\rangle_\omega - \langle \partial^H f, \overline{\partial}^H g\rangle_\omega)) + O(r^{-1})\\
&= T_p(\pi^*\omega(\xi_f^H, \xi_g^H)) + O(r^{-1}),
\end{aligned} \tag{5.6}
$$

where we used the fact that $\nabla^{\operatorname{End} E} T_\pi(f) = T_\pi(d^H f) + O(r^{-1})$ due to Ma and Zhang in [25, Theorem 0.8]. Theorem 2.14 thus states that the Lie bracket $[T_p(f), T_p(g)] = [T_{E_p}(F), T_{E_p}(G)]$ is a quantization of the Poisson bracket $\{\cdot,\cdot\}_p$ at the weak coupling limit when $p \to +\infty$. Note that this interpretation cannot be obtained as a consequence of Theorem 2.14 for $E = \mathbb{C}$ over $(M, \omega^M + p\pi^*\omega)$, since the 2-form $\pi^*\omega$ is degenerate along the fibers of $\pi : M \to X$, so that the limiting regime of $(M, \omega^M + p\pi^*\omega)$ as $p \to +\infty$ cannot be reduced to the usual semi-classical limit over $(M, p\widetilde{\omega})$ as $p \to +\infty$ for some symplectic form $\widetilde{\omega} \in \Omega^2(M, \mathbb{R})$.

Theorem 2.14 also shows that the Berezin–Toeplitz quantization of (E, h^E) over $(X, p\omega)$ as $p \to +\infty$ provides a deformation of the pointwise Lie bracket $[\cdot,\cdot]$ over $\mathscr{C}^\infty(X, \operatorname{Herm}(E))$, so that the coefficient (2.33) forms a degree-1 cocycle for this Lie algebra, which means that for all $F, G, H \in \mathscr{C}^\infty(X, \operatorname{Herm}(E))$, we have

$$\begin{aligned} &[C(F,G),H] + [C(H,F),G] + [C(G,H),F] \\ &\quad + C([F,G],H) + C([H,F],G) + C([G,H],F) = 0. \end{aligned} \tag{5.7}$$

In case $F, G \in \mathscr{C}^\infty(X, \operatorname{Herm}(E))$ are scalar endomorphisms in each fiber, Eq. (2.33) shows in addition that $C(F,G)$ coincides with the Poisson bracket on $\mathscr{C}^\infty(X, \mathbb{R})$ induced by ω. This is the quantum counterpart of the expansion (5.5), which provides a deformation of the vertical Poisson bracket $\{\cdot,\cdot\}$ over $\mathscr{C}^\infty(M, \mathbb{R})$. Specifically, the expansion (5.5) shows that the coefficient $c(f,g) := \pi^*\omega(\xi_f^H, \xi_g^H)$, for all $f, g \in \mathscr{C}^\infty(M, \mathbb{R})$, forms a degree-1 cocycle for this Poisson algebra, and coincides with the Poisson bracket induced by ω on the base of $\pi : M \to X$ when $f, g \in \mathscr{C}^\infty(M, \mathbb{R})$ are constant in the fibers. It would be interesting to figure out the algebraic properties of such cocycles, and deduce from them invariants of the mixed dynamics of a quantum-classical hybrid. In the following example, we describe a physically relevant situation where such mixed dynamics can be fully described.

Example 5.3 (Generalized Stern–Gerlach). Consider the physically relevant situation described in Sec. 5.2, where one considers a subalgebra of observables which are diagonal in a direct sum $E = \oplus_{j=1}^m E_j$ of holomorphic Hermitian subbundles of (E, h^E). Then the cocycle (2.33) preserves this subalgebra. Specifically, consider $F, G \in \mathscr{C}^\infty(X, \operatorname{Herm}(E))$ such that $F = \sum_{j=1}^m f_j P_j$ and $G = \sum_{j=1}^m g_j P_j$, for some $f_j, g_j \in \mathscr{C}^\infty(X, \mathbb{C})$ and $P_j \in \operatorname{End}(E)$ the orthogonal projector on E_j, for each $1 \leq j \leq m$. Then Eq. (2.33) gives

$$C(F,G) = \sum_{j=1}^m \{f_j, g_j\} P_j, \tag{5.8}$$

where $\{\cdot,\cdot\}$ denotes the usual Poisson bracket induced by ω on $\mathscr{C}^\infty(X, \mathbb{R})$. In particular, the time evolution of the observable F under the Hamiltonian flow generated by G over the quantum-classical hybrid is described by the ordinary differential equations

$$\dot{f}_j = \{g_j, f_j\}, \quad 1 \leq j \leq m. \tag{5.9}$$

Theorem 2.14 thus shows that the dynamics on the quantum-classical hybrid described by this subalgebra of observables reduces to the classical dynamics induced by a collection of Hamiltonians related to each different quantum state, represented by the corresponding eigenspace. This is precisely the case for the usual Stern–Gerlach experiment, in which case $E = X \times \mathbb{C}^2$ and a classical particle goes up or down depending on either its quantum spin is $(1,0)$ or $(0,1) \in \mathbb{C}^2$.

Remark 5.4. Let us write $(E_{(r)}, h^{E_{(r)}})$ for the quantum-classical hybrid associated with $\pi : (M, r\omega^M) \to (X, \omega)$, for every $r \in \mathbb{N}$. For the first line of (5.6), we used the fact that the Berezin–Toeplitz quantization $T_{\pi,r} : \mathscr{C}^\infty(M,\mathbb{R}) \to \mathscr{C}^\infty(X, \mathrm{Herm}(E_{(r)}))$ of the fibers of $\pi : (M, r\omega^M) \to (X,\omega)$ satisfies the quantum-classical correspondence of Theorem 2.14 for $E = \mathbb{C}$ in each fiber as $r \to +\infty$. For the second line of (5.6), we used furthermore a technical result of Ma and Zhang in [25, Theorem 0.4], where it is also shown that the trace-free part $R_0^{E_{(r)}} \in \Omega^2(X, \mathrm{End}\, E_{(r)})$ of the Chern curvature of $(E_{(r)}, h^{E_{(r)}})$ satisfies the following asymptotic expansion in the operator norm as $r \to +\infty$,

$$\frac{1}{r} R_0^{E_{(r)}} = T_{\pi,r}(\rho) + O(r^{-1}), \tag{5.10}$$

where $\rho \in \mathscr{C}^\infty(M, \Lambda^2 T^H M^*)$ is the *symplectic curvature* of the Hamiltonian fibration $\pi : (M, \omega^M) \to (X,\omega)$, as described for instance in [27]. This answers a question of Savelyev and Shelukhin in [28, Remark 7.2].

Acknowledgments

The first author was partially supported by the European Research Council Starting grant 757585 and second author was partially supported by the Israel Science Foundation grant 1102/20. The authors wish to thank the anonymous referee for helpful suggestions, which allowed great improvements in the presentation of this paper.

References

[1] W. Barth, C. Peters and A. Van de Ven, *Compact Complex Surfaces* (Springer, Heidelberg, 1984).

[2] F. A. Berezin, Quantization, *Izv. Akad. Nauk SSSR Ser. Mat.* **38** (1974) 1116–1175.

[3] N. Berline, E. Getzler and M. Vergne, *Heat Kernels and Dirac Operators*, Grundlehren Text Editions (Springer-Verlag, Berlin, 2004), corrected reprint of the 1992 original.

[4] J.-M. Bismut, Demailly's asymptotic Morse inequalities: A heat equation proof, *J. Funct. Anal.* **72** (1987) 263–278.

[5] J.-M. Bismut, H. Gillet and C. Soulé, Analytic torsion and holomorphic determinant bundles. II. Direct images and Bott–Chern forms, *Commun. Math. Phys.* **115**(1) (1988) 79–126.

[6] M. Bordemann, E. Meinrenken and M. Schlichenmaier, Toeplitz quantization of Kähler manifolds and gl(N), $N \to \infty$ limits, *Commun. Math. Phys.* **165**(2) (1994) 281–296.

[7] X. Dai, K. Liu and X. Ma, On the asymptotic expansion of Bergman kernel, *J. Differential Geom.* **72**(1) (2006) 1–41.
[8] S. K. Donaldson, Infinite determinants, stable bundles and curvature, *Duke Math. J.* **54**(1) (1987) 231–247.
[9] S. K. Donaldson, Scalar curvature and projective embeddings. I, *J. Differential Geom.* **59**(3) (2001) 479–522.
[10] S. K. Donaldson, Scalar curvature and projective embeddings. II, *Q. J. Math.* **56**(3) (2005) 345–356.
[11] S. K. Donaldson, Some numerical results in complex differential geometry, *Pure Appl. Math. Q.* **5**(2) (2009) 571–618, special Issue: In honor of Friedrich Hirzebruch. Part 1.
[12] H.-T. Elze, Linear dynamics of quantum-classical hybrids, *Phys. Rev. A* **85** (2012) 052109.
[13] J. Fine, Calabi flow and projective embeddings, *J. Differential Geom.* **84**(3) (2010) 489–523, with an appendix by Kefeng Liu and Xiaonan Ma.
[14] J. Fine, Quantization and the Hessian of Mabuchi energy, *Duke Math. J.* **161**(14) (2012) 2753–2798.
[15] V. Guillemin, E. Lerman and S. Sternberg, *Symplectic Fibrations and Multiplicity Diagrams* (Cambridge University Press, Cambridge, 1996).
[16] L. Ioos, Anticanonically balanced metrics over Fano manifolds, *Ann. Global Anal. Geom.* **62** (2022) 1–32, doi:10.1007/s10455-022-09834-4.
[17] L. Ioos, Balanced metrics for Kähler–Ricci solitons and quantized Futaki invariants, *J. Funct. Anal.* **282**(8) (2022) 109400.
[18] L. Ioos, V. Kaminker, L. Polterovich and D. Shmoish, Spectral aspects of the Berezin transform, *Ann. H. Lebesgue* **3** (2020) 1343–1387.
[19] J. Keller, J. Meyer and R. Seyyedali, Quantization of the Laplacian operator on vector bundles, I, *Math. Ann.* **366**(3–4) (2016) 865–907.
[20] K. Köhler, Equivariant analytic torsion on $P^n\mathbb{C}$, *Math. Ann.* **297** (1993) 553–565.
[21] G. Lebeau and L. Michel, Semi-classical analysis of a random walk on a manifold, *Ann. Probab.* **38**(1) (2010) 277–315.
[22] X. Ma and G. Marinescu, *Holomorphic Morse Inequalities and Bergman Kernels*, Progress in Mathematics, Vol. 254 (Birkhäuser Verlag, Basel, 2007).
[23] X. Ma and G. Marinescu, Toeplitz operators on symplectic manifolds, *J. Geom. Anal.* **18**(2) (2008) 565–611.
[24] X. Ma and G. Marinescu, Berezin–Toeplitz quantization on Kähler manifolds, *J. Reine Angew. Math.* **662** (2012) 1–56.
[25] X. Ma and W. Zhang, Superconnection and family Bergman kernels, *Math. Ann.* (2022) 1–47, doi:10.1007/s00208-022-02438-0.
[26] D. H. Phong and J. Sturm, Scalar curvature, moment maps, and the Deligne pairing, *Amer. J. Math.* **126**(3) (2004) 693–712.
[27] L. Polterovich, Gromovs K-area and symplectic rigidity, *Geom. Funct. Anal.* **6**(4) (1996) 726–739.
[28] S. Sabatini and D. Sepe, On topological properties of positive complexity one spaces, *Transform. Groups* **27**(2) (2022) 723–735.
[29] Y. Sano, Numerical algorithm for finding balanced metrics, *Osaka J. Math.* **43**(3) (2006) 679–688.
[30] R. Seyyedali, Numerical algorithm for finding balanced metrics on vector bundles, *Asian J. Math.* **13**(3) (2009) 311–321.
[31] G. Székelyhidi, *An Introduction to Extremal Kähler Metrics*, Graduate Studies in Mathematics, Vol. 152 (American Mathematical Society, Providence, RI, 2014).

[32] K. Uhlenbeck and S.-T. Yau, On the existence of Hermitian-Yang-Mills connections in stable vector bundles, *Comm. Pure Appl. Math.* **39** (1986) S257–S293, *Front. Math. Sci.* (New York, 1985).
[33] X. Wang, Canonical metrics on stable vector bundles, *Comm. Anal. Geom.* **13**(2) (2005) 253–285.